NORMAL COMPLEMENTS TO QUASICONVEX SUBGROUPS.

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ABSTRACT. Let Γ be a hyperbolic group. The normal topology on Γ is defined by taking all cosets of infinite normal subgroups as a basis. This topology is finer than the pro-finite topology, but it is not discrete. We prove that every quasiconvex subgroup $\Delta < \Gamma$ is closed in the normal topology. For a uniform lattice $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ we prove, using the tameness theorem of Agol and Calegary-Gabai, that every finitely generated subgroup of Γ is closed in the normal topology.

1. INTRODUCTION

In most infinite groups, it is virtually impossible to understand the lattice of all subgroups. Group theorists therefore focus their attention on special families of subgroups such as finite index subgroups, normal subgroups, or finitely generated subgroups. Of special interest, in the setting of word hyperbolic groups, is the family of quasiconvex subgroups.

Definition 1.1. A subgroup $\Delta < \Gamma$ of a word hyperbolic group is *quasiconvex* if a Γ geodesic between two elements of Δ stays within uniformly bounded distance from Δ.

Families of subgroups and especially the interconnections between such families, are often discussed in topological terms. If $\mathcal{N}$ is a collection of subgroups, which is invariant under conjugation and satisfies the condition

$$\text{for all } N_1, N_2 \in \mathcal{N} \ \ \exists\, N_3 \in \mathcal{N} \text{ such that } N_3 \leq N_1 \cap N_2, \tag{1}$$

then one can define an invariant topology on Γ in which the given family of groups constitutes a basis of open neighborhoods for the identity element. The most famous example, the *pro-finite* topology, is obtained by taking $\mathcal{N}$ to be the family of finite index subgroups. When the family of all infinite normal subgroups satisfies condition (1), we refer to the resulting topology as the *normal topology.*[1] The normal topology is well defined on any hyperbolic group Γ. In fact any two normal subgroups with trivial intersection commute. When the two normal subgroups are infinite one can find an infinite order element in each one, producing a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of Γ. This is impossible in a hyperbolic group.

With this terminology we can state our main theorem.

Theorem 1.2. *If Γ is a word hyperbolic group and $\Delta < \Gamma$ is a quasiconvex subgroup, then Δ is closed in the normal topology on Γ.*

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[1] In previous papers [1, 8] the normal topology is defined as the topology of all non-trivial normal subgroups. In this paper we use the slightly different notation which is especially suitable in the setting of hyperbolic groups.

In the special case where Γ is the fundamental group of a compact hyperbolic 3-orbifold, or in other words when Γ is a uniform lattice in $\mathrm{PSL}_2(\mathbb{C})$, we use Marden's tameness conjecture, recently proved by Agol [2] and by Calegari-Gabai [5], to prove:

Theorem 1.3. *Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a uniform lattice. Then every finitely generated subgroup of Γ is closed in the normal topology.*

Indeed, by the tameness theorem every finitely generated geometrically infinite subgroup of Γ is a virtual fiber, and therefore closed even in the coarser pro-finite topology. The proof of Theorem 1.3 is therefore reduced to geometrically finite subgroups. Since Γ is a uniform lattice it is word hyperbolic and the quasiconvex subgroups are the same as the geometrically finite subgroups; thus the result follows from Theorem 1.2.

Remark 1.4. Theorems 1.2 and 1.3 remain true when Γ is a non-uniform lattice in $\mathrm{PSL}_2(\mathbb{C})$. In fact once one translates the proof to the more geometric language of hyperbolic 3-manifolds it easily generalizes to the finite volume case. In an effort to keep this paper short and its style consistent, we omit the treatment of non-uniform lattices. A previous version of this paper, that deals only with hyperbolic 3-manifolds, but treats also finite volume manifolds is still available on the arXiv [10].

It is a well known question whether or not Theorem 1.3 remains true for the coarser pro-finite topology. Groups in which every finitely generated group is closed in the pro-finite topology are called *locally extended residually finite*, or *LERF* for short. The list of groups that are known to be LERF is rather short. It includes lattices in $\mathrm{PSL}_2(\mathbb{R})$ (see [14]) as well as special examples of lattices in $\mathrm{PSL}_2(\mathbb{C})$ (see [9]). The family of lattices in $\mathrm{PSL}_2(\mathbb{C})$ that are known to be LERF was significantly extended recently to include all Bianchi groups. In this case too, tameness provides the last ingredient of the proof. Separability of geometrically finite subgroups was already established by Agol, Long and Reid in [3]. It is worth noting that there are word hyperbolic groups that are not LERF (See [11]).

While the normal topology is not as popular as the pro-finite topology, it is a natural object of study. This topology captures information about all possible quotients of a given group. The normal topology of a group also plays an important role in the construction of its faithful primitive permutation representations in [8]. The main technical result of [8] is the construction of proper dense subgroups in the normal topology. These are called *pro-dense* subgroups for short. In particular it is shown that all the groups considered in this paper admit pro-dense subgroups. It was conjectured in [8, Conjecture 10.2] that proper pro-dense subgroups are never finitely generated. Theorem 1.3 above proves a strong version of this conjecture for lattices in $\mathrm{PSL}_2(\mathbb{C})$. Theorem 1.2 can considered as a first step towards the proof of the general conjecture.

For lattices in higher rank simple Lie groups the situation is drastically different. Here the normal topology coincides with the pro-finite topology by Margulis' normal subgroup theorem. Furthermore all such lattices are arithmetic by Margulis' arithmeticity theorem and therefore one can define congruence subgroups which give rise to the pro-congruence topology. The congruence subgroup property, which is proved in many cases including $\mathrm{PSL}_n(\mathbb{Z})$, implies that even this *a priori* coarser topology coincides with the first two. It is remarked by Alex Lubotzky that these groups are very far from being LERF. In fact the strong approximation theorem of Nori [13] and Weisfeiler [17] imply that if a subgroup is Zariski dense then its closure in any one of these topologies is open. Thus one can find many finitely generated subgroups that are dense.

Still one question remains open even in $\mathrm{PSL}_n(\mathbb{Z})$. Note that a maximal subgroup of infinite index is automatically dense. Margulis and Soĭfer asked if a maximal subgroup of infinite index in $\mathrm{PSL}_n(\mathbb{Z})$, $n \geq 3$, can possibly be finitely generated [12].

The same question can be asked about $\mathrm{PSL}_2(\mathbb{C})$. A maximal subgroup of infinite index is still pro-finitely dense but it need not be pro-dense. Thus the following does not follow formally from Theorem 1.3. Still the same method of proof shows

Corollary 1.5. *If $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ is a uniform lattice then a maximal subgroup of infinite index in Γ cannot be finitely generated.*

Note that the same corollary has other alternative proofs, once the tameness theorem of Agol Calegari-Gabai is assumed. For example it follows from tameness combined with the results of Arzhantseva in [4], or from tameness combined with Soĭfer's proof of Theorem 4.1 in [15].

After the proof of our main Lemma 2.1 it was mentioned to us by F. Haglund that a similar proof was sketched to him in an e-mail from T. Delzant. Still this very natural lemma does not seem to appear anywhere in the literature.

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2. QUASICONVEX SUBGROUPS.

We first state our main lemma and a corollary,

Lemma 2.1. *Let Γ be a hyperbolic group, $\Delta < \Gamma$ a quasiconvex subgroup of infinite index. Then there exists an infinite normal subgroup $N \lhd \Gamma$ such that $N \cap \Delta = \langle id \rangle$. In addition one can require that Γ/N be hyperbolic, that $N \cap B_r = \{e\}$ where B_r is a ball of a prescribed radius r, and that N will be a free group.*

Corollary 2.2. *Let Γ be a hyperbolic group and $\Delta_1, \Delta_2, \ldots, \Delta_n$ a finite number of quasiconvex subgroups. Then one can find a homomorphism onto an infinite group $\phi : \Gamma \to H$ such that $\ker \phi \neq \langle id \rangle$, but all the Δ_i's as well as a ball of any prescribed radius r are mapped injectively into H.*

Proof. Lemma 2.1 gives rise to $N_i \lhd \Gamma$ such that $N_i \cap \Delta_i = N_i \cap B_r = \langle id \rangle$. Since the normal topology is well defined on Γ we know that $\cap_{i=1\ldots n} N_i \neq \langle id \rangle$. □

Terminology. Let us fix a finite set of generators $\Gamma = \langle x_1, \ldots x_n \rangle$. The choice of generators determines a right Cayley graph $\mathcal{C}\Gamma$. The standard path metric on $\mathcal{C}\Gamma$ is δ-hyperbolic for some constant δ that will be fixed throughout. The distance from the identity in this metric will be denoted by $|\gamma| = d(\gamma, e)$. By abuse of notation, we will use the same notation for a group element $\gamma \in \Gamma$ and for the corresponding vertex in the Cayley graph $\gamma \in \mathcal{C}\Gamma$. Given a subset $S \subset \mathcal{C}\Gamma$ we will denote by $B_m(S) = \{\gamma \mid d(\gamma, S) < m\}$ its m-neighborhood. The left action of Γ on $\mathcal{C}\Gamma$ will be denoted by $\eta \mapsto \gamma \cdot \eta$. The *minimum translation length* of an element $\gamma \in \Gamma$ is defined by $\ell(\gamma) = \min\{x \in \mathcal{C}\Gamma | d(x, \gamma x)\}$.

Words and paths. Let $F_n = F(x_1, \dots, x_n)$ be the free group on the set of generators of Γ. We will refer to reduced words in F_n simply as *words*, the length of a word $w \in F_n$ will be denoted by $\|w\|$. For every choice of a vertex $\gamma \in \mathcal{C}\Gamma$ there is a one to one correspondence between words and paths without backtracking starting in the initial vertex $\mathcal{C}\Gamma$

$$\begin{array}{rcl} F_n \times \mathcal{C}\Gamma & \to & \mathrm{Paths}(\mathcal{C}\Gamma) \\ (w, \gamma) & \mapsto & \tilde{w}(\gamma). \end{array}$$

We will use the same notation for infinite rays corresponding to one sided infinite words.

Quasigeodesics. A finite path $\alpha : I \to \mathcal{C}\Gamma$ is called an (E, ϵ)-*quasigeodesic* for some $E \geq 1, \epsilon \geq 0$ if for every two points $a, b \in I$ in the interval,

$$\frac{1}{E} d(\alpha(a), \alpha(b)) - \epsilon \leq b - a \leq E d(\alpha(a), \alpha(b)) + \epsilon.$$

A one or two sided infinite path is called an (E, ϵ)-*quasigeodesic* if all of its finite subintervals are. If $w \in F_n$ is a word we say that it is an (E, ϵ)-*quasigeodesic* if one, and hence any, of its lifts to $\mathcal{C}\Gamma$ are. In a δ hyperbolic space the Hausdorff distance between two (E, ϵ)-quasigeodesics connecting the same two points (possibly boundary points) is bounded. In other words there exists a constant $K = K(\delta, E, \epsilon)$ such that if w_1, w_2 are two (E, ϵ)-quasigeodesics (possibly infinite on either side) connecting the same two points then

$$\mathrm{H.\,dist}(\tilde{w_1}(\cdot), \tilde{w_2}(\cdot)) \leq K = K(\delta, E, \epsilon). \tag{2}$$

One of the most important features of hyperbolic spaces is that being quasigeodesic is a local property. Thus if L is large enough, equation (2) holds, with a possibly larger constant $K(\delta, E, \epsilon, L)$, under the weaker assumption that w_i's are L-local (E, ϵ)-quasigeodesics. Namely that every subinterval of length $\leq L$ is an (E, ϵ)-quasigeodesic.

Cyclically reduced words. An element $\gamma \in \Gamma$ is called ϵ *close to being cyclically reduced* if it almost has minimal length within its conjugacy class. Namely $|\gamma| \leq \left|\eta\gamma\eta^{-1}\right| + \epsilon \ \forall \eta \in \Gamma$. A *cyclic conjugate* of a word $w \in F_n$ is just a cyclic permutation of its letters, or in other words the conjugate by some suffix of the word. Note that if $w \in F_n$ is a geodesic representation of an element of γ that is ϵ close to being cyclically reduced, then so is every cyclic conjugate of w. Thus the word w^N is a $\|w\|$-local quasigeodesic for every $N \in \mathbb{N}$. If $\|w\|$ is large enough, every power w^n will be quasigeodesic with quasigeodesic constants depending only on δ and ϵ (see also [7, Proposition 3.1]).

Facts about Hyperbolic groups. This section collects some lemmas on hyperbolic groups that are needed in the sequel. It is probably a good idea for the reader who is familiar with the theory to skip to the auxiliary Lemma 2.7 and refer to this section only if necessary.

Lemma 2.3. *Let Γ be a hyperbolic group $\Delta < \Gamma$ a quasiconvex subgroup of infinite index then the limit set $\mathcal{L}\Delta$ of Δ is a closed subset with empty interior in $\mathcal{L}\Gamma$.*

Proof. By definition, $\mathcal{L}\Delta$ is closed. Assume, by way of contradiction, that there exists an open subset $V \subset \mathcal{L}\Delta$. It follows from minimality of the action of Δ on $\mathcal{L}\Delta$ that $\mathcal{L}\Delta$ itself is open. Given $x \in \mathcal{L}\Delta$ there exists a $\delta \in \Delta$ such that $\delta \cdot x \in V$, and therefore $\delta^{-1}V \subset \mathcal{L}\Delta$ is an open neighborhood of x. Consequently $\overline{\Delta} = \Delta \cup \mathcal{L}\Delta$ is open in $\overline{\Gamma} = \Gamma \cup \mathcal{L}\Gamma$. By minimality of the action of Γ on $\mathcal{L}\Gamma$ we deduce that

$$\overline{\Gamma} = \bigcup_{\gamma \in \Gamma} \gamma \cdot \overline{\Delta}.$$

On the group itself this is just the coset decomposition. On the boundary, given any point $\xi \in \mathcal{L}\Gamma$ minimality provides an element $\gamma \in \Gamma$ such that $\xi \in \gamma^{-1} \cdot \mathcal{L}\Delta$. By compactness of $\overline{\Gamma}$ we pass to a finite subcover. Forgetting the boundary now, we obtain a finite cover of Γ by cosets of Δ, thus proving the lemma. □

Lemma 2.4. *There exists a constant C depending only on δ, such that if γ is a hyperbolic element of translation length $\ell(\gamma) > C$ and $l = (\gamma^-, \gamma^+)$ is an infinite geodesic line connecting its two fixed points then*

$$\big| |\gamma| - (\ell(\gamma) + 2d(e, l)) \big| < C.$$

Proof. Let $y \in \mathcal{C}\Gamma$ be a point realizing the minimal translation length, i.e. $d(y, \gamma y) = \ell(\gamma)$. And consider the infinite path $l' = \ldots [\gamma^{-1}y, y][y, \gamma y][\gamma y, \gamma^2 y] \ldots$. This is a γ invariant C-locally geodesic path from γ^- to γ^+. If we choose C large enough this will be an (E, ϵ)-quasigeodesic path for some constants (E, ϵ) that depend only on δ. Consequently $\mathrm{H.dist}(l, l') < K(\delta, E, \epsilon, C) = K(\delta)$. Note that we ignore the dependence on C because, after fixing some minimal value, increasing C can only improve K.

Every point on the path l' realizes the minimal translation distance so we obtain:

$$|\gamma| = d(e, \gamma e) \le d(e, l') + \ell + d(\gamma e, l') = 2d(e, l') + \ell \le 2d(e, l) + \ell + 2K(\delta)$$

which yields one side of the inequality once we choose $C \ge 2K(\delta)$.

Conversely assume that $y \in l'$ is such that $d(e, l') = d(e, y)$. Consider a geodesic triangle $\triangle(e, y, \gamma y)$ in $\mathcal{C}\Gamma$. By hyperbolicity there exists a point $z \in [e, \gamma y]$ that is δ-close simultaneously to both other edges. Let $v \in [e, y]$ be such that $d(v, z) \le \delta$, and therefore $d(v, [y, \gamma y]) \le d(v, z) + d(z, [y, \gamma y]) \le 2\delta$. By our choice of the point y we conclude that $d(v, y) \le 2\delta$ and hence $d(z, y) \le 3\delta$. This shows that

$$d(e, \gamma y) = d(e, z) + d(z, \gamma y) \ge d(e, y) - 3\delta + d(y, \gamma y) - 3\delta = d(e, l') + \ell - 6\delta.$$

An identical computation will show that

$$d(y, \gamma e) \ge d(e, l') + \ell - 6\delta.$$

Adding the last two equations and then using hyperbolicity on the square $(e, y, \gamma y, \gamma e)$ we conclude that:

$$\begin{aligned} 2d(e, l') + 2\ell - 12\delta &\le d(e, \gamma y) + d(y, \gamma e) \\ &\le \max\{d(e, y) + d(\gamma e, \gamma y), d(e, \gamma e) + d(y, \gamma y)\} + 2\delta \\ &= \max\{2d(e, l'), |\gamma| + \ell\} + 2\delta \end{aligned}$$

assuming that $\ell = \ell(\gamma) \ge C > 7\delta$ we can ignore the first option in the maximum function and deduce that

$$|\gamma| \ge 2d(e, l') + \ell - 14\delta \ge 2d(e, l) + \ell - 14\delta - 2K(\delta) \ge 2d(e, l) + \ell - C.$$

Where we have made the choice $C \ge 14\delta + 2K(\delta)$. This demonstrates the other side of the inequality and concludes the proof of the lemma. □

Corollary 2.5. *There exists a constant C satisfying the following. If γ is a hyperbolic element of translation length $\ell(\gamma) \geq C$, and a geodesic l connecting the fixed points of γ passes through the ball $B_r(e)$, then γ is $2r + 2C$ close to being cyclically reduced.*

Proof. Note that translation length is conjugation invariant. We take C to be the constant provided by Lemma 2.4. The latter provides an upper bound $|\gamma| \leq \ell(\gamma) + 2r + C$ for the length of γ as well as a lower bound on the length of elements in its conjugacy class $|\eta\gamma\eta^{-1}| \geq \ell(\eta\gamma\eta) - C = \ell(\gamma) - C$. The corollary follows. $\square$

Lemma 2.6. *There exists a constant C, depending only on δ satisfying the following. For any $M \in \mathbb{N}$ and any open dense subset of the boundary $V \subset \mathcal{L}\Gamma$ there exists a hyperbolic element $\gamma \in \Gamma$ such that (i) γ is C close to being cyclically reduced, (ii) $\gamma^+, \gamma^- \in V$, (iii) $\ell(\gamma) \geq M$.*

Proof. Let γ_1 be a hyperbolic element and $l_1 = (\gamma_1^-, \gamma_1^+)$ a geodesic connecting its two fixed points. Let $A^+ \subset V$ and $A^- \subset V$ be disjoint open sets that are small enough and close enough to γ_1^- and γ_1^+ so that every geodesic connecting a point in A^- to a point in A^+ passes in the ball $B_r(e)$, where r is a big enough constant to make this possible. Fix elements $\gamma', \gamma'' \in \Gamma$ satisfying $\gamma'\gamma_1^- \in A^-$ and $\gamma''\gamma_1^+ \in A^+$, and consider the element $\gamma = \gamma''\gamma_1^N\gamma'^{-1}$. By choosing N large enough we can make this element hyperbolic with arbitrarily large translation length. In particular if C_1 is the constant appearing in Corollary 2.5 we can see to it that $\ell(\gamma) \geq \max\{C_1, M\}$. Furthermore it is easy to check that when N is large the fixed points of γ will satisfy $\gamma^\pm \in A^\pm$. By our choice of the neighborhoods $A^\pm$ and by Corollary 2.5 we deduce that γ is $2r + 2C_1$ close to being cyclically reduced. We conclude by setting $C \geq 2r + 2C_1$. $\square$

An auxiliary lemma.

Lemma 2.7. *Let Γ be a hyperbolic group, $\Delta < \Gamma$ be a quasiconvex subgroup of infinite index, $j \in \mathbb{N}$ an integer and $O = B_j(\operatorname{Conv}\Delta) \subset \mathcal{C}\Gamma$ the j-neighborhood of the convex hull. Then there exists a word $w \in F_n$ such that if v is a power of a cyclic conjugate of w or of w^{-1}, then*

(1) *for every $\gamma \in \mathcal{C}\Gamma$ the path $\tilde{v}(\gamma)$ has a short intersection with O:*

$$|\tilde{v}(\gamma) \cap O| < \frac{\|v\|}{10} \qquad \forall \gamma \in \mathcal{C}\Gamma.$$

(2) *v is quasigeodesic and almost cyclically reduced, with all the constants depending only on δ.*

Proof. Fix a number $j' > j$ and set $O' = B_{j'}(\operatorname{Conv}\Delta) \subset \mathcal{C}\Gamma$; the constant j' will be determined later. Since Δ is finitely generated and quasiconvex there is a finite set of vertices K such that $O' \subset \Delta K$. After possibly enlarging K we can assume that it is symmetric (i.e. $k \in K \Rightarrow k^{-1} \in K$), and contains the identity element $e \in K$. Since Δ is quasiconvex of infinite index its limit set $\mathcal{L}\Delta$ is closed and nowhere dense in $\mathcal{L}\Gamma$ by Lemma 2.3.

The set $\bigcup_{k\in K} k \cdot (\mathcal{L}\Delta)$ is also closed and nowhere dense. Using Lemma 2.6 there is constant $C = C(\delta)$ such that we can find a hyperbolic element $\eta \in \Gamma$ which is C close to being cyclically reduced and whose attracting and repelling fixed points satisfy:

$$\{\eta^+, \eta^-\} \cap \left(\bigcup_{k \in K} k \cdot (\mathcal{L}\Delta) \right) = \emptyset.$$

We can furthermore require the translation length of η to be large enough that every power of η is an (E, ϵ)-quasigeodesic where E and ϵ depend only on δ (and on C which is itself a function of δ).

Let h be a geodesic word representing η. Consider the infinite sequences of letters $h^{+\infty} = hhh\dots$ and $h^{-\infty} = h^{-1}h^{-1}h^{-1}\dots$. Given any $\gamma \in \mathcal{C}\Gamma$ the infinite ray $\widetilde{h^{\pm\infty}}(\gamma)$ will start at the vertex $\gamma \in \mathcal{C}\Gamma$ and approach the boundary point $\gamma \cdot \eta^{\pm} \in \partial\Gamma$. By construction of η, whenever $\gamma \in K$, we have $\gamma \cdot \eta^{\pm} \notin \mathcal{L}\Delta$. We can therefore define

$$M = \max \left\{ m \in \mathbb{N} \left| \widetilde{h^m}(\gamma) \cap O' > \frac{m \, \|h\|}{20} \text{ for some } \gamma \in K \right. \right\} + 1.$$

Thus for every $m \geq M$ and for every $\gamma \in K$ at most 5% of the path $\widetilde{h^m}(\gamma)$ will pass inside O'. Replacing M by a bigger number if necessary we will also assume that $M \geq 20$. We claim that the word $w = h^M$ satisfies the conditions of the lemma.

Indeed let v be a cyclic conjugate of some power of w and assume by way of contradiction that

$$\tilde{v}(\gamma) \cap O \geq \tfrac{\|v\|}{10} \tag{3}$$

for some $\gamma \in \mathcal{C}\Gamma$. Either the word v or its inverse takes the form $v = shhhh\dots t$, where $h = ts$. As v is a power of h it will be quasigeodesic. By the observation on proximity of quasigeodesics given in equation (2) there is a bound $K(E, \epsilon, \delta)$ on the Hausdorff distance between any two (E, ϵ)-quasigeodesics connecting the same two points. Since the quasigeodesic constants for v depend only on δ, we can write $K(E, \epsilon, \delta) = K(\delta)$.

We now define $j' = j + K(\delta)$ so that if $\tilde{v}(\gamma)$ intersects O in two different points x, y then the whole segment of the path $\tilde{v}(\gamma)$ connecting x, y will be contained in O'. Thus if $\tilde{v}(\gamma) \cap O$ is large we can assume that $\tilde{v} \cap O'$ contains a long connected interval of length at least $\|v\|/10$. Replacing the word v by a cyclic conjugate, and replacing the point γ accordingly, we can assume without loss of generality that the first 10% of the path $\tilde{v}(\gamma)$ is contained in O'. Since by assumption $Mge20$ we have $\|v\| \geq 20 \cdot \|h\|$, thus replacing v again by its cyclic conjugate $s^{-1}vs$ and replacing γ accordingly we can assume that $v = h^M$ while still maintaining $|\widetilde{v}(\gamma) \cap O'| > \frac{\|v\|}{20}$. But O' is Δ invariant, so after replacing γ by some Δ translate we can assume that $\gamma \in K$. This gives a contradiction to equation (3) and completes the proof of the auxiliary lemma. Indeed, by our choice of M, for any $\gamma \in K$ we have

$$|\tilde{v}(\gamma) \cap O'| = \left| \widetilde{h^M}(\gamma) \cap O' \right| \leq \frac{M \, \|h\|}{20} = \frac{\|v\|}{20}.$$

$\square$

About the proof of Lemma 2.1. The proof uses Delzant's small cancellation theory for hyperbolic groups. Let w be the word provided by the auxiliary Lemma 2.7 and $\langle\langle w^M \rangle\rangle$. If the power is high enough then small cancellation theory applies. A typical element in $\langle\langle w^M \rangle\rangle$ assumes the form $\gamma = a_1 R_1 a_1^{-1} a_2 R_2 a_2^{-1} \dots a_l R_l a_l^{-1}$ where all the R_i's are cyclic conjugates of w^M. While this word will not be quasi-geodesic, Greendlinger's lemma assures us that large parts of some of the R_i's pass close to any equivalent geodesic word. If $\gamma \in \Delta$ this will mean that a large part of a cyclic conjugate of w^M lies close to the group Δ which stands in contradiction to the choice of w.

Proof of Lemma 2.1.

Proof. Given a quasiconvex subgroup $\Delta < \Gamma$, there is by definition a constant J such that every geodesic path connecting two points of Δ will pass inside the J-neighborhood of Δ. Choose a word $w \in F_n$ satisfying the conclusion of Lemma 2.7. We will determine the parameter j from this lemma later, taking care that our choice depends only on δ. Recall that j is the vicinity of $\mathrm{Conv}\,\Delta$ which does not contain long parts of w or its powers. For any given number m, let v_m be the element of minimal length in the conjugacy class of w^m, and let $\mathcal{R}_m$ be the collection of all cyclic conjugates of v_m and of v_m^{-1}. Note that by Lemma 2.7, w^m and all of its cyclic conjugates are quasigeodesic and almost cyclically reduced with constants that depend only on δ. Combining this with the estimate on proximity of quasigeodesics in equation (2) we conclude the following: There exists a constant $K(\delta)$ such that for every $R \in \mathcal{R}_m$ there exists a word $s \in F_n$ of length $\|s\| \leq K(\delta)$ and a cyclic conjugate w' of w, such that R and $sw'^m s^{-1}$ represent the same elements of γ, and furthermore

$$\mathrm{H.\,dist}(\tilde{R}(e), \widetilde{sw'^m s^{-1}}(e)) \leq K(\delta) \tag{4}$$

We can now specify the choice for j in Lemma 2.7

$$j = J + K(\delta) + 10\delta.$$

By [7, Theorem 3.2] there exists some very big integer M such that for all $k \in \mathbb{N}$ the set $\mathcal{R}_{kM}$ satisfies the small cancellation condition $C'(1/10k)$ as it is defined in [7, Section 2.1]. Fix some $k \in \mathbb{N}$ whose exact value will be determined later and set $v = v_{kM}$, $\mathcal{R} = \mathcal{R}_{kM}$. If k is big enough, the normal subgroup $N = \langle\langle \mathcal{R}_{kM} \rangle\rangle$ generated by $\mathcal{R}$ will satisfy all the conditions of the lemma.

Delzant's small cancellation theorem [7, Theorem 2.1] implies that Γ/N is infinite, word hyperbolic and that N is a free group. Furthermore the same theorem implies that if k is chosen so that $(1/4)\min_{R\in\mathcal{R}}(|R|) - 1000\delta \geq r$ then the ball B_r of radius r will inject into Γ/N.

The fact that $N \cap \Delta = \langle id \rangle$ follows from Delzant's version of Greendlinger's lemma. Assume by way of contradiction that $e \neq \gamma \in N \cap \Delta$. Since $\gamma \in N$ we can find a word representing γ of the form

$$u_1 = a_1 R_1 a_1^{-1} a_2 R_2 a_2^{-1} \ldots a_l R_l a_l^{-1}, \tag{5}$$

where each a_i is a word that does not contain more than half of a relation $R \in \mathcal{R}$ and each $R_i \in \mathcal{R}$. In other words each R_i is a cyclic conjugate of v or of v^{-1}. We can furthermore assume that the above presentation of γ is shortest in the sense that l is minimal. The path $\widetilde{u_1}(e)$, connecting e and γ is not quasigeodesic, however Greendlinger's Lemma [7, Lemma 2.4] combined with standard estimates in δ-hyperbolic spaces [7, Proposition 1.3.4] implies that there exists $1 \leq i_0 \leq l$, such that a portion of that path, within the section labelled R_{i_0}, of length at least $(1 - \frac{3}{10k})\,|R_{i_0}| - 300\delta$ lies in the 10δ-neighborhood of any geodesic connecting e and γ. We can always assume that k was chosen big enough so that

$$(1 - \tfrac{3}{10k})\,|R_{i_0}| - 300\delta > \tfrac{|R_{i_0}|}{2}.$$

Let s be a short word and w' a cyclic conjugate of w realizing the estimate given in equation (4). Since R_{i_0} and $sw'^{kN}s^{-1}$ represent the same element of γ, so do the two words u_1 and

$$u_1' = a_1 R_1 a_1^{-1} a_2 R_2 a_2^{-1} \dots a_{i_0} s w'^{kN_0} s^{-1} a_{i_0}^{-1} \dots a_l R_l a_l^{-1}. \tag{6}$$

More specifically the path $\widetilde{u_1'}(e)$ coincides with the path $\widetilde{u_1}(e)$ everywhere except along the section described by R_{i_0} in one word and by $hw'^{kM}h^{-1}$ in the other. Furthermore along this section the two paths are at most $K(\delta)$ apart. Thus we obtain a section of the word w'^M of length at least $\frac{|R_{i_0}|}{4}$ inside the $10\delta + K(\delta)$ vicinity of any geodesic connecting e with γ in $\mathcal{C}\Gamma$. Since both $e, \gamma \in \Delta$ this will be inside the $j = J + K(\delta) + 10\delta$ vicinity of the subgroup Δ. This contradiction to our choice of the word w concludes the proof of the lemma. □

When is a subgroup closed in the normal topology. Before beginning the proof of Theorem 1.2 let us recall why a subgroup is closed in the normal topology if and only if it is the intersection of open subgroups. One direction is obvious, an open subgroup is also closed because its complement is a union of open cosets. Conversely, applying the definition of the topology, a set $S \subseteq \Gamma$ is open if and only if

$$S = \bigcup_{N \lhd \Gamma, |N| = \infty} \ \bigcup_{\{\gamma \mid \gamma N \subset S\}} \gamma N.$$

(Pick an open neighborhood in S about each point in S to obtain the above union.) Thus Δ is closed if and only if it is of the form:

$$\Delta = \bigcap_{N \lhd \Gamma, |N| = \infty} \ \bigcup_{\{\gamma \mid \gamma N \cap \Delta \neq \emptyset\}} \gamma N,$$

When Δ is a subgroup,

$$\Delta_N \stackrel{\text{def}}{=} \bigcup_{\{\gamma \mid \gamma N \cap \Delta \neq \emptyset\}} \gamma N$$

is an open subgroup for every fixed N. Indeed if $\gamma_1 n_1, \gamma_2 n_2 \in \Delta_N$ then $(\gamma_1 n_1)^{-1} \gamma_2 n_2 \in \gamma_1^{-1} \gamma_2 N$. We can pick $n_1', n_2' \in N$ so that $\gamma_1 n_1', \gamma_2 n_2' \in \Delta$. But $(\gamma_1 n_1')^{-1} \gamma_2 n_2' \in \gamma_1^{-1} \gamma_2 N$. So the coset $\gamma_1^{-1} \gamma_2 N$ must appear in the union defining Δ_N. Finally the trivial coset N appears in Δ_N in order to account for the identity element in Δ.

Proof of Theorem 1.2.

Proof. Let Γ be a hyperbolic group and Δ a quasiconvex subgroup. We will show that Δ is closed in the normal topology.

If Δ happens to be of finite index then it is, by definition, open in the pro-finite topology which is more than what we need. We therefore restrict our attention to infinite index subgroups. In this case Δ cannot contain an infinite normal subgroup of Γ. This is because the limit set of a quasiconvex subgroup of infinite index has measure zero, whereas every infinite normal subgroup of Γ has a maximal limit set.

We will prove that

$$\Delta = \overline{\Delta} \stackrel{\text{def}}{=} \bigcap_{\{N \lhd \Gamma, N \nleq \Delta\}} \Delta N. \tag{7}$$

Assume by way of contradiction that there exists an element $\gamma \in \overline{\Delta} \setminus \Delta$. Applying Lemma 2.1, we find an infinite normal subgroup $N \lhd \Gamma$ such that $\Delta \cap N = \mathrm{id}$, and therefore $\Delta N = \Delta \ltimes N$. By the definition of $\overline{\Delta}$ we know that $\gamma \in \overline{\Delta} \leq \Delta N$. Let $\gamma = \delta n$ be the unique way of factoring γ into a product with $n \in N$ and $\delta \in \Delta$. We recall that every cyclic subgroup of a hyperbolic group is quasiconvex. Lemma 2.1 therefore gives rise to a non-trivial normal subgroup $N' \lhd \Gamma$ such that $N \cap \langle n \rangle = \langle id \rangle$. Consider the group $M = N \cap N'$. M is nontrivial, as the intersection of non-trivial normal subgroups, and it intersects Δ trivially because it is a subgroup of N. Using the definition of $\overline{\Delta}$ again, $\gamma \in \overline{\Delta} \leq \Delta M$ so we can write $\gamma = \delta' m$ with $\delta' \in \Delta, m \in M$. But by construction $n \notin M$, so $n \neq m$. This contradicts the uniqueness of the factorization $\gamma = \delta n$. □

3. Lattices in $\mathrm{PSL}_2(\mathbb{C})$

Two main theorems of hyperbolic geometry.

Theorem 3.1. *(Marden Conjecture)* [*Agol* [2], *Calegari-Gabai* [5]] *Let Δ be a discrete finitely generated torsion-free subgroup of $\mathrm{PSL}_2\,\mathbb{C}$. Then the hyperbolic manifold $\mathbb{H}^3/\Delta$ is topologically tame, i.e. it is homeomorphic to the interior of a compact manifold with boundary.*

Theorem 3.2. *(Covering Theorem)* [*Thurston* [16, Ch.9]/*Canary* [6]] *Let $\Gamma < \mathrm{PSL}_2\,\mathbb{C}$ be a torsion-free lattice. Let $\Delta < \Gamma$ be a subgroup such that $\mathbb{H}^3/\Delta$ is topologically tame. Then either*
(1) Δ is geometrically finite, or
(2) there exists a finite index subgroup $H \leq \Gamma$ and a subgroup $F \leq \Delta$ of index one or two such that H splits as a semidirect product $F \rtimes \mathbb{Z}$.

We may now combine Theorems 3.1 and 3.2 with Selberg's lemma to prove a strengthened version of the Covering Theorem for lattices with torsion.

Corollary 3.3. *Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a lattice. Let $\Delta < \Gamma$ be a finitely generated subgroup. Then either*
(1) Δ is geometrically finite, or
(2) there exists a torsion free finite index subgroup $H \leq \Gamma$ and a finite index normal subgroup $F \trianglelefteq \Delta$ such that H splits as a semidirect product $F \rtimes \mathbb{Z}$.

Proof. Assume Δ is not geometrically finite. Apply Selberg's lemma to obtain a torsion free finite index normal subgroup $\Gamma^{\mathrm{tf}} \leq \Gamma$. The subgroup $\Delta \cap \Gamma^{\mathrm{tf}}$ is normal torsion free and finite index in Δ. Apply Theorems 3.1 and 3.2 to the pair Γ^{tf} and $\Delta \cap \Gamma^{\mathrm{tf}}$.

We know $\Delta \cap \Gamma^{\mathrm{tf}}$ is not geometrically finite. So there is a finite index subgroup $H \leq \Gamma^{\mathrm{tf}}$ and a subgroup $F \leq \Delta \cap \Gamma^{\mathrm{tf}}$ of index one or two such that H splits as a semidirect product $F \rtimes \mathbb{Z}$. Since its index is at most two, F is in fact a characteristic subgroup of $\Delta \cap \Gamma^{\mathrm{tf}}$, implying it is a normal subgroup of Δ. This proves the corollary. □

Finitely generated subgroups of uniform lattices in $\mathrm{PSL}_2(\mathbb{C})$. We are now ready to prove Theorem 1.3, which we restate here for convenience.

Theorem 1.3. Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a uniform lattice. Then every finitely generated subgroup of Γ is closed in the normal topology.

Proof. Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a uniform lattice, $\Delta < \Gamma$ a finitely generated subgroup. If Δ is geometrically finite then it is quasiconvex and the result follows directly from theorem 1.2. Assume therefore that Δ is geometrically infinite. In this case we argue that Δ is in fact even pro-finitely closed in Γ, which is more than what we need.

Let us find groups H and F as in Corollary 3.3. Let $G = \langle H, \Delta \rangle$ be the group generated by H and Δ. Since G is finite index in Γ it is enough to show that Δ is closed in the pro-finite topology on G. Factoring out by the normal subgroup $F \lhd G$, it is enough to show that the finite subgroup Δ/F is pro-finitely closed in G/F. Note that G/F is virtually cyclic and in particular it is residually finite.

It remains only to observe that any finite subgroup is closed in the pro-finite topology on a residually finite group. Let $H_i \lhd (G/F)$ be finite index normal subgroups with trivial intersection. We claim that $\Delta/F = \cap_i H_i(\Delta/F)$. Indeed let $g = h_1\delta_1 = h_2\delta_2 = \ldots = h_i\delta_i = \ldots$ be any element of this intersection, where $h_i \in H_i$ and $\delta_i \in \Delta/F$. But there are only finitely many ways to write this element in the form $h_i\delta_i$, because Δ/F is finite. In particular there exists some $h \in G/F$ such that $h_i = h$ for infinitely many i's. But $\cap H_i = \langle id \rangle$ so $h = \mathrm{id}$ and $g \in \Delta/F$. $\square$

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