

# ON THE STABILITY, COMPLEXITY, AND DISTRIBUTION OF SIMILARITY CLASSES OF THE LONGEST EDGE BISECTION PROCESS FOR TRIANGLES

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ABSTRACT. The Longest Edge Bisection (LEB) of a triangle is performed by joining the midpoint of its longest edge to the opposite vertex. Applying this procedure iteratively produces an infinite family of triangles. Surprisingly, a classical result of Stynes (1980) shows that for any initial triangle, this infinite family falls into finitely many similarity classes.

While the set of classes is finite, we show that a far smaller, stable subset of “fat” triangles, called **terminal quadruples**, effectively dominates the final mesh structure. We prove the following asymptotic area distribution result: for every initial triangle, the portion of area occupied by terminal quadruples tends to one, with the convergence occurring at an exponential rate. In fact, we provide the precise distribution of triangles in every step. We introduce the **bisection graph** and use spectral methods to establish this result.

Given this dominance, we provide a complete characterization of triangles possessing a single terminal quadruple, while conversely exhibiting a sequence of triangles with an unbounded number of terminal quadruples. Furthermore, we reveal several fundamental geometric properties of the points of a terminal quadruple, laying the groundwork for studying the geometric distribution of the entire orbit. Our analysis leverages the hyperbolic geometry framework of Perdomo and Plaza (2014) and refines their techniques.

## 1. INTRODUCTION

**1.1. Background.** The **Longest Edge Bisection** (LEB) process is a straightforward iterative procedure formally initiated by Rosenberg and Stenger [RS75]. In each step, the median to the longest edge is added, splitting the triangle into two daughter triangles. Rosenberg and Stenger established the critical property of non-degeneracy, which guarantees geometric quality. They proved that for an initial triangle  $\Delta_0$  with smallest angle  $\alpha_0$ , every angle generated by the infinite process is bounded from below by  $\frac{\alpha_0}{2}$ . Additionally, a useful triangulation process requires the convergence of the mesh diameter to zero. Kearfott [Kea78] first proved this for LEB in arbitrary dimensions; later, Stynes [Sty79, Sty80] and Adler [Adl83] provided sharpened bounds on the convergence rates.

A landmark result, known as the **Finite Similarity Classes (FSC) Theorem**, was established by Stynes [Sty80, Corollary 1]. This theorem asserts that the infinite family of triangles generated by iteratively bisecting a single initial triangle falls into a finite number of similarity classes. This result was later reproved by Adler [Adl83] using an elegant inductive argument, and subsequently by Perdomo and Plaza [PP14], who recast

the analysis within the framework of hyperbolic geometry, a setting that proved to be particularly natural.

In their paper, Rosenberg and Stenger identified a specific region where triangles enter a periodic orbit of length 4 under the LEB process [RS75, Lemma 4]; we refer to these orbits as terminal sets. Building on this work and the FSC Theorem, Stynes [Sty80, Corollary 2] demonstrated that the proportion of the area occupied by triangles in these terminal sets converges to 1.

These results pave the way for a deeper, quantitative analysis of the LEB process as a dynamical system. While it is known that the variety of shapes is finite and that terminal sets asymptotically exhaust the area of the initial triangle, this does not describe the rate of convergence or the precise distribution of triangle shapes. Several fundamental questions naturally arise:

At what rate does the total area of the mesh become dominated by these stable shapes? What are the specific geometric properties of these terminal sets? Furthermore, how does the number of distinct terminal sets depend on the geometry of the initial triangle? Answering these questions is essential for a comprehensive understanding of the stability and complexity of the LEB process. We provide a precise formulation of these problems in §1.2 and address them as the main contribution of this paper. Partial results in these directions were previously noted in [GGR07].

**1.2. Preliminaries and main results.** The 2014 paper [PP14] by Perdomo and Plaza provided a new proof of the FSC theorem, employing the machinery of hyperbolic dynamics in a moduli space of triangle shapes, modeled in the hyperbolic upper half-plane  $\mathbb{H}$ , to analyze the bisection process as a dynamical system. Given a triangle  $\Delta_0$ , by

- (1) scaling the edge lengths so that the longest edge has length 1,
- (2) mapping the longest edge to the interval  $[0, 1]$  on the real axis, and
- (3) placing the third vertex  $z$  opposite to the longest edge, on the upper half plane, and so that the shortest edge is on the left,

we bijectively identify every similarity class of triangles with a unique point  $z$  in the domain  $D \subseteq \mathbb{H}$  defined by (see Figure 1a).

$$D = \left\{ z \in \mathbb{C} \mid \operatorname{Re}(z) \leq \frac{1}{2}, \operatorname{Im}(z) > 0, |z - 1| \leq 1 \right\}.$$

Relying on this identification, we refer to a point  $z \in D$  as a triangle. For such  $z \in D$ , the bisection by the median to the longest edge defines two function  $L, R : D \rightarrow D$ , where  $L(z)$  is the **left triangle** and  $R(z)$  is the **right triangle**. As explained in [PP14], the region  $D$  is subdivided into 6 subregions by the geodesics  $\operatorname{Re}(z) = \frac{1}{4}$ ,  $|z| = \frac{1}{2}$  and  $|z - \frac{1}{2}| = \frac{1}{2}$  (see Figure 1b), and the functions  $L$  and  $R$  are defined in a piecewise manner, by a Möbius or anti-Möbius transformation on each one of these regions. The formulas and the geometric interpretation of these transformations are presented in Table 1. An illustrative application presenting the geometry of the various transformations is available at <https://danielkalmanovich.wixsite.com/website/leb>. The reader is referred to

[Ahl78, §3] for more details on the action by Möbius transformations on the upper half-plane.

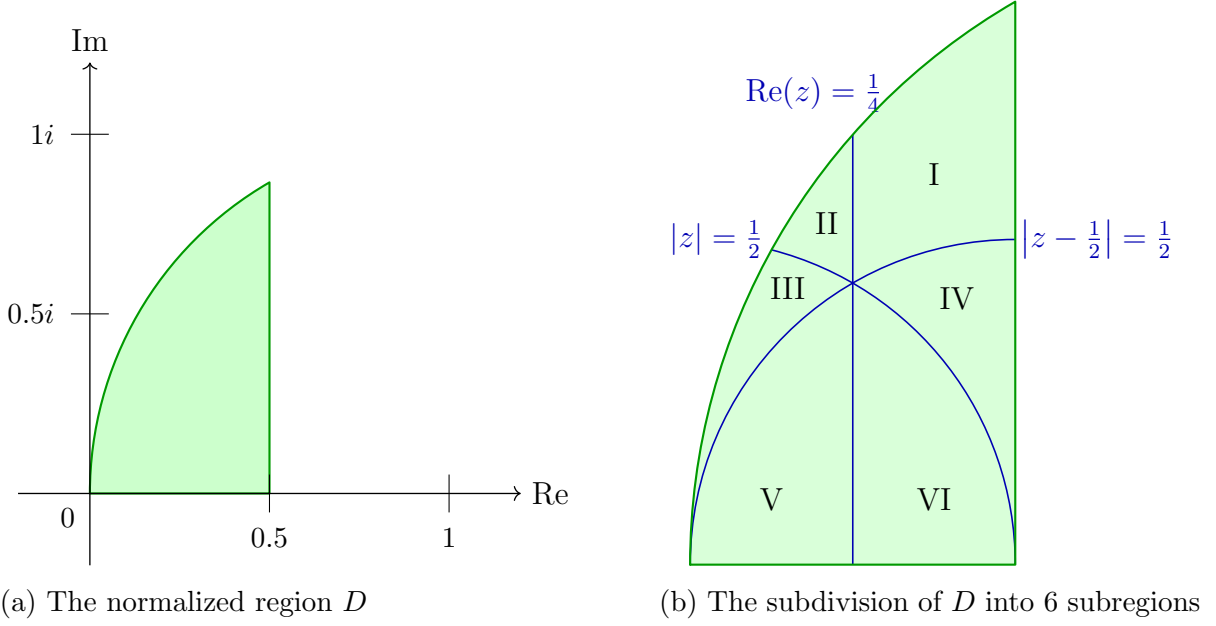


FIGURE 1. The normalized region  $D$  and its subdivision.

Region	Function	Geometric interpretation in $D$
I	$L(z) = \frac{1}{2\bar{z}}$	An inversion with respect to the circle $ z  = \frac{\sqrt{2}}{2}$ .
II	$L(z) = \frac{-1}{2z-1}$	An elliptic Möbius map with a fixed point at $\frac{1}{4} + \frac{\sqrt{7}}{4}i \in D$ . Points rotate counterclockwise by an angle of $\arccos(-\frac{3}{4})$ about the fixed point.
III	$L(z) = \frac{2z}{2\bar{z}-1}$	A hyperbolic anti-Möbius map. Points spiral from the repeller at $z = 0$ to the attractor at $z = \frac{3}{2}$ .
IV	$L(z) = \frac{2z-1}{2z}$	An elliptic Möbius map with a fixed point at $\frac{1}{2} + \frac{1}{2}i \in D$ . Points rotate clockwise by an angle of $\frac{\pi}{2}$ about the fixed point.
V	$L(z) = 2z$	A hyperbolic Möbius map that dilates by 2.
VI	$L(z) = 1 - 2\bar{z}$	A hyperbolic anti-Möbius map. A reflection across $\text{Re}(z) = \frac{1}{3}$ combined with a dilation by a factor of 2.
I $\cup$ II $\cup$ III	$R(z) = \frac{-1}{2z-2}$	An elliptic Möbius map with a fixed point at $\frac{1}{2} + \frac{1}{2}i \in D$ . Points rotate counterclockwise by an angle of $\frac{\pi}{2}$ about the fixed point.
IV $\cup$ V $\cup$ VI	$R(z) = \frac{2\bar{z}-1}{2\bar{z}-2}$	An inversion with respect to the circle $ z-1  = \frac{\sqrt{2}}{2}$ .

TABLE 1. The piecewise functions  $L(z)$  and  $R(z)$ .

As we will show, an exceptional property of the dynamics of the LEB process is that, regardless of the initial triangle, the process eventually produces *predominantly* triangles from a certain subregion of  $I \cup IV$  in  $D$ , with notable geometric properties. Furthermore, the orbit of these triangles is a periodic set with at most 4 elements. Generically<sup>1</sup>, these sets are of size exactly 4, thus we call them **terminal quadruples**. This analysis immediately raises several questions that motivate this paper.

- (1) How many distinct terminal quadruples can arise from a given initial triangle?
- (2) What proportion of the total area of the initial triangle  $z$  do terminal quadruple triangles occupy after  $k$  bisection steps?
- (3) Which geometric properties does the orbit of a triangle possess?

To answer these questions, we introduce the **bisection graph** (see §3) of a triangle, and use spectral methods to analyze it. Furthermore, we adopt and refine the methods used in [PP14] to study the geometric properties of the orbit of  $z$  under iterative LEB, denoted by  $\text{LEB}(z)$ . We also denote by  $q(z)$  the number of terminal quadruples that arise from  $z$ , and by  $l(z) = |\text{LEB}(z)|$ . Our main results are the following.

**Theorem 1.** *For a triangle  $z$  and  $j \in \mathbb{N}$  let  $w_j \in \mathbb{R}^{l(z)}$  be the probability vector that describes the partition of the area of  $z$  into triangles of different similarity classes, after  $j$  steps. Then, for every triangle  $z$  we have the following:*

- (1) *The limits  $w_{\text{even}} = \lim_{j \rightarrow \infty} w_{2j}$  and  $w_{\text{odd}} = \lim_{j \rightarrow \infty} w_{2j+1}$  exist, and the non-zero entries of  $w_{\text{even}}$  and  $w_{\text{odd}}$  are those corresponding to elements in terminal quadruples.*
- (2) *There exists  $\xi \in (0, 1)$  such that for every  $j \in \mathbb{N}$ , after  $j$  steps, the area of  $z$  that is occupied by triangles that do not belong to terminal quadruples is  $O(\xi^j)$ .*

In view of the theorem above, the area of any initial triangle  $z$  gets filled with triangles that belong to terminal quadruples, with an exponential rate as the LEB process progresses. Thus, the number of triangle shapes that one sees is essentially  $4q(z)$  and not  $l(z)$ . For instance, taking an acute isosceles triangle with apex angle  $\alpha$  we have  $l(z) = \Theta(\log(\frac{1}{\alpha}))$  [GGR07, Lemma 2], while according to Theorem 2 below we have  $q(z) = 1$ . In general, the following theorems shed light on the quantity  $q(z)$ , for  $z$  in the different regions.

**Theorem 2.** *For every  $z \in I \cup II \cup III \cup IV$  we have  $q(z) = 1$ .*

**Theorem 3.** *For every  $n \in \mathbb{N}$  there are triangles  $z$  in regions  $V$  and  $VI$  for which  $q(z) > n$ .*

The rest of the paper is structured as follows: in §2 we provide a classification of the non-generic triangles, a short discussion of geometric properties of terminal quadruples, a proof of Theorem 2, and a proof of Theorem 3. In §3 we introduce the bisection graph and prove our main result, Theorem 1, which motivates our study of the terminal quadruples in §2. Concluding remarks and open problems are presented in §4.

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<sup>1</sup>See §2.1.

## 2. TERMINAL QUADRUPLES

We start this section with a classification of the non-generic triangles in  $D$ , followed by a fundamental geometric property of terminal quadruples. In the last two subsections we prove Theorem 2 and Theorem 3.

**2.1. Generic triangles and triangles with orbits of size less than 4.** We begin with the definition of a generic triangle with respect to the LEB process. It can happen that  $L(z)$  is similar to  $R(z)$ , or that one of them is similar to  $z$ , thus we define:

**Definition 4.** A triangle  $z \in D$  is **generic** if for each  $w \in \text{LEB}(z)$ , the points  $w, L(w)$  and  $R(w)$  are all distinct.

The following proposition is a characterization of the non-generic triangles (see Figure 2a).

**Proposition 5.** *A triangle  $z \in D$  is generic if and only if  $\text{LEB}(z)$  does not intersect the following 3 geodesics:*

$$\text{Re}(w) = \frac{1}{2}, \quad |w| = \frac{\sqrt{2}}{2}, \quad |w - 1| = \frac{\sqrt{2}}{2}. \quad (1)$$

*Proof.* To help the reader follow the computations in the proof, we use subscripts to indicate which piece of the piecewise maps  $L$  and  $R$  is being applied.

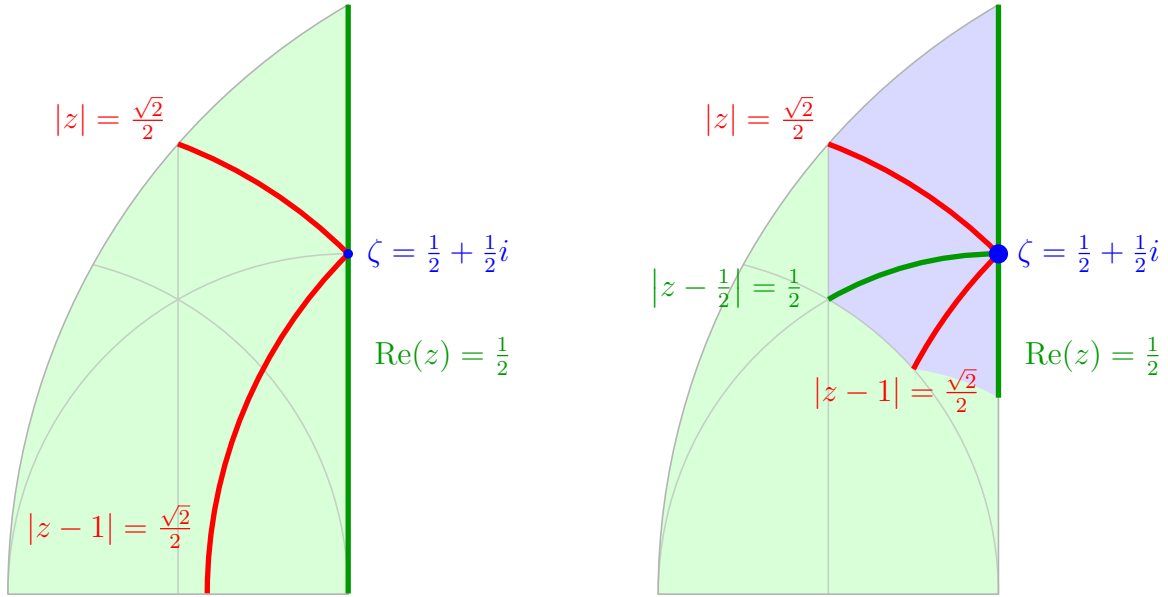
First, we address points  $w \in D$  that are fixed under either  $L$  or  $R$ . Since  $L_I$  is an inversion with respect to the geodesic  $|w| = \frac{\sqrt{2}}{2}$ , we have  $L_I(w) = w$  for points lying on this geodesic. Similarly,  $R_{IVUVUVI}$  is an inversion with respect to the geodesic  $|w - 1| = \frac{\sqrt{2}}{2}$  and so  $R_{IVUVUVI}(w) = w$  there. Each one of  $L_{II}, L_{IV}$  and  $R_{IUIUIII}$  has a unique fixed point, which is on one of these two geodesics, and  $L_{III}, L_V$  and  $L_{VI}$  have no fixed points in  $D$ . Thus  $w = R(w)$  or  $w = L(w)$  if and only if  $|w - 1| = \frac{\sqrt{2}}{2}$  or  $|w| = \frac{\sqrt{2}}{2}$ .

Next, we show that  $L(w) = R(w)$  if and only if  $\text{Re}(w) = \frac{1}{2}$ . Notice that for every  $w \in D$ , the triangle  $R(w)$  is not acute. In case  $L(w)$  is obtuse, one can easily see that  $L(w) \neq R(w)$  because the obtuse angle of  $L(w)$  is strictly smaller than that of  $R(w)$ . So we may assume that  $L(w)$  is not obtuse. Thus  $L(w) = R(w)$  if and only if  $L(w)$  and  $R(w)$  are the same right-angled triangle. Observe that if  $L(w) = R(w)$  then  $w$  is isosceles, and if the apex of an isosceles triangle  $w$  is  $< \frac{\pi}{3}$ , then its longest edge is its leg, thus  $L(w) \neq R(w)$  in that case. Hence, we conclude that  $L(w) = R(w)$  if and only if  $w$  is an isosceles triangle with apex  $\geq \frac{\pi}{3}$ . Thus  $L(w) = R(w)$  if and only if  $\text{Re}(w) = \frac{1}{2}$ .  $\square$

**Remark 6.** In view of Proposition 5, the set of non-generic triangles is contained in a countable union of arcs in  $D$ , i.e., the inverse images of the arcs described in (1) with respect to iterative applications of  $L$  and  $R$ , thus has measure zero. In particular, the set of generic triangles is of full measure.

Since any region I triangle generates at most 4 similarity classes and I is invariant under  $L$ , it follows that for any triangle  $z$  in the region

$$A = I \cup R(I) \subseteq I \cup IV$$



(a) The non-generic triangles in  $D$ : The triangles on the upper **red** curve satisfy  $z = L(z)$ , the triangles on the lower **red** curve satisfy  $z = R(z)$ . The triangles on the **green** curve satisfy  $L(z) = R(z)$ .

(b) The non-generic triangles in the shaded terminal region  $A$ : the **blue**  $\zeta$  generates 1 triangle. The triangles on the **red** curves generate 2 triangles. The triangles on the **green** curves generate 3 triangles.

FIGURE 2. The non-generic locus.

we have  $l(z) \leq 4$ . It is easy to verify that for any triangle  $z \in D \setminus A$  we have  $l(z) \geq 5$ , thus  $A$  is precisely the locus  $\{z \in D \mid l(z) \leq 4\}$ . We refer to  $A$  as the **terminal region** and the orbit  $\text{LEB}(z)$  of any  $z \in A$  is called a **terminal quadruple**. This terminology applies consistently to all  $z \in A$ : generic triangles produce orbits of 4 distinct elements, while non-generic triangles produce orbits of 4 elements, some of which are repeated. The precise characterization of these non-generic cases is provided in the following corollary.

**Corollary 7.** *Let  $z \in D$  and set  $\zeta = \frac{1}{2} + \frac{1}{2}i$ , then:*

- (a)  $l(z) = 1$  if and only if  $z = \zeta$ .
- (b)  $l(z) = 2$  if and only if  $z \in \left\{ w \in A \mid |w| = \frac{\sqrt{2}}{2} \text{ or } |w - 1| = \frac{\sqrt{2}}{2} \right\} \setminus \{\zeta\}$ .
- (c)  $l(z) = 3$  if and only if  $z \in \left\{ w \in A \mid |w - \frac{1}{2}| = \frac{1}{2} \text{ or } \text{Re}(w) = \frac{1}{2} \right\} \setminus \{\zeta\}$ .

*Proof.* Since the geodesic  $|w - \frac{1}{2}| = \frac{1}{2}$  in  $A$  is the image of the geodesic  $\text{Re}(w) = \frac{1}{2}$  in  $A$  under  $L$  and  $R$ , this is a direct consequence of Proposition 5 (see Figure 2b).  $\square$

**2.2. The geometric structure of the terminal quadruples.** While running simulations, we have observed many remarkable phenomena regarding the geometric position of the points of  $\text{LEB}(z)$  in  $D$ . The following result, which describes the geometric distribution of the points of a terminal quadruple, is a forerunner of a complete geometric distribution analysis of  $\text{LEB}(z)$  (see §4.1).

**Proposition 8.** For every  $z_1 \in A$ , the set  $\{z_1, z_2, z_3, z_4\}$  that is generated from  $z_1$  lies on a hyperbolic circle centered at  $\zeta = \frac{1}{2} + \frac{1}{2}i$ . In particular, every terminal quadruple lies on an Euclidean circle centered at a point on the line  $\text{Re}(z) = \frac{1}{2}$ .

*Proof.* For  $z \in I$ , the point  $L(z)$ , is obtained by an inversion with respect to the geodesic  $\left\{w \mid |w| = \frac{\sqrt{2}}{2}\right\}$ , that contains  $\zeta = \frac{1}{2} + \frac{1}{2}i$ . Since  $\{z_1, z_2, z_3, z_4\} \cap I \neq \emptyset$ , we may assume that  $z_1 \in I$ . The points  $z_1$  and  $L(z_1)$  are both in region I and have the same hyperbolic distance from  $\zeta$ . Furthermore, for  $z \in I$ , the point  $R(z) \in IV$  is obtained by a hyperbolic rotation of  $\frac{\pi}{2}$  around  $\zeta$ . Thus  $R(z_1)$  and  $RL(z_1)$  are also at the same hyperbolic distance from  $\zeta$  as  $z_1$ .  $\square$

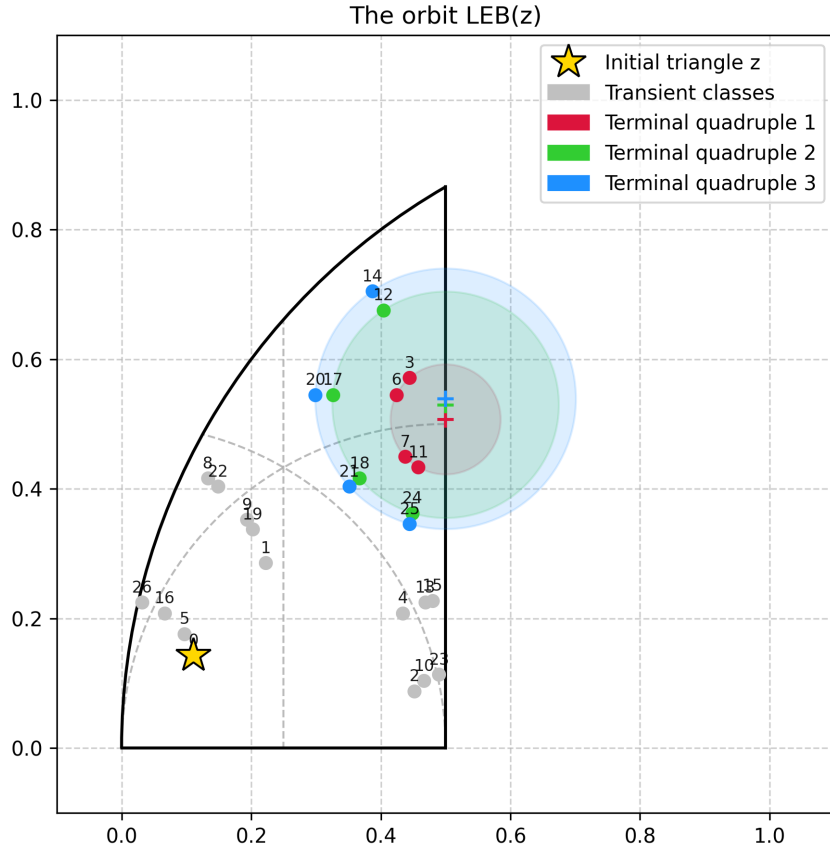


FIGURE 3. The points of  $\text{LEB}(z)$  for  $z = \frac{1}{9} + \frac{1}{7}i$ . There are 3 terminal quadruples, and the circles that they lie on are shaded.

Among other geometric properties that one might notice in Figure 3 is the fact that the points labeled 3, 12, 14 lie on a specific arc of a circle that is tangent to the real axis at 0. We explain this fact in §4.1.

**2.3. Regions with  $q(z) = 1$ .** We begin with some observations regarding compositions of our basic functions  $L$  and  $R$  in certain regions.

**Lemma 9.** *For every  $z \in I \cup II \cup III$  we have  $LR(z) = z$  and  $RR(z) = \frac{\bar{z}}{2\bar{z}-1}$ , which is a reflection with respect to the geodesic  $|w - \frac{1}{2}| = \frac{1}{2}$ .*

*Proof.* Recall that for  $z \in I \cup II \cup III$  we have  $R(z) = \frac{-1}{2z-2}$ . One easily checks that  $R$  maps the geodesics  $\operatorname{Re}(z) = \frac{1}{2}$ ,  $|z - 1| = 1$  and  $|z - \frac{1}{2}| = \frac{1}{2}$  to the geodesics  $|z - \frac{1}{2}| = \frac{1}{2}$ ,  $|z| = \frac{1}{2}$  and  $\operatorname{Re}(z) = \frac{1}{2}$ , respectively, and hence maps the region  $I \cup II \cup III$  to region IV. Since  $L(z) = \frac{2z-1}{2z}$  and  $R(z) = \frac{2\bar{z}-1}{2\bar{z}}$  for  $z \in IV$ , the assertion follows by a direct computation.  $\square$

**Lemma 10.** *For every  $z \in II \cup III$  we have  $L(z) = LRR(z)$ .*

*Proof.* By Lemma 9, for every  $z \in II$  we have  $RR(z) \in VI$  and thus

$$LRR(z) = 1 - 2 \overline{\left( \frac{\bar{z}}{2\bar{z}-1} \right)} = \frac{-1}{2z-1} = L(z).$$

For  $z \in III$ , notice that  $L(z) = \frac{2\bar{z}}{2\bar{z}-1} = 2RR(z)$ . Since  $RR(z) \in V$ , where  $L(z) = 2z$ , the proof is complete.  $\square$

**Lemma 11.** *For every  $z \in IV$  we have  $LL(z) = LR(z)$ .*

*Proof.* As before, we use subscripts to indicate which piece of the piecewise maps  $L$  and  $R$  is being applied.

Since both  $L$  and  $R$  can map  $z \in IV$  to different regions, we distinguish between 3 cases, determined by the geodesics  $|z - \frac{1}{3}| = \frac{1}{3}$  and  $|z - \frac{2}{3}| = \frac{1}{3}$  in region IV.

**Case 1:**  $|z - \frac{1}{3}| \geq \frac{1}{3}$ . One easily verifies that the inverse image of the geodesic  $|z| = \frac{1}{2}$  under  $R_{IVUVUVI}$  is the geodesic  $|z - \frac{1}{3}| = \frac{1}{3}$ , which is also the inverse image of  $\operatorname{Re}(z) = \frac{1}{4}$  under  $L_{IV}$ . Thus, for  $z \in IV$  that satisfies  $|z - \frac{1}{3}| \geq \frac{1}{3}$  we have  $L(z) \in I$  and  $R(z) \in IV$ . Therefore, in this case, we have

$$L_I(L_{IV}(z)) = \frac{\bar{z}}{2\bar{z}-1} = L_{IV}(R_{IVUVUVI}(z)).$$

**Case 2:**  $|z - \frac{1}{3}| < \frac{1}{3}$  **and**  $|z - \frac{2}{3}| \geq \frac{1}{3}$ . Here, the inverse image of the geodesic  $\operatorname{Re}(z) = \frac{1}{4}$  under  $R_{IVUVUVI}$  is the geodesic  $|z - \frac{2}{3}| = \frac{1}{3}$ , which is also the inverse image of  $|z| = \frac{1}{2}$  under  $L_{IV}$ . Thus, for  $z \in IV$  that satisfies  $|z - \frac{2}{3}| \geq \frac{1}{3}$  we have  $L(z) \in II$  and  $R(z) \in VI$ . Therefore

$$L_{II}(L_{IV}(z)) = \frac{-z}{z-1} = L_{VI}(R_{IVUVUVI}(z)).$$

**Case 3:**  $|z - \frac{1}{3}| < \frac{1}{3}$  **and**  $|z - \frac{2}{3}| < \frac{1}{3}$ . Finally, by the reasoning in case 2, for  $z \in IV$  that satisfies  $|z - \frac{1}{3}| < \frac{1}{3}$  we have  $L(z) \in III$  and  $R(z) \in V$ . Hence

$$L_{III}(L_{IV}(z)) = \frac{2\bar{z}-1}{\bar{z}-1} = L_V(R_{IVUVUVI}(z)).$$

$\square$

With these identities at hand we are ready to prove Theorem 2.

*Proof of Theorem 2.* We proceed by case analysis. Notice that for every triangle  $z$ , the orbit of  $z$  can be written as follows:

$$\begin{aligned} \text{LEB}(z) &= \{z\} \cup \text{LEB}(L(z)) \cup \text{LEB}(R(z)) \\ &= \{z\} \cup \text{LEB}(L(z)) \cup \text{LEB}(LR(z)) \cup \text{LEB}(RR(z)) \cup \{R(z)\}. \end{aligned} \quad (2)$$

**If  $z \in \mathbf{I}$ .** then  $z$  belongs to the terminal quadruple  $\text{LEB}(z) = \{z, L(z), R(z), RR(z)\}$  and clearly  $q(z) = 1$ .

**If  $z \in \mathbf{II}$ .** then  $RR(z) \in \mathbf{VI}$  by Lemma 9. Since  $R$  maps the region  $\mathbf{IV} \cup \mathbf{V} \cup \mathbf{VI}$  to itself, by an inversion with respect to  $|z - 1| = \frac{\sqrt{2}}{2}$ , we have  $RRR(z) = R(z)$ . By Lemma 9 we have  $LR(z) = z$  and by Lemma 10 we have  $LRR(z) = L(z)$ . Applying these identities to (2) yields

$$\begin{aligned} \text{LEB}(z) &= \{z\} \cup \text{LEB}(L(z)) \cup \text{LEB}(LRR(z)) \cup \{R(z)\} \cup \{RR(z)\} \\ &= \{z\} \cup \{R(z)\} \cup \{RR(z)\} \cup \text{LEB}(L(z)). \end{aligned} \quad (3)$$

Since the triangles  $z, R(z), RR(z)$  do not belong to a terminal quadruple and  $L(z) \in \mathbf{I}$ , it follows that  $q(z) = 1$ .

**If  $z \in \mathbf{III}$ .** then using the same identities as in the previous case, (3) holds. As before, the triangles  $z, R(z), RR(z)$  do not belong to a terminal quadruple. Thus, terminal quadruples can only arise from  $\text{LEB}(L(z))$ . Notice that  $L(z) \in \mathbf{I} \cup \mathbf{II} \cup \mathbf{III}$ . If  $L(z) \in \mathbf{I} \cup \mathbf{II}$  then we are done, by the previous cases. Otherwise,  $L(z) \in \mathbf{III}$ , and we may repeat the above argument. Namely, the singletons  $\{L(z)\}, \{RL(z)\}$  and  $\{RRL(z)\}$ , which do not belong to a terminal quadruple, are added to  $\text{LEB}(z)$ , and terminal quadruples can only arise from  $\text{LEB}(LL(z))$ . Since  $\text{LEB}(z)$  is finite, there exists some  $k \in \mathbb{N}$  for which  $L^k(z) \in \mathbf{I} \cup \mathbf{II}$ , and we are done.

**Finally, assume  $z \in \mathbf{IV}$ .** We have  $RR(z) = z$ , and so (2) in this case reads:

$$\text{LEB}(z) = \{z\} \cup \text{LEB}(L(z)) \cup \text{LEB}(LR(z)) \cup \{R(z)\}.$$

Since  $L(z) \in \mathbf{I} \cup \mathbf{II} \cup \mathbf{III}$  for  $z \in \mathbf{IV}$ , by the previous cases,  $\text{LEB}(L(z))$  contains a single terminal quadruple. By Lemma 11, we have  $\text{LEB}(LR(z)) = \text{LEB}(LL(z)) \subset \text{LEB}(L(z))$  and the proof is complete.  $\square$

**2.4. Arbitrarily large  $q(z)$ .** In this subsection, we prove Theorem 3, showing that the number of terminal quadruples that arise from an initial triangle can be arbitrarily large. For the proof, we focus on a subregion of  $\mathbf{V}$  defined by

$$\tilde{\mathbf{V}} = \left\{ z \in \mathbf{V} \mid \left| z - \frac{1}{3} \right| \leq \frac{1}{3} \right\}.$$

The geodesic  $|z - \frac{1}{3}| = \frac{1}{3}$  is the inverse image of the geodesic  $|z| = \frac{1}{2}$  under the function  $R$ . The relevance of this region will be made clear in the proof of Lemma 12.

For a triangle  $z \in \tilde{V}$  we denote by

$$h(z) = L \circ R(z)$$

and begin with some straightforward properties of the function  $h$ .

**Lemma 12.** *For  $z \in \tilde{V}$  we have*

(a)  $h(z) = \frac{-z}{z-1}$ .

(b) *The function  $h$  is a parabolic Möbius transformation, with a source and sink at the fixed point  $w = 0$ . Moreover, for every  $k \in \mathbb{N}$ , the point  $h^{k+1}(z)$  is obtained from  $h^k(z)$  by a counterclockwise rotation along the circle  $C_z$  through  $z$ , that is tangent to the real axis at 0.*

(c) *If  $\operatorname{Re}(z) = \operatorname{Im}(z)$  then the tangent line to  $C_z$  at  $z$  is vertical.*

*Proof.* First note that for every  $z \in \tilde{V}$  we have  $R(z) \in \text{VI}$ , thus property (a) follows directly from the formulas of  $R$  in region V and  $L$  in region VI. In view of (a), property (b) follows by standard analysis, see e.g. [Ah78, p.88]. For property (c), let  $o_z$  denote the center of  $C_z$ . Since  $C_z$  is tangent to the real axis at 0, the point  $o_z$  is on the imaginary axis. Since  $C_z$  passes through  $z$ , which satisfies  $\operatorname{Re}(z) = \operatorname{Im}(z)$ , one easily shows that the Euclidean distance from  $o_z$  to  $z$  is  $\operatorname{Im}(z)$  and from  $o_z$  to 0 is  $\operatorname{Re}(z)$ , which implies the assertion.  $\square$

**Lemma 13.** *For every  $z$  for which  $z, h(z), L(z) \in \tilde{V}$  we have  $L \circ h^2(z) = h \circ L(z)$ .*

*Proof.* By the previous lemma, and since  $z, h(z), L(z) \in \tilde{V}$ , one easily verifies that  $h^2(z) \in \text{V}$  and that

$$L \circ h^2(z) = \frac{-2z}{2z-1} = h \circ L(z),$$

implying the assertion.  $\square$

For  $k \in \mathbb{N}$  we consider the region  $B_k = \frac{1}{2^k}\text{I}$ , and set

$$B = \bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} \frac{1}{2^k}\text{I}.$$

Notice that  $B$  is comprised of shrinking copies of region I that approach the origin (see Figure 4), in particular,  $B \subseteq \text{V}$ . Furthermore, since  $\text{I} \subseteq \{z \mid |z-1| \leq 1\}$ , for each  $k \geq 1$  we have  $B_k \subseteq \{z \mid |z - \frac{1}{2^k}| \leq \frac{1}{2^k}\}$ , and in particular,  $B_k \subseteq \tilde{V}$  for every  $k \geq 2$ .

**Lemma 14.** *For  $\zeta = \frac{1}{2} + \frac{1}{2}i$ , and every  $m \in \mathbb{N}$  the point  $z_m = \frac{\zeta}{2^m} \in B_m$  satisfies*

$$\{h^j(z_m) \mid j \in \{0, \dots, 2^{m-1}\}\} \subseteq B_m.$$

*Proof.* We argue by induction on  $m$ . For  $m = 1$ , note that  $z_1 = \frac{1}{4} + \frac{1}{4}i \in \tilde{V}$ , then  $h(z_1) = \frac{1}{5} + \frac{2}{5}i \in B_1$  (since  $|h(z_1) - \frac{1}{2}| = \frac{1}{2}$  and the geodesic  $|w - \frac{1}{2}| = \frac{1}{2}$  bounds  $B_1$  from above, see Figure 4), as required. For  $z_m \in B_m$  suppose that  $\{h^j(z_m) \mid j \in \{0, \dots, 2^{m-1}\}\} \subseteq B_m$ . For the induction step, since  $B_k \subseteq \tilde{V}$  for every  $k \geq 2$ , by Lemma 13, we have

$$h^{2^m}(z_{m+1}) = (L^{-1} \circ h \circ L)^{2^{m-1}}(z_{m+1}) = L^{-1} \circ h^{2^{m-1}} \circ L(z_{m+1}) = \frac{1}{2}h^{2^{m-1}}(z_m) \in B_{m+1}, \quad (4)$$

where the relation on the right-hand side follows from  $h^{2^{m-1}}(z_m) \in B_m$ , which is obtained from the induction hypothesis. In particular,  $h^{2^m}(z_{m+1}) \in B_{m+1}$ . Moreover, iterating (4) yields

$$h^{2^m}(z_{m+1}) = \frac{1}{2^m}h(z_1).$$

Therefore,  $|h^{2^m}(z_{m+1}) - \frac{1}{2^{m+1}}| = \frac{1}{2^m}|h(z_1) - \frac{1}{2}| = \frac{1}{2^{m+1}}$ , which shows that  $h^{2^m}(z_{m+1})$  is on the top boundary of  $B_{m+1}$ , see Figure 4. Finally, denote by  $C_{m+1}$  the circle through

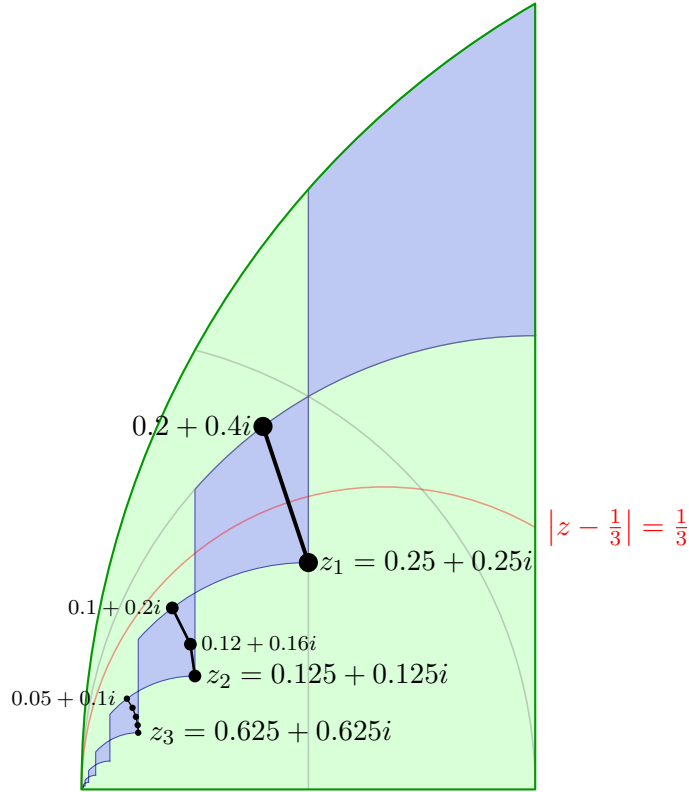


FIGURE 4. The region  $B$  is shaded in blue. The black points are the parts of the orbits of  $z_1 = \frac{1}{4} + \frac{1}{4}i$ ,  $z_2 = \frac{1}{8} + \frac{1}{8}i$  and  $z_3 = \frac{1}{16} + \frac{1}{16}i$  that belong to  $B$ .

0 and  $z_{m+1}$  that contains the  $h$ -orbit of  $z_{m+1}$ . By (c) of Lemma 12, the tangent to  $C_{m+1}$  at  $z_{m+1}$  is vertical. Since  $h^{2^m}(z_{m+1})$  is on the top boundary of  $B_{m+1}$ , the region  $B_{m+1}$  contains the entire arc of  $C_{m+1}$  that connects  $z_{m+1}$  and  $h^{2^m}(z_{m+1})$ . Since, by (b) of Lemma 12,  $h$  acts by moving a point counterclockwise on the circle  $C_{m+1}$ , we have  $\{h^j(z_{m+1}) \mid j \in \{0, \dots, 2^m\}\} \subseteq B_{m+1}$ , which completes the proof.  $\square$

We are now ready to deduce Theorem 3.

*Proof of Theorem 3.* We begin with finding such points  $z$  in region V. Given  $n \in \mathbb{N}$  we choose  $m \in \mathbb{N}$  that satisfies  $2^m \geq n$ , and set  $z = z_{m+2}$  from Lemma 14. Then

$$\{h^j(z) \mid j \in \{0, \dots, 2^{m+1}\}\} \subseteq B_{m+2} \subseteq B.$$

Note that by (b) of Lemma 12, the set  $\{h^j(z) \mid j \in \{0, \dots, 2^{m+1}\}\}$  contains  $2^{m+1} + 1$  distinct points. Also, notice that for every  $z \in B$  there exists a minimal  $k \in \mathbb{N}$  for which

$L^k(z) \in I$ , and in particular, for every  $j \in \{0, \dots, 2^{m+1}\}$  we have  $L^{m+2}(h^j(z)) \in I$ . These are  $2^{m+1} + 1$  distinct triangles in region I that are in  $\text{LEB}(z)$ . Each of these triangles in I has an orbit of length at most 4, which contains at most 2 elements from region I. It follows that  $q(z) \geq \frac{2^{m+1}+1}{2} > 2^m \geq n$ , as required.

To find such  $z$  in region VI, since  $R^2 = \text{Id}$  in regions V and VI, it suffices to show that  $R(z_m) = R\left(\frac{\zeta}{2^m}\right) \in \text{VI}$ , for  $m$  large enough. Since

$$R\left(\frac{\zeta}{2^m}\right) = 1 + \frac{1}{2} \frac{1}{\frac{\zeta}{2^m} - 1} = \left(\frac{1}{2} - \frac{2^{m+1} - 2}{2^{2m+2} - 2^{m+2}}\right) + i \left(\frac{1}{2^{m+1} - 2}\right),$$

we clearly have  $\left|R\left(\frac{\zeta}{2^m}\right) - \frac{1}{2}\right| < \frac{1}{2}$  and  $\text{Re}\left(R\left(\frac{\zeta}{2^m}\right)\right) > \frac{1}{4}$ , for every  $m \geq 2$ , and hence  $R\left(\frac{\zeta}{2^m}\right) \in \text{VI}$  and the proof is complete.  $\square$

### 3. THE BISECTION GRAPH AND ITS SPECTRAL PROPERTIES

In this section, we prove Theorem 1. The proof relies on known results for nonnegative matrices. We state them here for the completeness of the text. We use the notation  $\text{N}(A)$  for the null space of a matrix  $A$ , and  $[n] = \{1, 2, \dots, n\}$ . Recall that the **spectral radius**  $\rho(A)$  of a matrix  $A$  is the maximal absolute value of an eigenvalue of  $A$ . For a graph  $G$ , we use  $\rho(G)$  for the spectral radius of its adjacency matrix. By the **generalized eigenspace of  $A$  that belongs to an eigenvalue  $\lambda$**  we refer to the space  $\text{N}(A - \lambda I)^{d(\lambda)}$ , where  $d(\lambda)$  is the smallest natural number  $k$  that satisfies  $\text{N}(A - \lambda I)^k = \text{N}(A - \lambda I)^{k+1}$ . The non-zero vectors of the generalized eigenspace are called **generalized eigenvectors**.

**Theorem 15.** [HJ13, Theorem 8.1.22] *Let  $A = (a_{ij})$  be a nonnegative  $n \times n$  matrix, then*

$$\min_{1 \leq i \leq n} \sum_{j=1}^n a_{ij} \leq \rho(A) \leq \max_{1 \leq i \leq n} \sum_{j=1}^n a_{ij} \quad \text{and} \quad \min_{1 \leq j \leq n} \sum_{i=1}^n a_{ij} \leq \rho(A) \leq \max_{1 \leq j \leq n} \sum_{i=1}^n a_{ij}.$$

The following theorem is a particular case of [BP94, Theorem 3.20, p. 43].

**Theorem 16.** *Suppose that  $G = ([n], E)$  is a directed graph and  $A_G$  is the adjacency matrix of  $G$ . Let  $\alpha_1, \dots, \alpha_s$  be all the strongly connected components of  $G$  with  $\rho(\alpha_j) = \rho(A_G)$ , for every  $j$ . Then the dimension of the generalized eigenspace of  $\rho(A_G)$  is at most  $s$ .*

The following is a direct consequence of the Perron–Frobenius theorem (see [HJ13, Theorem 8.4.4]).

**Lemma 17.** *Let  $A \in M_n(\mathbb{R})$  be a nonnegative matrix with column sums equal to  $\lambda > 0$ . Let  $B \in M_k(\mathbb{R})$ , with  $2 \leq k < n$ , be an irreducible principal submatrix of  $A$  with at least one column sum which is strictly smaller than  $\lambda$ . Then  $\rho(B) < \lambda = \rho(A)$ .*

*Proof.* Since  $B$  is irreducible, by the Perron–Frobenius theorem, there is a positive eigenvector  $v = (c_1, \dots, c_k)^t$  (i.e.,  $c_j > 0$  for all  $j$ ) satisfying  $Bv = \rho(B)v$ . The assumptions on the column sums of  $A$  and  $B$  imply that the vector  $B^t \mathbf{1}$  satisfies the inequality  $B^t \mathbf{1} \leq \lambda \mathbf{1}$

entry-wise, with strict inequality in at least one component, and since  $v$  is positive, we have:

$$\langle B^t \mathbf{1}, v \rangle < \langle \lambda \mathbf{1}, v \rangle.$$

Thus we get:

$$\rho(B) \langle \mathbf{1}, v \rangle = \langle \mathbf{1}, \rho(B)v \rangle = \langle \mathbf{1}, Bv \rangle = \langle B^t \mathbf{1}, v \rangle < \langle \lambda \mathbf{1}, v \rangle = \lambda \langle \mathbf{1}, v \rangle,$$

dividing by the positive scalar  $\langle \mathbf{1}, v \rangle$ , we conclude that  $\rho(B) < \lambda = \rho(A)$ .  $\square$

We are now ready to introduce the bisection graph  $G_{\text{LEB}}(z)$ . We start with the generic case.

**Definition 18** (generic  $G_{\text{LEB}}(z)$ ). For a generic initial triangle  $z$ , the LEB process can be represented by a directed graph  $G_{\text{LEB}}(z) = (V, E)$  on the vertex set  $V = \text{LEB}(z)$ , with

$$E = \{(w, L(w)) \mid w \in \text{LEB}(z)\} \uplus \{(w, R(w)) \mid w \in \text{LEB}(z)\}.$$

We label each edge in  $G_{\text{LEB}}(z)$  naturally by  $L$  or  $R$ .

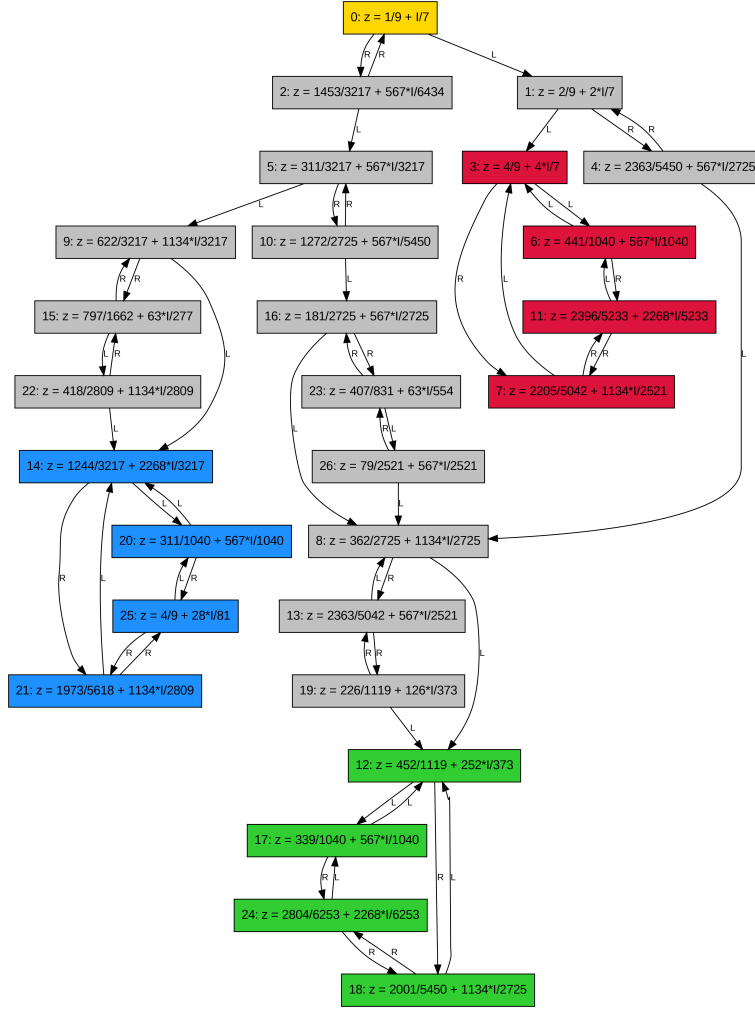


FIGURE 5. The graph  $G_{\text{LEB}}(z)$  for  $z = \frac{1}{9} + \frac{1}{7}i$ . The colors and numbering correspond to what is shown in Figure 3.

For non-generic  $z$ , the above graph has loops or double edges. From Proposition 5, we know that the set  $T = \left\{ w \in \text{LEB}(z) \mid |w| = \frac{\sqrt{2}}{2} \text{ or } |w - 1| = \frac{\sqrt{2}}{2} \right\}$  is the subset of vertices that have loops, and the set  $S = \left\{ w' \in \text{LEB}(z) \mid \text{Re}(w') = \frac{1}{2} \right\}$  is the subset of vertices that have (outgoing) double edges. Although the set of non-generic points has measure zero, it is desirable for  $G_{\text{LEB}}(z)$  to be a simple directed graph even in this case. To this end, we introduce two vertices to represent a “problematic” point, having the same outgoing edges, and treat the incoming edges as in the following definition. This procedure formally makes  $\text{LEB}(z)$  larger.

**Definition 19** (non-generic  $G_{\text{LEB}}(z)$ ). Let  $z \in D$  be a non-generic triangle.

- (1) **Resolving a single loop ( $T \setminus A$ ):** A point  $w$  on this curve has a loop  $R(w) = w$ . We replace  $w$  with two vertices  $w_1, w_2$ , and the loop is replaced by the 2-cycle: edges  $(w_1, w_2)$  labeled  $R$  and  $(w_2, w_1)$  labeled  $R$ . Incoming edges to  $w$  are directed to  $w_1$ . See Figure 6a.
- (2) **Resolving a pair of loops ( $T \cap A$ ):** Points in  $T \cap A$  come in pairs:  $w, w'$ , where each possesses a loop  $L(w) = w, R(w') = w'$ , and  $L(w') = w, R(w) = w'$ . We apply the splitting procedure (as in the previous item) to both vertices. We then connect the resulting pairs by duplicating the edges between  $w$  and  $w'$ : for  $i = 1, 2$ , we add edges  $(w_i, w'_i)$  labeled  $R$  and  $(w'_i, w_i)$  labeled  $L$ . Incoming edges to  $w$  (resp.  $w'$ ) are directed to  $w_1$  (resp.  $w'_1$ ). See Figure 6b.
- (3) **Resolving double edges ( $S$ ):** For a point  $w' \in S$  where  $L(w') = R(w') = w$ , we split  $w$  into two vertices  $w_L$  and  $w_R$ . All incoming edges labeled  $L$  are directed to  $w_L$ , and all incoming edges labeled  $R$  are directed to  $w_R$ . See Figure 6c.

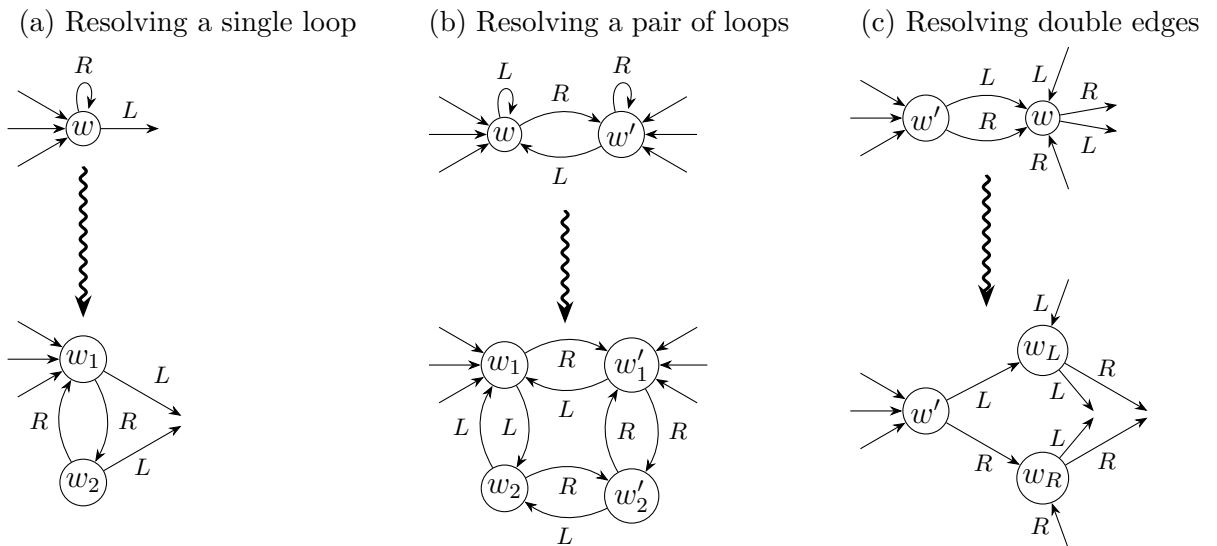


FIGURE 6. Adjustments of  $G_{\text{LEB}}(z)$  for vertices with loops or double edges.

Notice that with the above definition, even for non-generic points, a terminal quadruple corresponds to a directed  $K_{2,2}$  subgraph of  $G_{\text{LEB}}(z)$ .

For a triangle  $z$ , we enumerate the similarity classes in  $\text{LEB}(z)$  by  $[n]$ , setting the similarity class of  $z$  first, and consider the **adjacency matrix**  $A \in M_n(\mathbb{R})$  of  $G_{\text{LEB}}(z)$  defined by (Note that our definition of the adjacency matrix is the transpose of the common definition.)

$$A_{ij} = \begin{cases} 1, & (j, i) \in E \\ 0, & \text{otherwise} \end{cases}$$

We use the vector  $v_S \in \mathbb{Z}_+^n$  to encode a collection  $S$  of triangles in  $\text{LEB}(z)$ , where the  $i$ th coordinate of  $v_S$  represents the number of triangles of type  $i$  in  $S$ . It is clear that multiplying  $A$  by a vector in  $\mathbb{Z}_+^n$  models a single step of the LEB process. More precisely, for any collection of triangles  $S$ , the vector that represents the collection of triangles obtained after applying LEB once to all the triangles in  $S$  is  $A \cdot v_S$ . Furthermore, note that the LEB process partitions all triangles in each step into triangles of the same area. Thus, for every  $m \in \mathbb{N}$ , the probability vector  $\frac{A^m \cdot e_1}{\|A^m \cdot e_1\|}$  represents the distribution of areas occupied by the different classes in  $\text{LEB}(z)$ , after  $m$  steps.

**Lemma 20.** *For every triangle  $z$ , the matrix  $A$  satisfies the following:*

- (a)  $\lambda = 2$  is an eigenvalue of  $A$  and the spectral radius of  $A$ .
- (b) Let  $E_2$  and  $E_{-2}$  denote the eigenspaces of  $\lambda = 2$  and  $\lambda = -2$  respectively. Then  $\dim(E_2) = \dim(E_{-2}) = q(z)$ , the number of terminal quadruples in  $\text{LEB}(z)$ . Furthermore, the eigenvalues  $\pm 2$  do not have any non-trivial Jordan blocks.

*Proof.* (a) Since the column sums of  $A$  are all equal to 2, we have that  $\lambda = 2$  is an eigenvalue of  $A^t$ , with the all-ones vector  $\mathbf{1}$  as an eigenvector. Since the sets of eigenvalues of  $A$  and of  $A^t$  are equal, we know that  $\lambda = 2$  is an eigenvalue of  $A$ . By Theorem 15, we get that 2 is the spectral radius of  $A$  in this case.

(b) Let  $1 \leq k = q(z)$  be the number of terminal quadruples in  $G_{\text{LEB}}(z)$ , then we can find an  $n \times n$  permutation matrix  $P$  such that the matrix  $P^t A P$  has the following block triangular form:

$$P^t A P = \left( \begin{array}{ccc|cccc} & & & 0 & \cdots & \cdots & 0 \\ & & & \vdots & & & \vdots \\ & & C & 0 & \cdots & \cdots & 0 \\ \hline * & \cdots & * & B_1 & & & \\ \vdots & & \vdots & & B_2 & & \\ \vdots & & \vdots & & & \ddots & \\ * & \cdots & * & & & & B_k \end{array} \right)$$

where the first columns of  $A$  and of  $P^t A P$  are the same, and  $B_i = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

for each  $1 \leq i \leq k$ . Denote by  $e_j$  the  $j$ -th vector of the standard basis of  $\mathbb{R}^n$ . Thus,

for every  $1 \leq i \leq k$  the vector

$$v_i = e_{n-4(k-i)} + e_{n-4(k-i)-1} + e_{n-4(k-i)-2} + e_{n-4(k-i)-3} \quad (5)$$

is an eigenvector of  $P^t AP$  corresponding to the eigenvalue  $\lambda = 2$ , and the vectors  $v_1, \dots, v_k$  are linearly independent, so we obtain

$$\dim(E_2) \geq k.$$

In a similar way, for every  $1 \leq i \leq k$ , the vector

$$\tilde{v}_i = e_{n-4(k-i)} + e_{n-4(k-i)-1} - e_{n-4(k-i)-2} - e_{n-4(k-i)-3} \quad (6)$$

is an eigenvector of  $P^t AP$  corresponding to the eigenvalue  $\lambda = -2$ , and the vectors  $\tilde{v}_1, \dots, \tilde{v}_k$  are linearly independent, so

$$\dim(E_{-2}) \geq k.$$

For the reverse inequality, we apply Theorem 16 to  $A^2$ . The vectors  $v_1, \dots, v_k, \tilde{v}_1, \dots, \tilde{v}_k$  are  $2k$  linearly independent eigenvectors of  $A^2$  that correspond to the eigenvalue  $\lambda = 4$ , and thus its geometric multiplicity is at least  $2k$ .

From the dynamics of the LEB process, we know that from every vertex there is a path to a vertex in a terminal quadruple. In particular, any strongly connected component  $\alpha_j$  of  $G_{\text{LEB}}(z)$  that does not correspond to a terminal quadruple has at least one outgoing edge. This implies that its corresponding adjacency matrix  $B_j$  is irreducible (since  $\alpha_j$  is strongly connected) and has at least one column sum that is strictly smaller than 2. Applying Lemma 17, we deduce that any such component  $\alpha_j$  has  $\rho(\alpha_j) < 2$ . Thus, the strongly connected components of  $G_{\text{LEB}}(z)$  with spectral radius 2 are precisely the terminal quadruples.

Observe that, for the action of  $A^2$ , each terminal quadruple is partitioned into 2 sets of two terminal triangles each, which form all the strongly connected components. Hence, the fact that the dimension of the generalized eigenspace of  $\rho(A^2) = 4$  is  $2k$  follows from Theorem 16. In particular,  $\dim(E_2) = \dim(E_{-2}) = k$  and the eigenvalues  $\pm 2$  do not have any non-trivial Jordan blocks. □

**Lemma 21.** *Every generalized eigenvector  $u$  of  $A$  that corresponds to an eigenvalue  $\mu \neq 2$  satisfies  $u \in \mathbf{1}^\perp$ .*

*Proof.* Let  $(u_1, \dots, u_r)$  be a Jordan chain of  $A$  that corresponds to an eigenvalue  $\mu \neq 2$ . That is,  $Au_1 = \mu u_1$ ,  $u_1 \neq 0$ , and for every  $2 \leq i \leq r$  we have  $Au_i = \mu u_i + u_{i-1}$ .

We first observe that  $u_1 \in \mathbf{1}^\perp$ . Indeed,

$$\mu \langle u_1, \mathbf{1} \rangle = \langle \mu u_1, \mathbf{1} \rangle = \langle Au_1, \mathbf{1} \rangle = \langle u_1, A^t \cdot \mathbf{1} \rangle = \langle u_1, 2 \cdot \mathbf{1} \rangle = 2 \langle u_1, \mathbf{1} \rangle.$$

Since  $\mu \neq 2$  we have  $\langle u_1, \mathbf{1} \rangle = 0$ .

We proceed by induction. Suppose that  $u_i \in \mathbb{1}^\perp$ . Since  $Au_{i+1} = \mu u_{i+1} + u_i$ , we have

$$2\langle u_{i+1}, \mathbb{1} \rangle = \langle u_{i+1}, A^t \mathbb{1} \rangle = \langle Au_{i+1}, \mathbb{1} \rangle = \langle \mu u_{i+1} + u_i, \mathbb{1} \rangle = \mu \langle u_{i+1}, \mathbb{1} \rangle + \underbrace{\langle u_i, \mathbb{1} \rangle}_{=0},$$

and therefore  $\langle u_{i+1}, \mathbb{1} \rangle = 0$ . □

The proof of Theorem 1 is implied by the more detailed proposition below.

**Proposition 22.** *Let  $z$  be a triangle and  $A$  as above. Then*

- (a) *The limits  $w_{\text{even}} = \lim_{m \rightarrow \infty} \left(\frac{A}{2}\right)^{2m} e_1$  and  $w_{\text{odd}} = \lim_{m \rightarrow \infty} \left(\frac{A}{2}\right)^{2m+1} e_1$  exist.*
- (b) *The vectors  $w_{\text{even}}$  and  $w_{\text{odd}}$  are probability vectors, and there exist  $v \in E_2, \tilde{v} \in E_{-2}$  such that  $w_{\text{even}} = v + \tilde{v}$  and  $w_{\text{odd}} = v - \tilde{v}$ .*
- (c) *For  $j \in \mathbb{N}$  let  $w_j \in \mathbb{R}^{l(z)}$  be the probability vector that describes the partition of the area of  $z$  into triangles of different similarity classes, after  $j$  steps. Then there exists  $\xi \in (0, 1)$  such that*

$$\begin{cases} \|w_{\text{odd}} - w_j\| = O(\xi^j), & j \text{ is odd.} \\ \|w_{\text{even}} - w_j\| = O(\xi^j), & j \text{ is even.} \end{cases}$$

*Proof.* (a) It suffices to show that the first limit exists and then use  $w_{\text{odd}} = \frac{A}{2} \cdot w_{\text{even}}$ .

By Lemma 20, the spectral radius of  $A^2$  is equal to 4. Let  $4 = \lambda_1 > |\lambda_2| \geq \dots \geq |\lambda_s|$  be the distinct eigenvalues of  $A^2$ , in a decreasing order in absolute value. Since the space is a direct sum of the generalized eigenspaces, for each  $1 \leq i \leq s$  let  $u_i$  be a vector in the generalized eigenspace of  $\lambda_i$  such that

$$e_1 = \sum_{i=1}^s u_i.$$

Since  $e_1 \notin \mathbb{1}^\perp$ , by Lemma 21,  $u_1 \neq 0$ . By Lemma 20, the eigenvalues 2 and  $-2$  of  $A$  do not have non-trivial Jordan blocks, so we have

$$\left(\frac{A}{2}\right)^{2m} e_1 = \frac{(A^2)^m \cdot \sum_{i=1}^s u_i}{4^m} = u_1 + \frac{\sum_{i=2}^s (A^2)^m \cdot u_i}{4^m},$$

thus

$$\left\| \left(\frac{A}{2}\right)^{2m} e_1 - u_1 \right\| \leq \sum_{i=2}^s \frac{|\lambda_i|^m \cdot |g_i(m)| \cdot \|u_i\|}{4^m}, \quad (7)$$

where each  $g_i(m)$  is a fixed polynomial whose degree is at most the size of the maximal Jordan block of  $\lambda_i$ . Since  $|\lambda_2|, \dots, |\lambda_s| < 4$ , we have  $\lim_{m \rightarrow \infty} \left(\frac{A}{2}\right)^{2m} e_1 = u_1$ .

- (b) Since  $\frac{A}{2}$  is a non-negative matrix with all column sums equal to 1, it preserves the set of non-negative probability vectors. In particular,  $\left(\frac{A}{2}\right)^j e_1$  is a probability vector for every  $j \in \mathbb{N}$ , and hence so are  $w_{\text{even}}$  and  $w_{\text{odd}}$ . Note that the eigenspace of  $A^2$  corresponding to  $\lambda = 4$  is  $E_2 \oplus E_{-2}$ . Since  $w_{\text{even}} = u_1$ , there exist  $v \in E_2, \tilde{v} \in E_{-2}$  such that  $w_{\text{even}} = v + \tilde{v}$ . Furthermore, the equality  $w_{\text{odd}} = v - \tilde{v}$  follows from the fact that  $w_{\text{odd}} = \frac{A}{2} \cdot w_{\text{even}}$ .

(c) From the discussion above, it is clear that  $w_j = \left(\frac{A}{2}\right)^j e_1$ . Then for  $j = 2m$ , we have

$$\|w_{\text{even}} - w_j\| \stackrel{(7)}{=} \sum_{i=2}^s \frac{|\lambda_i|^m \cdot |g_i(m)| \cdot \|u_i\|}{4^m} = O(\xi^j),$$

where  $\xi$  is any number in the interval  $\left(\frac{\sqrt{|\lambda_2|}}{2}, 1\right)$ . The odd case is proven similarly, and this completes the proof.  $\square$

#### 4. CONCLUDING REMARKS AND OPEN PROBLEMS

**4.1. A remark on the geometric distribution of  $\text{LEB}(z)$ .** As we already mentioned in §2.2, we have observed several surprising properties of the geometric position of the points of  $\text{LEB}(z)$  in  $D$ , in particular, Proposition 8 characterized the geometric position of a terminal quadruple.

Furthermore, as we have seen in the proof of Lemma 14, the points

$$\{h^j(z_m) \mid j \in \{0, \dots, 2^{m-1}\}\} \subseteq V$$

lie on the circle  $C_m$ , which is tangent to the real axis at the origin. As in the proof of Theorem 3, applying  $L^m$  to this set produces  $2^{m-1} + 1$  distinct triangles in region I that lie on a circle that is tangent to the real axis at the origin. In fact, these  $2^{m-1} + 1$  points belong to distinct terminal quadruples: since  $L$  in region I is an inversion with respect to the geodesic  $|z| = \frac{\sqrt{2}}{2}$ , for every  $z \in \text{I}$ , the only circle that is centered at the imaginary axis and passes through the points  $z$  and  $L(z)$  is the one centered at  $\frac{\sqrt{2}}{2}i$ . A direct computation shows that this circle is not tangent to the real axis at 0, since the circle  $C_z$  whose center is at  $\frac{\sqrt{2}}{2}i$  does not intersect region I.

We conjecture that the whole orbit  $\text{LEB}(z)$  can be partitioned into a certain number of subsets, each lying on a certain arc that is tangent to the real axis at 0 or at  $\frac{1}{2}$ . The precise number and location of these arcs, and the precise number and location of points on these arcs requires a more refined analysis in the spirit that we have performed. We leave this for future research.

**4.2. A remark on continuity and small perturbations.** Since a small perturbation of the vertices of a triangle produces a nearby triangle, it is natural to consider the relation between  $\text{LEB}(z)$  and  $\text{LEB}(z')$  for close triangles  $z$  and  $z'$ . Let  $d : \mathbb{H} \times \mathbb{H} \rightarrow [0, \infty)$  be the hyperbolic metric. For a set  $F \subseteq \mathbb{H}$  and  $\varepsilon > 0$  we denote by

$$N_\varepsilon[F] = \bigcup_{x \in F} B(x, \varepsilon),$$

the  $\varepsilon$ -neighborhood of  $F$ . In [PP14, Lemma 3.2], the following non-increasing property of the metric  $d$  was proven. For every  $z, w \in D$  we have

$$d(L(w), L(z)) \leq d(w, z) \quad \text{and} \quad d(R(w), R(z)) \leq d(w, z).$$

We mention the following continuity result of  $\text{LEB}$ , which is an immediate corollary of the non-increasing property.

**Corollary 23.** *For every  $z \in D$  we have the following: for every  $\varepsilon > 0$  and every  $w \in D$  with  $d(z, w) < \varepsilon$  we have  $\text{LEB}(w) \subseteq N_\varepsilon[\text{LEB}(z)]$ .*

This phenomenon is exemplified in Figure 7 below, where one can readily see that triangles with low complexity can have nearby triangles with high complexity.

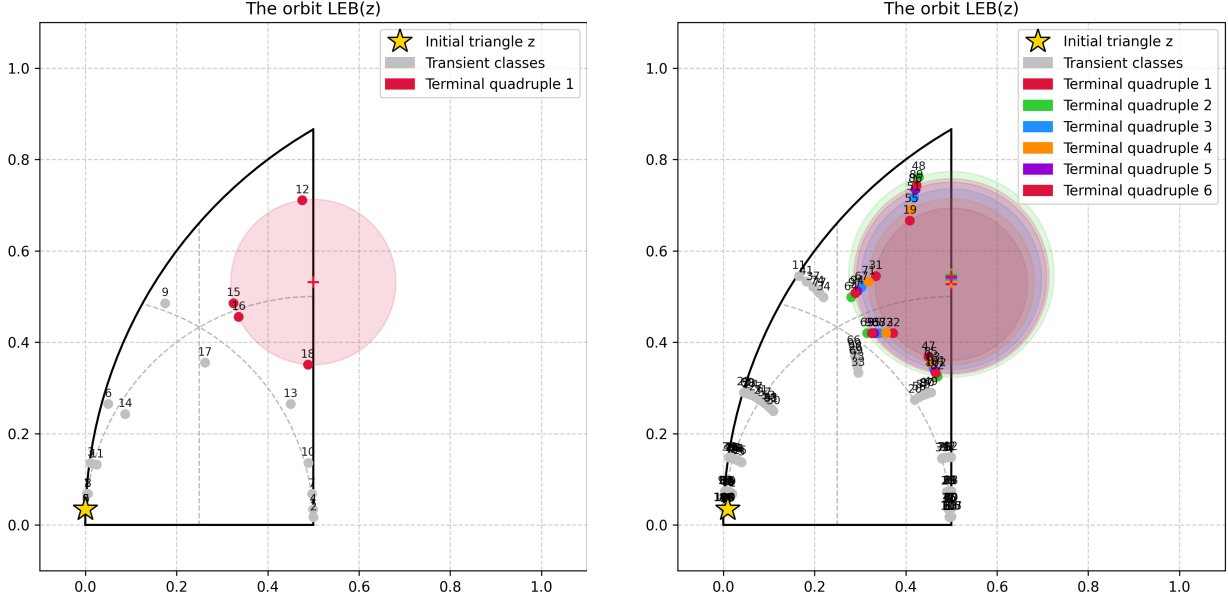


FIGURE 7. On the left, we see the orbit of the isosceles triangle  $z = \frac{1}{1700} + \frac{\sqrt{3399}}{1700}i$ , which has a total of 19 similarity classes and 1 terminal quadruple. On the right, we see the orbit of the nearby triangle  $z' = \frac{1}{100} + \frac{\sqrt{3399}}{1700}i$ , which has a total of 109 similarity classes and 6 terminal quadruples.

**4.3. Open problems.** In view of computational observations about  $G_{\text{LEB}}(z)$  we ask the following.

- (1) Is it true that the spectrum of  $G_{\text{LEB}}(z)$  is contained in  $\{\pm 2, \pm\sqrt{2}, \pm 1, 0\}$ ?
- (2) Furthermore, is it true that  $G_{\text{LEB}}(z)$  is bipartite?
- (3) Classify the cases where the adjacency matrix of  $G_{\text{LEB}}(z)$  is diagonalizable.

In view of Theorem 1, an important complexity measure of the mesh is the number  $q(z)$  of terminal quadruples, thus we propose:

- (4) Find a formula for  $q(z)$  in terms of  $z$ .

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