

PERIODICITY OF JOINT CO-TILES IN \mathbb{Z}^d

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ABSTRACT. An old theorem of Newman asserts that any tiling of \mathbb{Z} by a finite set is periodic. A few years ago, Bhattacharya proved the periodic tiling conjecture in \mathbb{Z}^2 . Namely, he proved that for a finite subset F of \mathbb{Z}^2 , if there exists $A \subseteq \mathbb{Z}^2$ such that $F \oplus A = \mathbb{Z}^2$ then there exists a periodic $A' \subseteq \mathbb{Z}^2$ such that $F \oplus A' = \mathbb{Z}^2$. The recent refutation of the periodic tiling conjecture in high dimensions due to Greenfeld and Tao motivates finding different generalizations of Newman's theorem and of Bhattacharya's theorem that hold in arbitrary dimension d . In this paper, we formulate and prove such generalizations. We do so by studying the structure of joint co-tiles in \mathbb{Z}^d . Our generalization of Newman's theorem states that for any $d \geq 1$, any joint co-tile for d independent tiles is periodic. For a $(d-1)$ -tuple of finite subsets of \mathbb{Z}^d that satisfy a certain technical condition that we call property (\star) , we prove that any joint co-tile decomposes into disjoint $(d-1)$ -periodic sets. Consequently, we show that for a $(d-1)$ -tuple of finite subsets of \mathbb{Z}^d that satisfy property (\star) , the existence of a joint co-tile implies the existence of periodic joint co-tile. Conversely, we prove that if a finite subset F in \mathbb{Z}^d admits a periodic co-tile A , then there exist $(d-1)$ additional tiles that together with F are independent and admit A as a joint co-tile, so that the first $(d-2)$ of these tiles together with F satisfy property (\star) . Combined, our results give a new necessary and sufficient condition for a subset of \mathbb{Z}^d to tile periodically. We also discuss tilings and joint tilings in other countable abelian groups.

1. INTRODUCTION

Let Γ be a countable abelian group. Given $A, B \subseteq \Gamma$ the *sumset* $A + B$ is given by

$$A + B = \{a + b : a \in A, b \in B\}.$$

We write $A \oplus B = C$ if every $c \in C$ has a unique representation as $c = a + B$ with $a \in A$ and $b \in B$. We denote the cardinality of a set F by $|F|$. We write $F \Subset \Gamma$ to indicate that F is a finite subset of Γ .

One says that F *tiles* Γ if there exists a collection of disjoint union of translates of F whose union is equal to Γ . That is, F *tiles* Γ if there exists a set $A \subseteq \Gamma$ such that

$$F \oplus A = \Gamma. \tag{1}$$

In that case, we say that A is a *co-tile* for the *tile* F . Let $g, h : \Gamma \rightarrow \mathbb{R}$, where Γ is a countable abelian group. When at least one of the functions $g, h : \Gamma \rightarrow \mathbb{R}$ is finitely supported, the *convolution* $g * h : \Gamma \rightarrow \mathbb{R}$ is well defined and given by

$$g * h(x) = \sum_{y \in \Gamma} g(y) \cdot h(x - y).$$

Using this notation, equation (1) is equivalent to $\mathbf{1}_F * \mathbf{1}_A = 1$, where $\mathbf{1}_X$ denotes the indicator function of the set X .

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Elements $g_1, \dots, g_k \in \Gamma$ of the abelian group Γ are called *independent* if the only integers $n_1, \dots, n_k \in \mathbb{Z}$ that satisfy $\sum_{j=1}^k n_j g_j = 0$ are $n_1 = \dots = n_k = 0$. The *rank* of an abelian group is the maximal size of an independent set.

Suppose that Γ is an abelian group of rank d and that $k \leq d$. A set $C \subseteq \Gamma$ is called *k-periodic* if there exists a subgroup $L \leq \Gamma$, with $\text{rank}(L) \geq k$, such that $C + L = C$. In the case that $k = d$ we will also say that C is *periodic* instead of *d-periodic*. We say that a tile set $F \Subset \Gamma$ *tiles Γ periodically* if there exists a periodic co-tile for F . If F tiles Γ but does not admit a periodic co-tile, then the set F is called *aperiodic*.

Newman [New77] proved that any tiling of $\Gamma = \mathbb{Z}$ by a finite set is periodic. Already for $\Gamma = \mathbb{Z}^2$, it is not difficult to find tilings of Γ by a finite set that are not even 1-periodic. See [GT21, §1.3] for some examples and a brief discussion. Still, it is natural to ask for different generalizations of Newman’s theorem to higher-rank abelian groups. It has been conjectured for some time that for any $F \Subset \mathbb{Z}^d$, if there exists $A \subseteq \mathbb{Z}^d$ such that $F \oplus A = \mathbb{Z}^d$ then there exists a periodic $A' \subseteq \mathbb{Z}^d$ such that $F \oplus A' = \mathbb{Z}^d$ [LW96], [GS87]. This conjecture became known as the *periodic tiling conjecture*. The periodic tiling conjecture can be interpreted as an attempt to generalize Newman’s theorem. The \mathbb{Z}^2 case of the periodic tiling conjecture was proved several years ago by Bhattacharya [Bha20]. Other instances of the periodic tiling conjecture have been proved, under additional assumptions [BN91, Khe22, Ken92, Sze98, WvL84]. The periodic tiling conjecture has recently been disproved for sufficiently large d by Greenfeld and Tao [GT22].

The recent refutation of the periodic tiling conjecture in high dimensions motivates finding different generalizations of Newman’s theorem and of Bhattacharya’s theorem that hold in arbitrary dimension d . In this paper, we formulate and prove such generalizations. Our approach is to study the structure of sets $A \subseteq \mathbb{Z}^d$ that satisfy

$$F_j \oplus A = \mathbb{Z}^d \text{ for all } j = 1, \dots, k, \quad (2)$$

for subsets $F_1, \dots, F_k \Subset \mathbb{Z}^d$. We refer to such an A as a *joint co-tile* for F_1, \dots, F_k .

In [GT23], sets $A \subseteq \mathbb{Z}^d$ satisfying (2) have been referred to as solutions to the *system of tiling equations*. It is not difficult to see that the existence of a joint co-tile for $F_1, \dots, F_k \Subset \mathbb{Z}^d$ implies that $|F_1| = |F_2| = \dots = |F_k|$ (see Proposition 2.5).

Observe that whenever $A \subseteq \mathbb{Z}^d$ is a co-tile for $F \Subset \mathbb{Z}^d$, for every k there exists tiles $F_1, \dots, F_k \Subset \mathbb{Z}^d$ admitting A as a joint co-tile. For instance, for every $v_1, \dots, v_k \in \mathbb{Z}^d$, the set A is a co-tile for $F_1, \dots, F_k \Subset \mathbb{Z}^d$, where $F_i = F + v_i$, $i = 1, \dots, k$. Furthermore, by the Dilation lemma [GT22, Lemma 3.1] (see also Lemma 3.2 below) for any $F \Subset \mathbb{Z}^d$, and any $k \in \mathbb{N}$ there exists $r_1, \dots, r_k \in \mathbb{N}$, such that any co-tile of F is also a co-tile for F_1, \dots, F_k , where $F_i := r_i F = \{r_i f : f \in F\}$. Thus, in general, joint co-tiles for k -tuples of tile F_1, \dots, F_k have no more structure than co-tiles for a single tile F . However, it turns out that additional structure on a $A \subset \mathbb{Z}^d$ can be concluded from the fact that it is joint co-tile for k tiles that are “sufficiently different”, in a natural sense that in particular excludes the stations where these k tiles cannot be obtained from a single tile by translations and dilations.

As with ordinary systems of linear equations, it makes sense to introduce a notion of independence in this setup. For $F \Subset \mathbb{Z}^d$ we denote

$$F^* := F \setminus \{0\}.$$

We say that (F_1, \dots, F_k) is an *independent tuple* of tiles (or *k independent tiles*) if each F_j is a finite subset of \mathbb{Z}^d , with $0 \in F_j$, and for every choice of $v_1 \in F_1^*, \dots, v_k \in F_k^*$, the k -tuple (v_1, \dots, v_k) is independent (equivalently here, linearly independent vectors over \mathbb{Q} , or

similarly over \mathbb{R} or \mathbb{C}), see Figure 1. Notice that if (F_1, \dots, F_k) is an independent tuple of tiles then $k \leq d$.

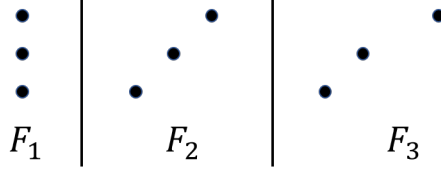


FIGURE 1. Here $F_1, F_2, F_3 \subset \mathbb{Z}^2$ and the lower point in all three of them is $(0, 0)$. Each one of the pairs (F_1, F_2) and (F_1, F_3) is independent, but the pair (F_2, F_3) is not. Also note that if the points in each of the sets lay on the lines $\{y = 0\}$, $\{y = 1\}$ and $\{y = 2\}$, then $A = \mathbb{Z} \times 3\mathbb{Z}$ is a joint co-tile for F_1, F_2, F_3 .

Building on methods developed in [Bha20], [GT21] and earlier work, we prove the following:

Theorem 1.1. *For every $1 \leq k \leq d$, the indicator function of any joint co-tile for k independent tiles in \mathbb{Z}^d is equal, to a sum of $[0, 1]$ -valued k -periodic functions plus an integer constant.*

The case $k = 1$ of Theorem 1.1 was proven in [GT21]. As a direct consequence of Theorem 1.1, we obtain the following multidimensional generalization of Newman’s result:

Theorem 1.2. *For any independent tuple of tiles in \mathbb{Z}^d , the set of joint co-tiles is finite. In particular, any joint co-tile for d independent tiles in \mathbb{Z}^d is d -periodic. Furthermore, if (F_1, \dots, F_d) is an independent tuple of tiles in \mathbb{Z}^d and $f : \mathbb{Z}^d \rightarrow \mathbb{Z}$ is a bounded function that satisfies $\mathbf{1}_{F_i} * f = 1$ for every $1 \leq i \leq d$, then f is d -periodic.*

We discuss further generalizations of Newman’s theorem in Section 4 and particularly to groups of the form $\mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})$ in Proposition 4.6.

We say that a set $A \subseteq \mathbb{Z}^d$ is *piecewise k -periodic* if there exist $A_1, \dots, A_r \subset \mathbb{Z}^d$ such that $A = \biguplus_{j=1}^r A_j$ and each A_j is k -periodic. Note that [Bha20], [GT21] and [Khe22] used *weakly periodic* for piecewise 1-periodic. In [GT21] it was shown that any $A \subseteq \mathbb{Z}^2$ satisfying $F \oplus A = \mathbb{Z}^2$ is piecewise 1-periodic, whereas in [Bha20] it was shown that almost every solution to $F \oplus A = \mathbb{Z}^2$ is piecewise 1-periodic, with respect to any invariant measure on the space of solutions. The apriori weaker “almost everywhere” result sufficed to prove the \mathbb{Z}^2 periodic tiling conjecture. The following result shows that the existence of piecewise $(d - 1)$ -periodic joint co-tiles implies the existence of d -periodic joint co-tiles. For $k = 1$ and $d = 2$ it coincides with the results in [Bha20], [GT21], deducing 2-periodicity from piecewise 1-periodicity.

Theorem 1.3. *Let k and d be positive integers and let $F_1, \dots, F_k \in \mathbb{Z}^d$. If F_1, \dots, F_k admit a piecewise $(d - 1)$ -periodic joint co-tile, then they admit a d -period joint co-tile.*

We now define an additional condition on a tuple of tiles, that is needed for the formulation of a certain generalization of Bhattacharya’s and Greenfeld-Tao’s theorems to $d > 2$:

Definition 1.4. Let (F_1, \dots, F_{d-1}) be a tuple of tiles in \mathbb{Z}^d , $d \geq 2$. We say that (F_1, \dots, F_{d-1}) has *property (\star)* if it is an independent tuple and for every $(v_1, \dots, v_{d-1}), (w_1, \dots, w_{d-1}) \in F_1^* \times \dots \times F_{d-1}^*$ such that

$$\text{span}(v_1, \dots, v_{d-1}) = \text{span}(w_1, \dots, w_{d-1}),$$

we have $v_i = w_i$ for all $1 \leq i \leq d - 2$.

Example 1.5. Consider the tiles

$$F_1 = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0)\} \text{ and } F_2 = \{(0, 0, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}.$$

Then (F_1, F_2) has property (\star) . Indeed it is easy to see that (F_1, F_2) is an independent tuple of tiles in \mathbb{Z}^3 , and for any $v_1 \in F_1^*$ and $v_2 \in F_2^*$ the intersection of $\text{span}(v_1, v_2)$ with the plane $V = \{(a, b, 0) : a, b \in \mathbb{R}\}$ is equal to $\text{span}(v_1)$. Also, for $A = 2\mathbb{Z} \times 2\mathbb{Z} \times \mathbb{Z}$ we have $F_1 \oplus A = F_2 \oplus A = \mathbb{Z}^3$, so A is a (periodic) joint co-tile.

Theorem 1.6. *Let (F_1, \dots, F_{d-1}) be a tuple of tiles in \mathbb{Z}^d that has property (\star) . Then any joint co-tile for F_1, \dots, F_{d-1} is piecewise $(d - 1)$ -periodic.*

The next statement follows immediately from [Theorem 1.6](#) together with [Theorem 1.3](#).

Corollary 1.7. *Let (F_1, \dots, F_{d-1}) be a tuple of tiles in \mathbb{Z}^d that has property (\star) . If (F_1, \dots, F_{d-1}) admits a joint co-tile then it admits a d -periodic joint co-tile.*

Note that for $d = 2$, property (\star) is vacuous, hence [Theorem 1.6](#) reduces to the statement that any co-tile for a finite subset of \mathbb{Z}^2 is piecewise 1-periodic (Greenfeld-Tao's theorem) and [Corollary 1.7](#) reduces to the statement that any finite subset of \mathbb{Z}^2 that admits a co-tile also admits a periodic co-tile (Bhattacharya's theorem). Hence for $d \geq 3$, it is natural to ask whether property (\star) is a necessary condition for the existence of a periodic joint co-tile of $(d - 1)$ tiles of \mathbb{Z}^d .

We note a particular application of our methods, although not directly related to our main results:

Theorem 1.8. *Suppose that \mathbb{Z}^d decomposes into $(d - 1)$ -periodic subsets $A_1, \dots, A_r \subset \mathbb{Z}^d$, where at least one of them is not d -periodic. Then there exists $\Gamma \leq \mathbb{Z}^d$ of rank $d - 1$ so that $A_j + \Gamma = A_j$ for all $1 \leq j \leq r$.*

On the other hand, we obtain the following converse results for [Theorem 1.2](#) and [Corollary 1.7](#).

Theorem 1.9. *Suppose that $\{0\} \subsetneq F \in \mathbb{Z}^d$ admits a periodic tiling $A \subseteq \mathbb{Z}^d$, then there exist $F_1, \dots, F_{d-1} \in \mathbb{Z}^d$ with $0 \in F_j$ and $F_j \oplus A = \mathbb{Z}^d$ for all $1 \leq j \leq d$, such that*

- (a) (F_1, \dots, F_{d-1}, F) is a d -tuple of independent tiles.
- (b) (F_1, \dots, F_{d-2}, F) has property (\star) .

Combining [Corollary 1.7](#) and [Theorem 1.9](#) (b) we obtain the following:

Corollary 1.10. *A finite set $\{0\} \subsetneq F \in \mathbb{Z}^d$ tiles \mathbb{Z}^d periodically if and only if there exists $F_1, \dots, F_{d-2} \in \mathbb{Z}^d$ and $A \subset \mathbb{Z}^d$ such that (F_1, \dots, F_{d-2}, F) has property (\star) , $F \oplus A = \mathbb{Z}^d$ and $F_j \oplus A = \mathbb{Z}^d$ for all $1 \leq j \leq d - 2$. In particular, $F \in \mathbb{Z}^3$ tiles \mathbb{Z}^3 periodically if and only if there is another co-tile F' for A such that (F', F) has property (\star) .*

The structure of the paper is as follows. [Section 2](#) contains basic background and definitions. In [Section 3](#) we prove [Theorem 3.1](#), a periodic decomposition theorem for joint co-tiles, which is a refinement of [Theorem 1.1](#). From [Theorem 3.1](#), we directly deduce [Theorem 1.1](#) and [Theorem 1.2](#). In [Section 4](#), we discuss generalizations of [Theorem 3.1](#), [Theorem 1.1](#) and [Theorem 1.2](#) to countable abelian groups. This allows us to extend Newman's Theorem to tilings of the group $\mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})$. In [Section 5](#) we prove [Theorem 1.6](#), which asserts that

property (\star) implies piecewise $(d-1)$ -periodicity of joint co-tiles. Then in [Section 6](#) we prove [Theorem 1.8](#) and deduce [Theorem 1.3](#). [Section 7](#) is dedicated to the proof of [Theorem 1.9](#). Finally, [Section 8](#) contains concluding remarks and related questions.

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2. PRELIMINARIES

A function $f : \mathbb{Z}^d \rightarrow \mathbb{R}$ is called L -periodic, where $L \leq \mathbb{Z}^d$, if for every $x \in \mathbb{Z}^d$ and $v \in L$ we have $f(x+v) = f(x)$. Recall that a set $A \subseteq \mathbb{Z}^d$ is *piecewise k -periodic* if A is the disjoint union of k -periodic sets.

Definition 2.1. Let Γ_1, Γ_2 be abelian groups. For $f : \Gamma_1 \rightarrow \Gamma_2$ and $v \in \Gamma_1$, we define the *discrete derivative of f in direction v* , $D_v f : \Gamma_1 \rightarrow \Gamma_2$, by

$$D_v f(w) := f(w) - f(w - v).$$

A function $P : \Gamma_1 \rightarrow \Gamma_2$ is called *a polynomial map of degree at most r* if

$$\forall v_1, \dots, v_{r+1} \in \Gamma_1 : D_{v_1} \dots D_{v_{r+1}} P = 0$$

(where for consistency $P = 0$, is a polynomial of degree -1). Given a subgroup $\Gamma_3 < \Gamma_1$, one says that $P : \Gamma_1 \rightarrow \Gamma_2$ is *a polynomial map of degree at most r with respect to Γ_3* if

$$\forall v_1, \dots, v_{r+1} \in \Gamma_3 : D_{v_1} \dots D_{v_{r+1}} P = 0.$$

The following basic facts about polynomials will be useful for us. [Lemma 2.2](#) below is due to Leibman [[Lei02](#), Prop. 1.21]. We include a short proof for the reader's convenience.

Lemma 2.2. *Let $P : \mathbb{Z}^d \rightarrow \mathbb{R}$ be a polynomial map with respect to a finite index subgroup $L \leq \mathbb{Z}^d$, which is bounded, then P is constant on cosets of L .*

Proof. Let $r \in \mathbb{N}$ denote the degree of P , as a polynomial with respect to L . It is clear from [Definition 2.1](#) that if r is equal to 0, then the restriction of P to each coset of L is a constant. Similarly, if the degree of P is equal to 1, then the restriction of P to each coset of L is a constant plus a non-trivial homomorphism (see e.g. [[Lei02](#)]). For contradiction, we may assume that $r \geq 1$. Observe that since P is bounded and for every $v \in L$ we have $D_v P \subseteq P(\mathbb{Z}^d) - P(\mathbb{Z}^d)$, thus $D_v P$ is bounded. Therefore, for every $v_1, \dots, v_{r-1} \in L$ the function $D_{v_1} \dots D_{v_{r-1}} P$ is a bounded polynomial map of degree exactly one, with respect to L . But non-trivial homomorphisms into \mathbb{R} are unbounded, a contradiction. \square

Definition 2.3. We say that a bounded function $f : \mathbb{Z}^d \rightarrow \mathbb{R}$ has mean m if

$$\lim_{n \rightarrow \infty} \frac{1}{|B_n|} \sum_{v \in B_n} f(v) = m, \tag{3}$$

where $B_n = \{-n, \dots, n\}^d$.

One says that $f : \mathbb{Z}^d \rightarrow \mathbb{R}/\mathbb{Z}$ is equidistributed in \mathbb{R}/\mathbb{Z} if

$$\lim_{n \rightarrow \infty} \frac{1}{|B_n|} \sum_{v \in B_n} g(f(v)) = \int_0^1 g(x) dx \tag{4}$$

holds for every continuous function $g : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$, where we identify $g : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ with $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g(x+1) = g(x)$ for all $x \in \mathbb{R}$.

We will use the following version of Weyl's equidistribution theorem for polynomials in several variables [Wey16, Theorem 20]. See also [Tao12, Chapter 1.1] and [Yif22, Theorem 1.2].

Theorem 2.4 (Weyl's equidistribution theorem for polynomials in several variables). *Let $P : \mathbb{Z}^d \rightarrow \mathbb{R}/\mathbb{Z}$ be a polynomial map with respect to a finite index subgroup Γ of \mathbb{Z}^d . Then on every coset $v + \Gamma$ of Γ , the restriction of P to $v + \Gamma$ is either equidistributed in \mathbb{R}/\mathbb{Z} or periodic.*

We implicitly rely on the following basic observation (compare [Kol04, Lemma 1.1]).

Proposition 2.5. *Let $F \subseteq \mathbb{Z}^d$. Suppose that $F \subset B_{n_0}$ for some $n_0 \in \mathbb{N}$ and that $f : \mathbb{Z}^d \rightarrow \mathbb{R}$ is a bounded function satisfying $\mathbf{1}_F * f = 1$. Denote by $C = |F|(\max f - \min f)$. Then for every $n > n_0$ one has*

$$|B_{n-n_0}| - C |B_{n+n_0} \setminus B_{n-n_0}| \leq |F| \sum_{w \in B_n} f(w) \leq |B_{n-n_0}| + C |B_{n+n_0} \setminus B_{n-n_0}|, \quad (5)$$

and thus the function f has mean $\frac{1}{|F|}$. In particular, if $F_1, F_2 \subseteq \mathbb{Z}^d$ satisfy $\mathbf{1}_{F_1} * f = \mathbf{1}_{F_2} * f = 1$, then $|F_1| = |F_2|$.

Proof. Pick $n_0 \in \mathbb{N}$ such that $F \subset B_{n_0}$. Observe that $\mathbf{1}_F * f = 1$ implies that for every $n > n_0$ we have

$$\mathbf{1}_{B_{n-n_0}} - C \cdot \mathbf{1}_{B_{n+n_0} \setminus B_{n-n_0}} \leq \mathbf{1}_F * f|_{B_n} \leq \mathbf{1}_{B_{n-n_0}} + C \cdot \mathbf{1}_{B_{n+n_0} \setminus B_{n-n_0}},$$

where $f|_{B_n}$ denotes the restriction of f to B_n . Taking the sum of the values of these functions over all $z \in \mathbb{Z}^d$ implies that (5) holds for every $n > n_0$. Since $\lim_{n \rightarrow \infty} \frac{|B_{n-n_0}|}{|B_n|} = 1$ and $\lim_{n \rightarrow \infty} \frac{|B_{n+n_0} \setminus B_{n-n_0}|}{|B_n|} = 0$, dividing (5) by $|F| \cdot |B_n|$ and letting $n \rightarrow \infty$ yields the assertion. \square

Remark 2.6. The mean of a function $f : \Gamma \rightarrow \mathbb{R}$ is defined similarly, using (3), for any countable amenable group Γ , in which case B_n is replaced by a Følner sequence in Γ , and an analogue of Proposition 2.5 holds in this more general context as well. In Section 8, we implicitly apply Proposition 2.5 for countable abelian groups Γ , which are in particular amenable.

2.1. Shifts of finite type. The space of co-tiles for a given finite set $F \subset \mathbb{Z}^d$, or more generally, the space of joint co-tiles for a given collection of sets, can naturally be endowed with the structure of a compact topological space on which \mathbb{Z}^d acts by homeomorphisms. Topological dynamical systems of this kind are called \mathbb{Z}^d -subshifts, more specifically *subshifts of finite type*. We include here relevant terminology and basic facts from the field of symbolic dynamics, particularly regarding shifts of finite type. We refer to [LM95] for a comprehensive introduction to symbolic dynamics.

Let Σ be a finite set (alphabet) and Γ a finitely generated abelian group. The set of functions from Γ to Σ , denoted Σ^Γ , is called *the full Γ -shift*. For $x \in \Sigma^\Gamma$ and $v \in \Gamma$, we use x_v to denote the value of x at v (this is an element of Σ). Also for $x \in \Sigma^\Gamma$ and $v \in \Gamma$ we denote by $\sigma_v(x) \in \Sigma^\Gamma$ the *shift of x by v* , which is given by

$$\sigma_v(x)_w = x_{v+w}.$$

Endowing Σ^Γ with the product topology, where the topology on Σ is the discrete topology, makes Σ^Γ a compact Γ -space. A closed, non-empty and Γ -invariant subset $X \subseteq \Sigma^\Gamma$ is called a Γ -*subshift*. For $x \in \Sigma^\Gamma$, the *stabilizer of x* is defined to be

$$\text{stab}(x) = \{v \in \Gamma : \sigma_v(x) = x\},$$

which is a (possibly trivial) subgroup of Γ . A point $x \in \Sigma^\Gamma$ is called *k-periodic* if $\text{stab}(x)$ is a subgroup of rank k . When $\Gamma = \mathbb{Z}$, we say that $x \in \Sigma^\mathbb{Z}$ is *periodic* if it has a non-trivial stabilizer.

Definition 2.7. A Γ -subshift $X \subseteq \Sigma^\Gamma$ is called a *subshift of finite type (SFT)* if there exists a finite set $W \subset \Gamma$ and a set $\mathcal{F} \subseteq \Sigma^W$ such that

$$X = \{x \in \Sigma^\Gamma : \forall v \in \Gamma, \sigma_v(x)|_W \notin \mathcal{F}\}.$$

For every $F \in \mathbb{Z}^d$ the space of co-tiles for F is a subshift of finite type, under the natural identification of the space of co-tiles for F with

$$X_F := \left\{x \in \{0, 1\}^{\mathbb{Z}^d} : \mathbf{1}_F * x = 1\right\}.$$

To see that X_F is indeed an SFT, take $W = -F$ and

$$\mathcal{F} = \left\{p \in \{0, 1\}^W : \sum_{w \in W} p(w) \neq 1\right\},$$

and then

$$X_F = \left\{x \in \{0, 1\}^{\mathbb{Z}^d} : \forall v \in \mathbb{Z}^d, \sigma_v(x)|_W \notin \mathcal{F}\right\}.$$

Since a non-empty intersection of SFTs is also an SFT, it follows that the space of joint co-tiles for a collection of tiles is an SFT (unless it is empty).

The following simple result is based on a pigeonhole argument. The proof is well-known and standard, see for instance [LM95, Exercise 2.2.10]. We include a proof for completeness.

Lemma 2.8. *Every \mathbb{Z} -subshift of finite type admits a periodic point.*

Proof. Let $X \subseteq \Sigma^\mathbb{Z}$ be a \mathbb{Z} -subshift of finite type, where Σ is a finite set. Then by definition, there exists a finite set $W \in \mathbb{Z}$ and $\mathcal{F} \subseteq \Sigma^W$ such that

$$X = \{x \in \Sigma^\mathbb{Z} : \forall v \in \mathbb{Z}, \sigma_v(x)|_W \notin \mathcal{F}\},$$

and $X \neq \emptyset$. Fix $x \in X$, and let $N \in \mathbb{N}$ be an integer bigger than $\max(W) - \min(W)$. Since the set $\Sigma^{\{1, \dots, N\}}$ is finite, by the pigeonhole principle there exist integers $0 \leq i < j \leq |\Sigma|^N$ such that

$$x|_{\{i, \dots, i+N-1\}} = x|_{\{j, \dots, j+N-1\}}.$$

Let $p = j - i$ and define $\hat{x} \in \Sigma^\mathbb{Z}$ by

$$\hat{x}_n = x_{i+(n \bmod p)}.$$

Then \hat{x} is a periodic point, and for every $n \in \mathbb{Z}$ there exists $t \in \{i, \dots, j-1\}$ such that $\hat{x}|_{W+n} = x|_{W+t}$. Hence, $\hat{x} \in X$, which proves that X admits a periodic point. \square

We recall the following result in multidimensional symbolic dynamics.

Lemma 2.9. *Let Γ be a finitely generated abelian group, $\Gamma_0 \leq \Gamma$ a subgroup, and $X \subseteq \Sigma^\Gamma$ a Γ -subshift. Let*

$$X_{\Gamma_0} := \{x \in X : \Gamma_0 \leq \text{stab}(x)\}. \quad (6)$$

If $X_{\Gamma_0} \neq \emptyset$ then it is a Γ -subshift. Furthermore, if X is a subshift of finite type then X_{Γ_0} is also a subshift of finite type.

Proof. First, we show that X_{Γ_0} is a subshift. Since Γ is abelian, for every $v \in \Gamma$, $v_0 \in \Gamma_0$ and $y \in X_{\Gamma_0}$ we have

$$\sigma_{v_0}(\sigma_v(y)) = \sigma_v(\sigma_{v_0}(y)) = \sigma_v(y).$$

This shows $\sigma_v(y) \in X_{\Gamma_0}$ for all $v \in \Gamma$ hence X_{Γ_0} is Γ -invariant. To see that X_{Γ_0} is a closed subset of Σ^Γ , consider a sequence $(y_n)_{n \in \mathbb{N}} \in X_{\Gamma_0}$ such that

$$\lim_{n \rightarrow \infty} y_n = y \in \Sigma^\Gamma$$

in the product topology. Since each $y_n \in X_{\Gamma_0} \subseteq X$ and X is a closed subset of Σ^Γ , we get $y \in X$. Note that for any $v_0 \in \Gamma_0$,

$$\sigma_{v_0}(y) = \sigma_{v_0} \left(\lim_{n \rightarrow \infty} y_n \right) = \lim_{n \rightarrow \infty} (\sigma_{v_0}(y_n)) = \lim_{n \rightarrow \infty} (y_n) = y,$$

which shows $y \in X_{\Gamma_0}$ and hence X_{Γ_0} is a subshift. Now assuming that X is an SFT we show that X_{Γ_0} is also an SFT. Observe that $X_{\Gamma_0} = X \cap Y$ where

$$Y = \{x \in \Sigma^\Gamma : \Gamma_0 \leq \text{stab}(x)\}.$$

Since Γ_0 is a subgroup of a finitely generated abelian group it is also finitely generated. Let $\{\gamma_1, \dots, \gamma_r\}$ be a finite generating set for Γ_0 . Then

$$Y = \bigcap_{i=1}^r \{x \in \Sigma^\Gamma : \forall v \in \Gamma, x_{v+\gamma_i} = x_v\}.$$

To see that Y is an SFT, let $W = \{0, \gamma_1, \dots, \gamma_r\}$ and

$$\mathcal{F} = \{w \in \Sigma^W : \exists 1 \leq i \leq r \text{ s.t. } w_0 \neq w_{\gamma_i}\}.$$

Then

$$Y = \{x \in \Sigma^\Gamma : \forall v \in \Gamma, \sigma_v(x)|_W \notin \mathcal{F}\}.$$

Hence Y is an SFT, which completes the argument. □

From [Lemma 2.9](#) we deduce the following conclusion:

Lemma 2.10. *Let Γ be a finitely generated abelian group of rank d . If $X \subseteq \Sigma^\Gamma$ is a Γ -subshift of finite type that admits a $(d-1)$ -periodic point then it admits a d -periodic point.*

Proof. Suppose $X \subseteq \Sigma^\Gamma$ is a Γ -subshift of finite type that admits a $(d-1)$ -periodic point, namely a point $z \in X$ and a subgroup $\Gamma_0 \leq \Gamma$ of rank $d-1$ such that $\text{stab}(z) = \Gamma_0$. Let X_{Γ_0} be given by (6). Then X_{Γ_0} is non-empty, and by [Lemma 2.9](#) it is a subshift of finite type. Because $\text{rank}(\Gamma_0) = d-1$, it follows that $\text{rank}(\Gamma/\Gamma_0) = 1$. Let $v \in \Gamma$ be a vector such that $k \cdot v \notin \Gamma_0$ for all $k \in \mathbb{N}$. Then $\Gamma_0 \oplus \mathbb{Z}v$ is a finite index subgroup of Γ . Let $D \subseteq \Gamma$ be a fundamental domain for $\Gamma_0 \oplus \mathbb{Z}v$, namely a finite set such that $\Gamma_0 \oplus \mathbb{Z}v \oplus D = \Gamma$. Because $D \oplus \mathbb{Z}v$ is a fundamental domain for Γ_0 in Γ , it follows that the restriction map $\rho : X_{\Gamma_0} \rightarrow \Sigma^{D \oplus \mathbb{Z}v}$ is injective, where ρ is given by $\rho(x) = x|_{D \oplus \mathbb{Z}v}$.

Indeed, the inverse $\rho^{-1} : \rho(X_{\Gamma_0}) \rightarrow X_{\Gamma_0}$ is given by $\rho^{-1}(\tilde{x})_u = (\tilde{x})_{u'}$ for $u \in \Gamma$, where u' is the unique element in $(D \oplus \mathbb{Z}v)$ that satisfies $u - u' \in \Gamma_0$. Using the natural identification $\Sigma^{D \oplus \mathbb{Z}v} \cong (\Sigma^D)^{\mathbb{Z}}$, we can view $\rho(X_{\Gamma_0})$ as a subset of $(\Sigma^D)^{\mathbb{Z}}$, which we denote by \tilde{X} .

Let us show that \tilde{X} is a \mathbb{Z} -subshift of finite type. Because X is a Γ -subshift of finite type, there exists a finite set $W \subset \Gamma$ and $\mathcal{F} \subset \Sigma^W$ such that X_{Γ_0} is equal to the set of $x \in \Sigma^\Gamma$ satisfying $\sigma_v(x) = x$ and $\sigma_v(x)|_W \notin \mathcal{F}$ for all $v \in \Gamma_0$. We can assume without loss of generality that W is a subset of $\mathbb{Z}v \oplus D$, because $\mathbb{Z}v \oplus D$ is a fundamental domain for Γ_0 . Let $\tilde{W} = \{n \in \mathbb{Z} : (nv + D) \cap W \neq \emptyset\}$. Then $W = \biguplus_{n \in \tilde{W}} (W \cap (nv + D))$. Thus, there is a natural bijection between Σ^W and $(\Sigma^D)^{\tilde{W}}$. Let $\tilde{\mathcal{F}}$ denote the image of \mathcal{F} under this bijection. Then it follows directly that

$$\tilde{X} = \left\{ x \in (\Sigma^D)^{\mathbb{Z}} : \forall v \in \mathbb{Z} : \sigma_v(x)|_{\tilde{W}} \notin \tilde{\mathcal{F}} \right\}.$$

This proves that \tilde{X} is indeed a \mathbb{Z} -subshift of finite type.

Since \tilde{X} is a \mathbb{Z} -subshift of finite type, by [Lemma 2.8](#) there exists a periodic point \tilde{z} in \tilde{X} . Let $x = \rho^{-1}(\tilde{z})$, then $x \in X$ is a d -periodic point. \square

Remark 2.11. The simple argument behind [Lemma 2.10](#) has been applied in [\[Bha20\]](#) for the case $\Gamma = \mathbb{Z}^2$, and also in a number of older references. See for instance [\[War94\]](#) and [\[Fri97\]](#). We had difficulties finding a precise reference to the statement of [Lemma 2.10](#) in full generality.

3. THE PERIODIC DECOMPOSITION THEOREM

The following theorem asserts a certain decomposition for a joint co-tile of k -tuple of tiles in \mathbb{Z}^d . The case where $k = 1$ and f is $\{0, 1\}$ -valued essentially coincides with [\[GT21, Theorem 1.7\]](#), which is closely related to [\[Bha20, Theorem 3.3\]](#). In the particular case that the tuple of tiles is independent, [Theorem 1.1](#) is a direct consequence. Namely, the indicator function of any joint co-tile of k independent tiles is a sum of k -periodic functions, each taking values in $[0, 1]$. The goal of this section is to prove the periodic decomposition theorem for joint co-tiles and to deduce [Theorem 1.1](#) and [Theorem 1.2](#).

Theorem 3.1 (Periodic decomposition theorem). *Let $F_1, \dots, F_k \subseteq \mathbb{Z}^d$, with $0 \in F_i$ for all $1 \leq i \leq k$, and let $f : \mathbb{Z}^d \rightarrow \mathbb{Z}$ be a bounded function that satisfies $\mathbf{1}_{F_i} * f = 1$ for all $1 \leq i \leq k$. We denote by $S := |F_1| = \dots = |F_k|$ (see [Proposition 2.5](#)). Then for every $1 \leq i \leq k$ and every $(v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*$ there exists a function $\phi_{v_1, \dots, v_i} : \mathbb{Z}^d \rightarrow [\min f, \max f]$ with the following properties:*

(a) For $i < k$ we have

$$\phi_{v_1, \dots, v_i} = 1 - \sum_{v_{i+1} \in F_{i+1}^*} \phi_{v_1, \dots, v_i, v_{i+1}}.$$

(b)

$$f = (-1)^i \sum_{(v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*} \phi_{v_1, \dots, v_i} + \sum_{j=1}^i (-(S-1))^{j-1}.$$

(c) Let q denote the product of all primes less than or equal to $(\max f - \min f)S$, then

$$(\mathbb{Z}qv_1 + \dots + \mathbb{Z}qv_i) \leq \text{stab}(\phi_{v_1, \dots, v_i}),$$

(d) $\mathbf{1}_{F_j} * \phi_{v_1, \dots, v_i} = 1$ for all $1 \leq j \leq k$. In particular, ϕ_{v_1, \dots, v_i} has mean $1/S$.

There are various extensions of [Theorem 3.1](#). Some of these generalizations have further applications. For the sake of readability, we do not state the most general form and instead indicate certain generalizations in the following sections, at the expense of some repetition.

The proof of [Theorem 3.1](#) relies on [Lemma 3.2](#) below. Various versions of this lemma, which is referred to as the dilation lemma, have been proved e.g. in [[GT21](#), Lemma 3.1], [[Bha20](#), Proposition 3.1] for $\Gamma = \mathbb{Z}^d$, $d \geq 1$. We also refer our readers to [[Tij95](#), Theorem 1] where this lemma is proved for integers. The proof is based on some elementary commutative algebra and it easily extends to countable abelian groups. For the sake of self-containment, we include a sketch of the proof below. The proof below is nearly identical to [[GT21](#), Lemma 3.1], except that we apply the assumption that r is co-prime to the order of torsion elements directly before [eq. \(7\)](#).

Lemma 3.2 (Dilation lemma). *Let Γ be a countable abelian group. Let $0 \in F \subseteq \Gamma$, $\ell \in \mathbb{N}$ and $f : \Gamma \rightarrow \mathbb{Z}$ a bounded function satisfying*

$$\mathbf{1}_F * f = \ell.$$

Let q_1 be the product of all primes less than or equal to $(\max f - \min f)|F|$, let q_2 be the product of all the orders of the torsion elements in $(F - F)$, and set $q = q_1 q_2$. Then

$$\mathbf{1}_{rF} * f = \ell,$$

for all $r \in \mathbb{N}$ such that $r \equiv 1 \pmod{q}$.

Proof. We use the notation $f^{*p} = \underbrace{f * \dots * f}_{\times p}$. For any prime p we have

$$\mathbf{1}_F^{*p} = \left(\sum_{v \in F} \delta_v \right)^{*p} = \sum_{v \in F} \delta_v^{*p} \pmod{p},$$

where the last equality holds by the Frobenius identity $(f + g)^{*p} = f^{*p} + g^{*p} \pmod{p}$. For integers p that are co-prime to q_2 we have that $p(v_1 - v_2) \neq 0$ for any $v_1 \neq v_2 \in F$, so the function $v \mapsto pv$ is injective on F . Thus:

$$\sum_{v \in F} \delta_v^{*p} = \sum_{v \in F} \delta_{pv} = \mathbf{1}_{pF}. \tag{7}$$

Now convolving both sides of $\mathbf{1}_F * f = \ell$ by $\mathbf{1}_F^{*(p-1)}$ yields $\mathbf{1}_F^{*p} * f = \ell |F|^{p-1}$. Combining the above, for primes p that are co-prime to q_2 we obtain $\mathbf{1}_{pF} * f = \ell |F|^{p-1} \pmod{p}$. If additionally p is co-prime to $|F|$ by Fermat little theorem $|F|^{p-1} \equiv 1 \pmod{p}$, thus

$$\mathbf{1}_{pF} * f = \ell \pmod{p}.$$

Note that both $\mathbf{1}_F * f$ and $\mathbf{1}_{pF} * f$ take values in $[|F| \min f, |F| \max f]$. Recall that $\ell = \mathbf{1}_F * f$, so $\ell \in [|F| \min f, |F| \max f]$. Thus, for p that is also greater than the size of that interval, the above equality holds without the \pmod{p} , namely $\mathbf{1}_{pF} * f = \ell$. Finally, for $r \equiv 1 \pmod{q}$, r is a product of primes that satisfy the conditions above, and the result follows by iterating the equation $\mathbf{1}_{pF} * f = \ell$. \square

Proof of [Theorem 3.1](#). For $1 \leq i \leq k$, $(v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*$ and $N \in \mathbb{N}$ denote:

$$\phi_{v_1, \dots, v_i}^{(N)} := \frac{1}{N^i} \sum_{n_1, \dots, n_i=1}^N \delta_{(1+n_1q)v_1 + \dots + (1+n_iq)v_i} * f. \tag{8}$$

Let q be the product of all primes less than or equal to $(\max f - \min f)S$. By applying [Lemma 3.2](#) for F_j with $\Gamma = \mathbb{Z}^d$ and $\ell = 1$ we get $\mathbf{1}_{rF_j} * f = 1$ for every $r \in q\mathbb{N} + 1$. Since $0 \in F_j$ we obtain

$$f = 1 - \sum_{v \in F_j^*} \delta_{rv} * f \text{ for every } 1 \leq j \leq k.$$

For every $N \in \mathbb{N}$, setting $r = 1 + nq$ for $n \in \{1, \dots, N\}$ and taking average we conclude that for every $1 \leq j \leq k$ we have

$$f = 1 - \sum_{v \in F_j^*} \frac{1}{N} \sum_{n=1}^N \delta_{(1+nq)v} * f. \quad (9)$$

Since $\phi_{v_1}^{(N)} = \frac{1}{N} \sum_{n=1}^N \delta_{(1+nq)v_1} * f$ this gives (with $j = 1$):

$$f = 1 - \sum_{v_1 \in F_1^*} \phi_{v_1}^{(N)}. \quad (10)$$

For $1 \leq i < k$, choose any $(v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*$ and $1 \leq n_1, \dots, n_i \leq N$. Setting $j = i + 1$ in (9) and convolving both sides of the equation by $\delta_{(1+n_1q)v_1 + \dots + (1+n_iq)v_i}$ we obtain

$$\delta_{(1+n_1q)v_1 + \dots + (1+n_iq)v_i} * f = 1 - \sum_{v_{i+1} \in F_{i+1}^*} \frac{1}{N} \sum_{n_{i+1}=1}^N \delta_{(1+n_1q)v_1 + \dots + (1+n_iq)v_i + (1+n_{i+1}q)v_{i+1}} * f.$$

By averaging over $1 \leq n_1, \dots, n_i \leq N$ and applying the definition in (8) we obtain that

$$\phi_{v_1, \dots, v_i}^{(N)} = 1 - \sum_{v_{i+1} \in F_{i+1}^*} \frac{1}{N^{i+1}} \sum_{n_1, \dots, n_{i+1}=1}^N \delta_{(1+n_1q)v_1 + \dots + (1+n_{i+1}q)v_{i+1}} * f = 1 - \sum_{v_{i+1} \in F_{i+1}^*} \phi_{v_1, \dots, v_{i+1}}^{(N)}. \quad (11)$$

Since $|F_i^*| = S - 1$ for $1 \leq i \leq k$, using (10), (11) and an inductive argument we obtain that for every $N \in \mathbb{N}$ and $1 \leq i \leq k$ we have

$$f = \sum_{j=1}^i (-S + 1)^{j-1} + (-1)^i \sum_{(v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*} \phi_{v_1, \dots, v_i}^{(N)} \quad (12)$$

Notice that the functions $\delta_{(1+n_1q)v_1 + \dots + (1+n_iq)v_i} * f$ are bounded between $\min f$ and $\max f$, thus by (8), the functions $\phi_{v_1, \dots, v_i}^{(N)}$ are bounded between $\min f$ and $\max f$ for every $(v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*$. In particular, for every $(v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*$ the sequence of functions $(\phi_{v_1, \dots, v_i}^{(N)})_{N \in \mathbb{N}}$ is uniformly bounded, hence by Arzelà–Ascoli theorem (or by a Cantor diagonalization argument), it converges along a subsequence. We denote the limit by ϕ_{v_1, \dots, v_i} . Then for every $(v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*$ we have

$$\min f \leq \phi_{v_1, \dots, v_i} \leq \max f,$$

and in view of (11) and (12) we have achieved (a) and (b).

To see (c), using (8), a standard telescoping argument shows that for every $w \in \mathbb{Z}^d$, $v = (v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*$ and every $1 \leq j \leq i$

$$|\phi_{v_1, \dots, v_i}^{(N)}(w + qv_j) - \phi_{v_1, \dots, v_i}^{(N)}(w)| \leq \frac{2N^{i-1}}{N^i} (\max(f) - \min(f)) = \frac{2}{N} (\max(f) - \min(f)).$$

Thus for every $(v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*$ the function ϕ_{v_1, \dots, v_i} is qv_j -periodic for every $1 \leq j \leq i$. It remains to show (d). Clearly, since $\mathbf{1}_{F_j} * f = 1$, for every $1 \leq i, j \leq k$, $(v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*$ and $n_1, \dots, n_i \in \mathbb{N}$ we have $\mathbf{1}_{F_j} * (\delta_{(1+n_1q)v_1 + \dots + (1+n_iq)v_i} * f) = 1$. Thus, by (8), $\mathbf{1}_{F_j} * \phi_{v_1, \dots, v_i}^{(N)} = 1$ for every $N \in \mathbb{N}$ and therefore $\mathbf{1}_{F_j} * \phi_{v_1, \dots, v_i} = 1$ for every $1 \leq i, j \leq k$. In particular, by Proposition 2.5, ϕ_{v_1, \dots, v_i} has mean $1/S$. \square

Remark 3.3. Under the assumption that f is $\{0, 1\}$ -valued, it directly follows from Theorem 3.1, part (a), that for every $1 \leq i < k$ and every $(v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*$, the sum $\sum_{v_{i+1} \in F_{i+1}^*} \phi_{v_1, \dots, v_i, v_{i+1}}$ is a $[0, 1]$ -valued function. Theorem 3.1, where $k = 1$ and f is $\{0, 1\}$ -valued, coincides with [GT21, Theorem 1.7]. We will not make use of the property that $\mathbf{1}_{F_j} * \phi_{v_1, \dots, v_i} = 1$ in this paper. We mention it only for completeness and possibly for future reference. The fact that the functions ϕ_{v_1} each have mean $1/S$ played an implicit role in [Bha20].

Using the assumption that the tuple of tiles is independent Theorem 1.1 is an immediate corollary of Theorem 3.1, with f being a $\{0, 1\}$ -valued function. The proof of Theorem 1.2 is straightforward.

Proof of Theorem 1.2. Suppose that (F_1, \dots, F_d) is an independent tuple of tiles in \mathbb{Z}^d and that $f : \mathbb{Z}^d \rightarrow \mathbb{Z}$ is a bounded function satisfying $\mathbf{1}_{F_i} * f = 1$ for all $1 \leq i \leq d$. By Proposition 2.5, we have $|F_1| = \dots = |F_d| := S$. Let q be the product of all primes less than or equal to $(\max f - \min f)S$ and let

$$L = \bigcap_{(v_1, \dots, v_d) \in F_1^* \times \dots \times F_d^*} q\mathbb{Z}v_1 + \dots + q\mathbb{Z}v_d.$$

Apply Theorem 3.1 with $k = d$. It follows that f is a sum of functions whose stabilizers are rank d -subgroups, more precisely,

$$f = (-1)^d \sum_{(v_1, \dots, v_d) \in F_1^* \times \dots \times F_d^*} \phi_{v_1, \dots, v_d} + \sum_{j=1}^d (-(S-1))^{j-1},$$

and for each $(v_1, \dots, v_d) \in F_1^* \times \dots \times F_d^*$ we have that $q\mathbb{Z}v_1 + \dots + q\mathbb{Z}v_d \leq \text{stab}(\phi_{v_1, \dots, v_d})$. By the above, $\text{stab}(f)$ contains the intersection of $\text{stab}(\phi_{v_1, \dots, v_d})$ over $(v_1, \dots, v_d) \in F_1^* \times \dots \times F_d^*$, that in turn contains L .

By the assumption that the tuple (F_1, \dots, F_d) is independent, $q\mathbb{Z}v_1 + \dots + q\mathbb{Z}v_d$ is a finite index subgroup of \mathbb{Z}^d for every $(v_1, \dots, v_d) \in F_1^* \times \dots \times F_d^*$. Since L is an intersection of finitely many finite index subgroups, L is also a finite index subgroup. In particular, this proves that f is d -periodic. In particular, any joint co-tile $A \subseteq \mathbb{Z}^d$ for (F_1, \dots, F_d) is L -periodic. For any finite-index subgroup L of \mathbb{Z}^d there are finitely many L -periodic subsets, so the set of joint co-tiles for (F_1, \dots, F_d) is finite. \square

Remark 3.4. It is clear that in any system of tiling equations that admits a finite number of solutions, the stabilizer of solutions is a finite index subgroup. It turns out that the converse is also true: If every solution of a certain system of tiling equations has a finite index stabilizer, it follows that the system admits only finitely many solutions [BDJ08]. See also [MS19] for further generalizations of this fact.

4. JOINT CO-TILINGS IN FINITELY GENERATED ABELIAN GROUPS

It is natural to ask which of the results about tilings generalize from \mathbb{Z}^d to more general groups. An inspection of the proof of [Theorem 3.1](#) reveals that the statement still holds, and the same proof applies if we replace \mathbb{Z}^d by an arbitrary countable abelian group Γ , and change the value of q in [Theorem 3.1 \(c\)](#) by multiplying it with the product of the orders of all torsion elements in $F - F$.

Recall that elements g_1, \dots, g_k in a countable abelian group Γ are called *independent* if the equation $\sum_{j=1}^k n_j g_j = 0$, with $n_1, \dots, n_k \in \mathbb{Z}$, implies that $n_1 = \dots = n_k = 0$. With this definition, [Theorem 1.1](#) extends directly as follows:

Theorem 4.1. *Let Γ be a countable abelian group. For every $k \in \mathbb{N}$, the indicator function of any joint co-tile for k independent tiles in Γ is equal to a sum of $[0, 1]$ -valued functions whose stabilizer has rank at least k , plus an integer constant.*

Remark 4.2. The structure of a general countable abelian group can be quite complicated. In contrast, *finitely generated* abelian groups have a simple structure theorem: Any finitely generated abelian group is isomorphic to $\mathbb{Z}^d \times G$, for some integer $d \geq 0$ and some finite abelian group G (which is a product of cyclic groups of the form $(\mathbb{Z}/m\mathbb{Z})$). However, the following simple observation allows one to reduce statements about tilings of countable abelian groups by a finite set to the finitely generated case: Let Γ be a countable abelian group and let $F \subseteq \Gamma$ with $0 \in F$. Let Γ_0 denote the group generated by the difference set $F - F$. The assumption $0 \in F$ implies that $F \subseteq \Gamma_0$. Then for any co-tile A of F we have that $A \cap \Gamma_0$ is a co-tile of F in Γ_0 , and tilings of Γ by F decompose into tilings of cosets of Γ_0 in Γ . A corresponding statement is true also for a tuple of tiles (F_1, \dots, F_k) and a joint co-tile.

[Theorem 1.2](#) can be extended to finitely generated abelian groups as follows:

Theorem 4.3. *Let Γ be a finitely generated abelian group of rank d . Any joint co-tile for d independent tiles in Γ has a finite orbit.*

Remark 4.4. Using [Remark 4.2](#) one can directly conclude the following property for joint co-tiles of an arbitrary countable group: Let Γ be a countable abelian group and let (F_1, \dots, F_d) be an independent tuple of tiles in Γ so that the group Γ_0 generated by $\bigcup_{i=1}^d (F_i - F_i)$ has rank d . Then there are finitely many set $A_1, \dots, A_m \subset \Gamma_0$ such that the any joint co-tile $A \subseteq \Gamma$ of (F_1, \dots, F_d) satisfies $A \cap \Gamma_0 = A_i$ for some $1 \leq i \leq m$.

A quick remark about the condition of independence for a tuple of tiles for finitely generated abelian groups with non-trivial torsion:

Remark 4.5. If Γ is of the form $\Gamma = \mathbb{Z}^d \times G$ where G is a finite abelian group and (F_1, \dots, F_k) is an independent tuple of tiles in Γ , then the only torsion element in each of the sets F_i is 0. For this reason, Newman's theorem (i.e. any tiling of \mathbb{Z} by a finite set is periodic) does not hold in abelian groups Γ that are finite extensions of \mathbb{Z} .

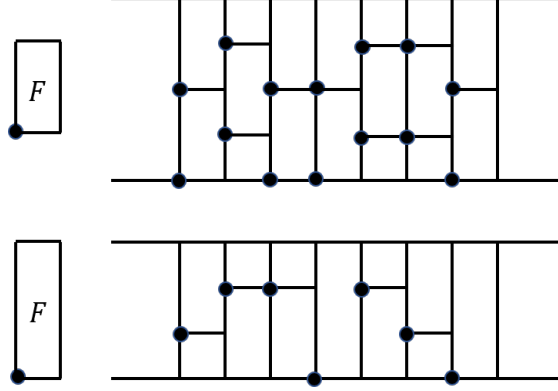


FIGURE 2. Non-periodic co-tiles of $\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ for $n = 4$ and $n = 3$ for the tile $F = \{1\} \times G$, where G is a non-trivial subgroup of $\mathbb{Z}/n\mathbb{Z}$.

Indeed, let $\Gamma = \mathbb{Z} \times G$, where G is a finite abelian group. Consider the tile $F = \{1\} \times G \in \Gamma$. The co-tiles of F are all the sets $A \subset \Gamma$ of the following form:

$$A = \{(n, g_n) : n \in \mathbb{Z}\},$$

for some sequence $(g_n)_{n \in \mathbb{Z}}$ of elements in G . In particular, it is no longer true that any co-tile of F must be periodic, unless G is trivial. Nonetheless, if G is a finite cyclic group of prime order, then the only obstructions to extending Newman's theorem are of this form. See [GT21, Lemma 5.1 and Remark 5.2] for related statements.

Proposition 4.6. *If $\Gamma = \mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})$ for some prime number p and $F \in \Gamma$ is a finite set, then every co-tile of F is periodic, unless F is of the form $F = \tilde{F} \times (\mathbb{Z}/p\mathbb{Z})$ for some finite tile $\tilde{F} \in \mathbb{Z}$, in which case the co-tiles of F are all of the form*

$$A = \{(n, g_n) : n \in \tilde{A}\}, \quad g_n \in \mathbb{Z}/p\mathbb{Z}, \quad (13)$$

where \tilde{A} is a co-tile of $\tilde{F} \in \mathbb{Z}$, which by Newman's theorem must be periodic.

The proof of the proposition relies on the following generalization of Theorem 3.1.

Theorem 4.7. *Let Γ be a countable abelian group, $F_1, \dots, F_k \in \Gamma$ such that $|F_i| = S$, and $0 \in F_i$ for all $1 \leq i \leq k$, and let $f : \Gamma \rightarrow \mathbb{Z}$ be a bounded function that satisfies $\mathbf{1}_{F_i} * f = 1$ for all $1 \leq i \leq k$. For every $1 \leq i \leq k$, let F_i^{Tor} denote the intersection of F_i with the torsion subgroup of Γ , and let $F_i^* = F_i \setminus F_i^{\text{Tor}}$. Then for every $1 \leq i \leq k$ and every $(v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*$ there exists a function $\phi_{v_1, \dots, v_i} : \Gamma \rightarrow [\min f, \max f]$ with the following properties:*

(a) *For $i < k$ we have*

$$\mathbf{1}_{F_{i+1}^{\text{Tor}}} * \phi_{v_1, \dots, v_i} = 1 - \sum_{v_{i+1} \in F_{i+1}^*} \phi_{v_1, \dots, v_i, v_{i+1}}.$$

(b) *For every $1 \leq i \leq k$ there is an integer constant C_i such that*

$$\mathbf{1}_{F_1^{\text{Tor}}} * \dots * \mathbf{1}_{F_i^{\text{Tor}}} * f = (-1)^i \sum_{(v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*} \phi_{v_1, \dots, v_i} + C_i.$$

(c) Let q_1 be the product of all primes less than or equal to $(\max f - \min f)S$, let q_2 be the product of all the orders of the torsion elements in the sets $F_i - F_i$, for $1 \leq i \leq k$, and set $q = q_1 q_2$. Then

$$(\mathbb{Z}q v_1 + \dots + \mathbb{Z}q v_i) \leq \text{stab}(\phi_{v_1, \dots, v_i}),$$

(d) $\mathbf{1}_{F_j} * \phi_{v_1, \dots, v_i} = 1$ for all $1 \leq j \leq k$. In particular, ϕ_{v_1, \dots, v_i} has mean $1/S$.

The proof of [Theorem 4.7](#) below is a minor adaptation of the proof of [Theorem 3.1](#). Note that in the case where Γ is a torsion free abelian group, $F_i^{\text{Tor}} = \{0\}$. In particular, when $\Gamma = \mathbb{Z}^d$, [Theorem 4.7](#) coincides with [Theorem 3.1](#).

Proof. By applying [Lemma 3.2](#) for F_i with $\ell = 1$ and q as in (c) we get $\mathbf{1}_{rF_i} * f = 1$ for every $r \in q\mathbb{N} + 1$. Because $r = 1 \pmod q$, we have $rF_i^{\text{Tor}} = F_i^{\text{Tor}}$. Since $F_i = F_i^{\text{Tor}} \uplus F_i^*$ we have

$$\mathbf{1}_{F_i^{\text{Tor}}} * f = 1 - \sum_{v \in F_i^*} \delta_{rv} * f \text{ for every } 1 \leq i \leq k.$$

For every $N \in \mathbb{N}$, setting $r = 1 + nq$ for $n \in \{1, \dots, N\}$ and taking average we conclude that for every $1 \leq j \leq k$ we have

$$\mathbf{1}_{F_j^{\text{Tor}}} * f = 1 - \sum_{v_j \in F_j^*} \frac{1}{N} \sum_{n_j=1}^N \delta_{(1+n_jq)v_j} * f. \quad (14)$$

Applying (14) with $j = i + 1$, convolving both sides by $\delta_{(1+n_1q)v_1 + \dots + (1+n_iq)v_i}$ and taking average over $\frac{1}{N^i} \sum_{n_1, \dots, n_i=1}^N$ yields

$$\begin{aligned} \mathbf{1}_{F_{i+1}^{\text{Tor}}} * \left[\frac{1}{N^i} \sum_{n_1, \dots, n_i=1}^N \delta_{(1+n_1q)v_1 + \dots + (1+n_iq)v_i} * f \right] = \\ 1 - \sum_{v_{i+1} \in F_{i+1}^*} \frac{1}{N^{i+1}} \sum_{n_1, \dots, n_i, n_{i+1}=1}^N \delta_{(1+n_1q)v_1 + \dots + (1+n_iq)v_i + (1+n_{i+1}q)v_{i+1}} * f. \end{aligned}$$

Defining $\phi_{v_1, \dots, v_i}^{(N)} = \frac{1}{N^i} \sum_{n_1, \dots, n_i=1}^N \delta_{(1+n_1q)v_1 + \dots + (1+n_iq)v_i} * f$, as in (8), we obtain

$$\mathbf{1}_{F_{i+1}^{\text{Tor}}} * \phi_{v_1, \dots, v_i}^{(N)} = 1 - \sum_{v_{i+1} \in F_{i+1}^*} \phi_{v_1, \dots, v_i, v_{i+1}}^{(N)}. \quad (15)$$

Note that (14) with $j = 1$ becomes $\mathbf{1}_{F_1^{\text{Tor}}} * f = 1 - \sum_{v_1 \in F_1^*} \phi_{v_1}^{(N)}$. Convolving both sides by $\mathbf{1}_{F_2^{\text{Tor}}}$ and using (15) with $i = 1$ gives

$$\mathbf{1}_{F_1^{\text{Tor}}} * \mathbf{1}_{F_2^{\text{Tor}}} * f = |F_2^{\text{Tor}}| - \sum_{v_1 \in F_1^*} \mathbf{1}_{F_2^{\text{Tor}}} * \phi_{v_1}^{(N)} = |F_2^{\text{Tor}}| - \sum_{v_1 \in F_1^*} \left(1 - \sum_{v_2 \in F_2^*} \phi_{v_1, v_2}^{(N)} \right).$$

By an inductive argument we obtain that for every $N \in \mathbb{N}$ and $1 \leq i \leq k$ there is a constant $C_i \in \mathbb{Z}$, that does not depend on N , such that

$$\mathbf{1}_{F_1^{\text{Tor}}} * \dots * \mathbf{1}_{F_i^{\text{Tor}}} * f = C_i + (-1)^i \sum_{(v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*} \phi_{v_1, \dots, v_i}^{(N)}. \quad (16)$$

Items (a) and (b) follow from (15) and (16) respectively. The rest of the proof is completely identical to the proof of [Theorem 3.1](#) and therefore omitted. \square

Lemma 4.8. *Let p be a prime number and let $\emptyset \neq F_0 \subsetneq \mathbb{Z}/p\mathbb{Z}$. Then $\mathbf{1}_{F_0}$ is an invertible element of the ring $\mathbb{Q}^{\mathbb{Z}/p\mathbb{Z}}$, where multiplication in the ring is convolution. In other words, there exists $g \in \mathbb{Q}^{\mathbb{Z}/p\mathbb{Z}}$ such that $g * \mathbf{1}_{F_0} = \delta_0$.*

Proof. Consider the ring $\mathbb{Q}[x]/\langle x^p - 1 \rangle$ (with operations of addition and multiplication of polynomials). It is easy to check that this ring is isomorphic as a ring to $\mathbb{Q}^{\mathbb{Z}/p\mathbb{Z}}$, with the operations of pointwise addition and convolution. The isomorphism is given by identifying an element

$$\sum_{i=0}^{p-1} a_i x^i + \langle x^p - 1 \rangle \in \mathbb{Q}[x]/\langle x^p - 1 \rangle$$

with the function $f \in \mathbb{Q}^{\mathbb{Z}/p\mathbb{Z}}$ given by $f(i + p\mathbb{Z}) = a_i$.

Let $F_0 \subset \mathbb{Z}/p\mathbb{Z}$ be a non-empty proper subset of $\mathbb{Z}/p\mathbb{Z}$. Then $\mathbf{1}_{F_0} \in \mathbb{Q}^{\mathbb{Z}/p\mathbb{Z}}$ is naturally identified with the coset of the polynomial $P(x) = \sum_{(i+p\mathbb{Z}) \in F_0} x^i$ in $\mathbb{Q}[x]/\langle x^p - 1 \rangle$. Then the assumption that F_0 is a non-empty proper subset of $\mathbb{Z}/p\mathbb{Z}$ implies that the polynomial P is co-prime to the cyclotomic polynomial of order p , $\Phi_p = \sum_{i=0}^{p-1} x^i$. Since $P(1) = |F_0| \neq 0$ it follows that P is co-prime to $x - 1$. Because $x^p - 1 = \Phi_p(x)(x - 1)$, it follows that P is co-prime to $x^p - 1$. Hence there exists polynomials $Q_1, Q_2 \in \mathbb{Q}[x]$ such that

$$1 = Q_1(x)P(x) + Q_2(x)(x^p - 1).$$

This means that in the ring $\mathbb{Q}[x]/\langle x^p - 1 \rangle$, the coset of $Q_1(x)P(x)$ is the same as the coset of the polynomial 1. Since the coset of the polynomial 1 in $\mathbb{Q}[x]/\langle x^p - 1 \rangle$ corresponds to $\delta_0 \in \mathbb{Q}^{\mathbb{Z}/p\mathbb{Z}}$, this implies that $g * \mathbf{1}_{F_0} = \delta_0$, where $g \in \mathbb{Q}^{\mathbb{Z}/p\mathbb{Z}}$ is the element corresponding to the coset of Q_1 . \square

Proof of Proposition 4.6. Let p be a prime number and $F \subseteq \mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})$ be a finite set. Suppose $A \subset \mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})$ satisfies $\mathbf{1}_F * \mathbf{1}_A = 1$. Applying Theorem 4.7 with $\Gamma = \mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})$, $k = 1$, $F_1 = F$ and $f = \mathbf{1}_A$, we conclude that $\mathbf{1}_{F^{\text{Tor}}} * \mathbf{1}_A$ is a sum functions having infinite stabilizer, hence $\mathbf{1}_{F^{\text{Tor}}} * \mathbf{1}_A$ is periodic.

First, assume that there is a set $\tilde{F} \subseteq \mathbb{Z}$ such that $F = \tilde{F} \times \mathbb{Z}/p\mathbb{Z}$. So $\mathbf{1}_F = \mathbf{1}_{\tilde{F} \times \{0\}} * \mathbf{1}_{\{0\} \times (\mathbb{Z}/p\mathbb{Z})}$. Thus $\mathbf{1}_{\tilde{F} \times \{0\}} * \mathbf{1}_{\{0\} \times (\mathbb{Z}/p\mathbb{Z})} * \mathbf{1}_A = 1$. This implies that $\mathbf{1}_{\{0\} \times (\mathbb{Z} \times p\mathbb{Z})} * \mathbf{1}_A \leq 1$, so for every $n \in \mathbb{Z}$ there exists at most one element $g_n \in \mathbb{Z}/p\mathbb{Z}$ such that $(n, g_n) \in A$. Hence, in this case, A is of the form (13) for some set $\tilde{A} \subset \mathbb{Z}$. It follows that $\mathbf{1}_{\tilde{F}} * \mathbf{1}_{\tilde{A}} = 1$, where the convolution here is with respect to the group \mathbb{Z} .

Now suppose that F is not of the above form. This means that there exists $n \in \mathbb{Z}$ such that $F \cap (\{n\} \times \mathbb{Z}/p\mathbb{Z})$ is a non-empty proper subset of $\{n\} \times (\mathbb{Z}/p\mathbb{Z})$. By translating F we can assume without loss of generality that F^{Tor} is neither empty nor equal to $\{0\} \times (\mathbb{Z}/p\mathbb{Z})$. Then there exists a non-empty proper subset $F_0 \subset \mathbb{Z}/p\mathbb{Z}$ such that $F^{\text{Tor}} = \{0\} \times F_0$. In this case, by Lemma 4.8, there exists $g : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Q}$ such that $g * \mathbf{1}_{F_0} = \delta_0$, where the convolution is in $(\mathbb{Z}/p\mathbb{Z})$. Let $\tilde{g} : \mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Q}$ be given by $\tilde{g}(0, i) = g(i)$ for $i \in \mathbb{Z}/p\mathbb{Z}$ and $g(n, i) = 0$ for every $n \in \mathbb{Z} \setminus \{0\}$ and $i \in \mathbb{Z}/p\mathbb{Z}$. Then $\tilde{g} * \mathbf{1}_{F^{\text{Tor}}} = \delta_0$, where this time the convolution is in $\mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})$. Since $\mathbf{1}_{F^{\text{Tor}}} * \mathbf{1}_A$ is periodic, so is $\tilde{g} * \mathbf{1}_{F^{\text{Tor}}} * \mathbf{1}_A = \mathbf{1}_A$.

We have thus shown that in the case that F is not of the form $F = \tilde{F} \times (\mathbb{Z}/p\mathbb{Z})$ for some $\tilde{F} \subseteq \mathbb{Z}$, every co-tile is periodic. \square

5. PROPERTY (\star) IMPLIES $(d - 1)$ -PIECEWISE PERIODICITY

In this section, we use property (\star) to deduce [Theorem 1.6](#). To this end, we will use [Theorem 2.4](#), which is a version of Weyl's equidistribution theorem for polynomials in several variables. The relevance of Weyl's equidistribution theorem to our setting comes from [Lemma 5.1](#) below. We note that similar arguments have appeared earlier in [\[Bha20\]](#), [\[KS20\]](#), and [\[GT21\]](#).

Lemma 5.1. *Suppose $g, g_1, \dots, g_m : \Gamma_1 \rightarrow \Gamma_2$ are functions, where Γ_1, Γ_2 are abelian groups, such that $\sum_{i=1}^m g_i = g$. Suppose g is a polynomial of degree at most $r \in \mathbb{N}$ with respect to a subgroup $\Gamma_0 \leq \Gamma_1$. For any $1 \leq i < j \leq m$ define the group $L_{i,j} = \text{stab}(g_i) + \text{stab}(g_j)$, and let $L = \bigcap_{1 \leq i < j \leq m} L_{i,j} \cap \Gamma_0$. Then each g_i is a polynomial of degree at most $\max\{m - 1, r\}$ with respect to L . In particular, if Γ_0 and $L_{i,j}$ has finite index in Γ_1 for every $1 \leq i < j \leq m$, then L has finite index in Γ_1 , and each g_i is a polynomial with respect to a finite index subgroup of Γ_1 .*

Proof. We prove the claim by induction on m . If $m = 1$ then $g_1 = g$, so the claim holds. For $m > 1$, take $v \in L$, then in particular $v \in L_{1,2} \cap \Gamma_0$ and thus $v = v_1 + v_2$ for some $v_1 \in \text{stab}(g_1)$ and $v_2 \in \text{stab}(g_2)$. Note that for every function $f : \Gamma_1 \rightarrow \Gamma_2$, the identity $D_v f = D_{v_1} f \circ \sigma_{v_2} + D_{v_2} f$ holds, where $\sigma_u : \Gamma_1 \rightarrow \Gamma_1$ denotes the shift by u , $\sigma_u(w) = w - u$. Since $D_{v_1} g_1 = 0$, applying this identity to $g_1 = -\sum_{i=2}^m g_i + g$ yields

$$D_v g_1 = D_{v_2} g_1 = -D_{v_2} \left(\sum_{i=2}^m g_i - g \right).$$

Since $D_{v_2} g_2 = 0$ we have

$$D_v g_1 + \sum_{i=3}^m D_{v_2} g_i = D_{v_2} g. \quad (17)$$

Note that $v_2 \in \Gamma_0$, hence $D_{v_2} g$ is a polynomial of degree at most $r - 1$ with respect to Γ_0 . So by the induction hypothesis, each summand on the left-hand side in (17) is a polynomial of degree at most $\max\{m - 2, r - 1\}$ with respect to a subgroup L' , defined in a similar way to L using the functions $D_v g_1, D_{v_2} g_3, \dots, D_{v_2} g_m$. In particular, for every $v \in L$ the function $D_v g_1$ is a polynomial of degree at most $\max\{m - 2, r - 1\}$ with respect to L' .

Now observe that for every $f : \Gamma_1 \rightarrow \Gamma_2$ and $v \in \Gamma_1$ we have $\text{stab}(f) \subseteq \text{stab}(D_v f)$, thus $L \leq L'$ and for every $v \in L$ we, in particular, have that $D_v g_1$ is a polynomial of degree at most $\max\{m - 2, r - 1\}$ with respect to L . In a similar way for $2 \leq i \leq m$ and every $v \in L$, each $D_v g_i$ is a polynomial of degree at most $\max\{m - 2, r - 1\}$ with respect to L , which completes the proof. \square

Lemma 5.2. *Suppose $g : \mathbb{Z}^d \rightarrow [0, 1]$ is a function such that:*

- (1) $g \bmod 1$ is a polynomial with respect to a finite index subgroup of \mathbb{Z}^d .
- (2) g is a sum of finitely many non-negative $(d - 1)$ -periodic functions.

Then there exists a finite index subgroup $\Gamma \leq \mathbb{Z}^d$ such that the restriction of g to each coset of Γ is $(d - 1)$ -periodic.

Proof. Suppose $g = \sum_{i=1}^m g_i$, where $g_i : \mathbb{Z}^d \rightarrow [0, 1]$ and $\text{rank}(\text{stab}(g_i)) \geq d - 1$. In case that $\text{rank}(\bigcap_{i=1}^m \text{stab}(g_i)) \geq d - 1$, the function g is $(d - 1)$ -periodic and the assertion follows. Otherwise, by summing together some of the g_i 's we can assume without loss of generality

that $\text{stab}(g_i) + \text{stab}(g_j)$ is a finite index subgroup of \mathbb{Z}^d , for every $i \neq j$. By [Lemma 5.1](#), because g modulo 1 is a polynomial with respect to a finite index subgroup, we conclude that each of the g_i 's modulo 1 are polynomials with respect to a finite index subgroup $\Gamma_0 \leq \mathbb{Z}^d$. Let

$$\Gamma = \Gamma_0 \cap \bigcap_{i \neq j} (\text{stab}(g_i) + \text{stab}(g_j)).$$

We will show that g is $(d-1)$ -periodic on each coset of Γ . Since $\Gamma \leq \Gamma_0$, each g_i modulo 1 is also a polynomials with respect to Γ . Hence by Weyl's equidistribution theorem ([Theorem 2.4](#)), every g_i modulo 1 is either equidistributed or periodic, on each coset of Γ .

Fix $u \in \mathbb{Z}^d$. Let $g^{(u)} : (u + \Gamma) \rightarrow [0, 1]$ denote the restriction of g to this coset. We consider 3 cases:

- (1) Suppose there exists $1 \leq i \leq m$ and $v \in (u + \Gamma)$ such that $g_i(v) = 1$. Then because $0 \leq g(v) \leq 1$ and $g_j(v) \geq 0$, we conclude that $g_j(v) = 0$ for all $j \neq i$. But $g_i(v) = 1$ implies that $g_i(v + w_1) = 1$ for all $w_1 \in \text{stab}(g_i)$ so by the same argument $g_j(v + w_1) = 0$ for all $w_1 \in \text{stab}(g_i)$. Thus, $g_j(v + w_1 + w_2) = 0$ for all $w_1 \in \text{stab}(g_i)$ and $w_2 \in \text{stab}(g_j)$. Since $\Gamma \leq \text{stab}(g_i) + \text{stab}(g_j)$, we conclude that g_j is zero on the coset $u + \Gamma$, for all $j \neq i$. This shows that in this case $g^{(u)} = g_i$ on $u + \Gamma$, and in particular $g^{(u)}$ is $(d-1)$ -periodic. So in the remaining cases we can assume that none of the g_i 's are equal to one, hence the g_i 's obtain values in the interval $[0, 1)$.
- (2) Suppose there exists $1 \leq i \leq m$ such that g_i is equidistributed modulo 1 on $u + \Gamma$. Let $0 < \epsilon < 1$ be smaller than all the non-zero values obtained by the (possibly empty) set of g_j that are periodic modulo 1. Because g_i is equidistributed modulo 1 on $u + \Gamma$, there exists $v \in u + \Gamma$ such that $g_i(v) > 1 - \epsilon$. Thus, $g_j(v) < \epsilon$ for all $j \neq i$. As in the previous part, using $\Gamma \leq \text{stab}(g_i) + \text{stab}(g_j)$, we conclude that $g_j(w) < \epsilon$ for all $j \neq i$ and all $w \in u + \Gamma$. This tells us that in particular that g_j is not equidistributed modulo 1 on $u + \Gamma$. By the choice of ϵ , $g_j(w) = 0$ for every periodic $j \neq i$ and every $w \in u + \Gamma$. We conclude also in this case that $g = g_i$ on $u + \Gamma$ and particular $g^{(u)}$ is $(d-1)$ -periodic.
- (3) The remaining case is that for every $i = 1, \dots, m$, the function g_i takes values in $[0, 1)$ and g_i modulo 1 is periodic on $u + \Gamma$. Since they take values in $[0, 1)$, the g_i 's themselves are all d -periodic. It follows in this case that $g^{(u)}$ is d -periodic, as the sum of d -periodic functions (and in particular $(d-1)$ -periodic).

□

Proof of [Theorem 1.6](#). We conveniently assume $d > 2$, because the case $d = 2$ is covered by [\[GT21\]](#). Suppose that $A \subset \mathbb{Z}^d$ satisfies $F_i \oplus A = \mathbb{Z}^d$ for all $1 \leq i \leq d-1$, where (F_1, \dots, F_{d-1}) is a tuple of tiles in \mathbb{Z}^d that has property (\star) , see [Definition 1.4](#). Let $\phi_{v_1, \dots, v_{d-1}} : \mathbb{Z}^d \rightarrow [0, 1]$ be as in [Theorem 3.1](#), applied for $k = d-1$ and $f = \mathbf{1}_A$. Given $(v_1, \dots, v_{d-2}) \in F_1^* \times \dots \times F_{d-2}^*$ and a $(d-1)$ -dimensional subspace $V < \mathbb{R}^d$ such that $v_1, \dots, v_{d-2} \in V$, define

$$\psi_V = \sum_{w_{d-1} \in F_{d-1}^* \cap V} \phi_{v_1, \dots, v_{d-2}, w_{d-1}}.$$

Note that by the independence of (F_1, \dots, F_{d-1}) , every $(d-1)$ -tuple in $F_1^* \times \dots \times F_{d-1}^*$ spans a $(d-1)$ -dimensional subspace. Denote by H the set (counted without multiplicity) of all $(d-1)$ -dimensional subspaces of \mathbb{R}^d spanned by $(d-1)$ -tuples in $F_1^* \times \dots \times F_{d-1}^*$, and for $(v_1, \dots, v_{d-2}) \in F_1^* \times \dots \times F_{d-2}^*$ let $H(v_1, \dots, v_{d-2}) \subset H$ be the set of such subspaces

of dimension $(d-1)$ that contain v_1, \dots, v_{d-2} . Thus, for every fixed tuple $(v_1, \dots, v_{d-2}) \in F_1^* \times \dots \times F_{d-2}^*$ we have

$$\sum_{w_{d-1} \in F_{d-1}^*} \phi_{v_1, \dots, v_{d-2}, w_{d-1}} = \sum_{V \in H(v_1, \dots, v_{d-2})} \psi_V. \quad (18)$$

By property (\star) , $\{H(v_1, \dots, v_{d-2}) : (v_1, \dots, v_{d-2}) \in F_1^* \times \dots \times F_{d-2}^*\}$ is a partition of H , therefore

$$\sum_{(v_1, \dots, v_{d-1}) \in F_1^* \times \dots \times F_{d-1}^*} \phi_{v_1, \dots, v_{d-1}} = \sum_{V \in H} \psi_V. \quad (19)$$

It follows that the functions ψ_V possess the following three properties:

(i)

$$1 - \phi_{v_1, \dots, v_{d-2}} = \sum_{V \in H(v_1, \dots, v_{d-2})} \psi_V.$$

(ii) $\text{stab}(\psi_V)$ is a rank $(d-1)$ subgroup of $V \cap \mathbb{Z}^d$.

(iii) ψ_V modulo 1 is a polynomial with respect to a finite index subgroup of \mathbb{Z}^d .

Indeed, property (i) is a direct consequence of [Theorem 3.1](#) part (a) with $i = d-1$, combined with (18). Property (ii) follows from [Theorem 3.1](#) part (c). Setting $\widetilde{\psi}_V = \psi_V \bmod 1$, the equation in [Theorem 3.1](#) part (b) (with $f = \mathbf{1}_A$ and $i = d-1$), combined with (19), yields that $\sum_{V \in H} \widetilde{\psi}_V = 0$. By property (ii), $\text{stab}(\widetilde{\psi}_V) + \text{stab}(\widetilde{\psi}_{V'})$ is a finite index subgroup of \mathbb{Z}^d whenever $V, V' \in H$ and $V \neq V'$. Thus property (iii) follows from [Lemma 5.1](#).

In view of these three properties, [Lemma 5.2](#) can be applied to $g = 1 - \phi_{v_1, \dots, v_{d-2}}$, for any $(v_1, \dots, v_{d-2}) \in F_1^* \times \dots \times F_{d-2}^*$. This implies that there is a finite index subgroup $\Gamma_{d-2} \leq \mathbb{Z}^d$ such that each $\phi_{v_1, \dots, v_{d-2}}$ is a polynomial with respect to Γ_{d-2} , and its restriction to every coset $u + \Gamma_{d-2}$ is $(d-1)$ -periodic.

Next, we iterate the above argument using the recursion formula in part (a) of [Theorem 3.1](#) combined with [Lemma 5.2](#). In turn, this yields a finite index subgroup $\Gamma_1 \leq \mathbb{Z}^d$ such that each ϕ_{v_1} is a polynomial with respect to Γ_1 , and its restriction to every coset $u + \Gamma_1$ is $(d-1)$ -periodic. By part (b) of [Theorem 3.1](#) with $i = 1$ we have that

$$1 - \mathbf{1}_A = \sum_{v_1 \in F_1^*} \phi_{v_1}.$$

So applying [Lemma 5.2](#) to $g = 1 - \mathbf{1}_A$, we obtain a finite index subgroup $\Gamma \leq \mathbb{Z}^d$ such that the restriction of $1 - \mathbf{1}_A$ to each coset of Γ is $(d-1)$ -periodic. Hence the restriction of $\mathbf{1}_A$ to each coset of Γ is $(d-1)$ -periodic. Thus, if u_1, \dots, u_r are cosets representatives of Γ in \mathbb{Z}^d , setting $A_{u_i} = A \cap (u_i + \Gamma) \subset \mathbb{Z}^d$ yields a decomposition $A = A_{u_1} \uplus \dots \uplus A_{u_r}$ of A into finitely many $(d-1)$ -periodic sets, as required. \square

6. FROM PIECEWISE $(d-1)$ -PERIODICITY TO d -PERIODICITY

The following lemma extracts an idea that appears within the proof of [[GT21](#), Theorem 5.4].

Lemma 6.1. *Suppose that $f_1, \dots, f_r, f : \mathbb{Z}^d \rightarrow \mathbb{R}$ are bounded functions satisfying $f = \sum_{i=1}^r f_i$. Assume additionally that:*

- (1) $\text{stab}(f_i) + \text{stab}(f_j)$ is a finite index subgroup of \mathbb{Z}^d for all $1 \leq i < j \leq r$.
- (2) $\text{stab}(f)$ is a finite index subgroup of \mathbb{Z}^d .

Then, for each $1 \leq j \leq r$, the group $\text{stab}(f_j)$ is of finite index in \mathbb{Z}^d .

Proof. Let $g_1 = f_1 - f$ and $g_j = f_j$ for $2 \leq j \leq r$. Then $g_1 + \dots + g_r = 0$ and $\text{stab}(g_i) + \text{stab}(g_j)$ is a finite index subgroup of \mathbb{Z}^d for all $1 \leq i < j \leq r$. Using the fact that 0 is a polynomial, and applying [Lemma 5.1](#), we get that each g_i is a polynomial with respect to a finite index subgroup of \mathbb{Z}^d . But each g_i is bounded. By [Lemma 2.2](#), a polynomial with respect to a finite index subgroup of \mathbb{Z}^d that is bounded must be constant on cosets of this finite index subgroup. This implies that for each $1 \leq j \leq r$ the group $\text{stab}(f_j)$ is of finite index in \mathbb{Z}^d . \square

[Theorem 1.8](#) is a direct consequence of the above lemma, as shown below.

Proof of [Theorem 1.8](#). Set $f_j = \mathbf{1}_{A_j}$, then $\sum_{j=1}^r f_j = 1$. Let $L_j \leq \mathbb{Z}^d$ be the subgroups of rank at least $d - 1$ that stabilizes A_j . Note that for every two such subgroups $L_{j_1}, L_{j_2} \leq \mathbb{Z}^d$, either their intersection has rank $d - 1$ or their sum has finite index in \mathbb{Z}^d . Assume by contradiction that the intersection of all L_j 's is of rank less than $d - 1$. By unifying some of the A_j 's we can assume without loss of generality that $L_{j_1} + L_{j_2}$ is a finite index subgroup of \mathbb{Z}^d for all $1 \leq j_1 < j_2 \leq r$. By [Lemma 6.1](#), for every $1 \leq j \leq r$ the group $L_j = \text{stab}(f_j)$ is of finite index in \mathbb{Z}^d . Since an intersection of finitely many finite index subgroups of \mathbb{Z}^d has rank d , the assumption that $\text{rank}\left(\bigcap_{j=1}^r L_j\right) < d - 1$ is false. \square

The following statement is a direct generalization of [[Bha20](#), Theorem 2.3]. The argument used in our proof below is quite similar to the one appearing in [BPeriodicity2020](#).

Lemma 6.2. *Suppose that $\Sigma \Subset \mathbb{R}$ is a finite set of real numbers, $g_1, \dots, g_r : \mathbb{Z}^d \rightarrow \mathbb{R}$ are finitely supported functions and $f : \mathbb{Z}^d \rightarrow \Sigma$ is a $(d - 1)$ -periodic function such that $g_j * f$ is d -periodic for every $1 \leq j \leq r$. Then there exists a d -periodic function $\tilde{f} : \mathbb{Z}^d \rightarrow \Sigma$ such that $g_j * f = g_j * \tilde{f}$ for every $1 \leq j \leq r$.*

Proof. Consider the space

$$X = \{\tilde{x} \in \Sigma^{\mathbb{Z}^d} : \forall 1 \leq j \leq r, g_j * \tilde{x} = g_j * f\},$$

and let $\Gamma = \bigcap_{j=1}^r \text{stab}(g_j * f)$. Then X is a Γ -shift of finite type, and by definition $f \in X$ is a $(d - 1)$ -periodic point in X . Apply [Lemma 2.10](#) to conclude that there exists $\tilde{f} \in X$ that is d -periodic. Any such point \tilde{f} satisfies the conclusion of the lemma. \square

At this stage, we are prepared to present the proof of [Theorem 1.3](#).

Proof of [Theorem 1.3](#). Suppose that $A \subseteq \mathbb{Z}^d$ is a piecewise $(d - 1)$ -periodic joint co-tile for $F_1, \dots, F_k \Subset \mathbb{Z}^d$. That is, there exists functions $f_1, \dots, f_r : \mathbb{Z}^d \rightarrow \{0, 1\}$, each f_j is $(d - 1)$ -periodic, and $\mathbf{1}_A = \sum_{j=1}^r f_j$. Consider first the case that $\text{rank}\left(\bigcap_j \text{stab}(f_j)\right) \geq d - 1$. Let $X = \bigcap_{i=1}^k \{x \in \{0, 1\}^{\mathbb{Z}^d} : \mathbf{1}_{F_i} * x = 1\}$. The elements of X are precisely the indicator functions of joint co-tiles for (F_1, \dots, F_k) . By the discussion following [Definition 2.7](#), X is a shift of finite type, and $\mathbf{1}_A = \sum_{j=1}^r f_j \in X$ is a $(d - 1)$ periodic point. Thus by [Lemma 2.10](#) X contains a d -periodic point.

It remains to consider the case where $\text{rank}\left(\bigcap_j \text{stab}(f_j)\right) < d - 1$: Note that for every two subgroups $L_1, L_2 \leq \mathbb{Z}^d$ having rank at least $d - 1$, either their intersection has rank at least $d - 1$ or $L_1 + L_2$ has finite index in \mathbb{Z}^d . So as before, by possibly summing some of the f_j 's we can assume without loss of generality that $\text{stab}(f_i) + \text{stab}(f_j)$ is a finite index subgroup of \mathbb{Z}^d

for all $1 \leq l < j \leq r$. Now consider the functions $\mathbf{1}_{F_i} * f_j$. Observe that for every $1 \leq i \leq k$ we have $\sum_{j=1}^r \mathbf{1}_{F_i} * f_j = 1$, and for every $1 \leq j \leq r$ we have $\text{stab}(f_j) \leq \text{stab}(\mathbf{1}_{F_i} * f_j)$. Thus setting $\Lambda_{i,j} := \text{stab}(\mathbf{1}_{F_i} * f_j)$ yields that $\text{rank}(\Lambda_{i,j}) \geq d - 1$ and $\Lambda_{i,l} + \Lambda_{i,j}$ is a finite index subgroup of \mathbb{Z}^d , for every $1 \leq i \leq k$ and $1 \leq l < j \leq r$. Applying [Lemma 6.1](#) for each $1 \leq i \leq k$ separately we see that each $\Lambda_{i,j}$ is a finite index subgroup of \mathbb{Z}^d . That is, each one of the functions $\mathbf{1}_{F_i} * f_j$ is d -periodic. For any fixed $1 \leq j \leq r$, applying [Lemma 6.2](#) with $g_i = \mathbf{1}_{F_i}$ and $f = f_j$ and $\Sigma = \{0, 1\}$, yields a d -periodic function $\tilde{f}_j : \mathbb{Z}^d \rightarrow \{0, 1\}$ that satisfies $\mathbf{1}_{F_i} * \tilde{f}_j = \mathbf{1}_{F_i} * f_j$, for all $1 \leq i \leq k$. In particular, the function $f : \mathbb{Z}^d \rightarrow \mathbb{Z}$ defined by $f := \sum_{j=1}^r \tilde{f}_j$ is bounded, d -periodic, and it satisfies

$$\forall 1 \leq i \leq k : \quad \mathbf{1}_{F_i} * f = \mathbf{1}_{F_i} * \left(\sum_{j=1}^r \tilde{f}_j \right) = \sum_{j=1}^r \mathbf{1}_{F_i} * \tilde{f}_j = \sum_{j=1}^r \mathbf{1}_{F_i} * f_j = 1.$$

Since $f = \sum_{j=1}^r \tilde{f}_j$ is a sum of $\{0, 1\}$ -valued functions and $\mathbf{1}_{F_i} * f = 1$, it follows that f itself is $\{0, 1\}$ -valued, hence f is an indicator of a set \tilde{A} that satisfies $F_i \oplus \tilde{A} = \mathbb{Z}^d$, for $1 \leq i \leq k$. Since each \tilde{f}_j is d -periodic, so is \tilde{A} . This completes the proof. \square

7. CONSTRUCTING INDEPENDENT TILES WITH PROPERTY (\star) FOR A PERIODIC CO-TILE

In this section, we prove [Theorem 1.9](#). We repeatedly rely on the following basic fact:

Lemma 7.1. *Let $L \leq \mathbb{Z}^d$ be a finite index subgroup and let $U_1, \dots, U_r \subset \mathbb{R}^d$ be affine subspaces of dimension strictly smaller than d . Then the set $L \setminus \bigcup_{i=1}^r U_i$ is infinite.*

Proof. For $n \in \mathbb{N}$ let $B_n = \{-n, \dots, n\}^d$. Then there exist $c, c_1, \dots, c_r > 0$ such that $|B_n \cap L| \geq cn^d$ while $|B_n \cap U_i| \leq c_i n^{\dim U_i} \leq c_i n^{d-1}$. In particular, $|B_n \cap (L \setminus \bigcup_{i=1}^r U_i)|$ tends to infinity as n tends to infinity. \square

Lemma 7.2. *Let $F \in \mathbb{Z}^d$, let $A \subseteq \mathbb{Z}^d$ such that $F \oplus A = \mathbb{Z}^d$ and let $L \leq \mathbb{Z}^d$ be a subgroup satisfying $A + L = A$. Then for every function $f : F \rightarrow L$ the tile set*

$$F_f := \{v + f(v) : v \in F\}$$

satisfies $F_f \oplus A = \mathbb{Z}^d$.

Proof. Given a function $f : F \rightarrow L$, we show that $F_f \oplus A = \mathbb{Z}^d$. The condition $F \oplus A = \mathbb{Z}^d$ can be rewritten as $\mathbb{Z}^d = \bigsqcup_{v \in F} (v + A)$. Since $A + L = A$ and $f(v) \in L$ for every $v \in F$, it follows that $f(v) + A = A$. Thus,

$$\mathbb{Z}^d = \bigsqcup_{v \in F} (v + A) = \bigsqcup_{v \in F} (v + f(v) + A) = \bigsqcup_{\tilde{v} \in F_f} (\tilde{v} + A).$$

This proves that $F_f \oplus A = \mathbb{Z}^d$. \square

Let us introduce some ad-hoc notation:

Definition 7.3. Given $d, m \in \mathbb{N}$, an m -tuple of vectors $T = (v_1, \dots, v_m)$ with $v_i \in \mathbb{Z}^d$ for $1 \leq i \leq m$, a function $g : \{1, \dots, m\} \rightarrow \mathbb{Z}^d$, a subset $J \subseteq \{1, \dots, m\}$ and a subspace $W < \mathbb{R}^d$, let $V_W(T, g, J)$ denote the subspace of \mathbb{R}^d/W obtained by projecting $\text{span}\{v_j + g(j) : j \in J\}$ into \mathbb{R}^d/W via the map $v \mapsto v + W$.

Lemma 7.4. *Suppose we are given $d, m \in \mathbb{N}$, an m -tuple of vectors $T = (v_1, \dots, v_m)$ with $v_i \in \mathbb{Z}^d$ for $1 \leq i \leq m$ and finite index subgroup $L \leq \mathbb{Z}^d$. Then for every finite collection \mathcal{W} of proper subspaces of \mathbb{R}^d there exists a function $g : \{1, \dots, m\} \rightarrow L$ so that for every $J \subseteq \{1, \dots, m\}$ and every $W \in \mathcal{W}$ we have*

$$\dim(V_W(T, g, J)) = \min\{d - \dim(W), |J|\}.$$

Proof. We fix a finite collection \mathcal{W} of proper subspaces of \mathbb{R}^d and prove the claim by induction on m . For $m = 1$, we only need to choose $g(1) \in L$ such that $v_1 + g(1) \notin W$ for any $W \in \mathcal{W}$. This is possible by Lemma 7.1. Assume by induction that $g(1), \dots, g(m) \in L$ have been defined so that the conclusion holds for every $J \subseteq \{1, \dots, m\}$ and every $W \in \mathcal{W}$. Using Lemma 7.1 we can choose $g(m+1) \in L$ that is not contained in any affine hyperplane of the form

$$U := -v_{m+1} + \text{span}\{v_j + g(j) : j \in J\} + W,$$

where $W \in \mathcal{W}$ and J ranges over subsets of $\{1, \dots, m\}$ of size at most $d - \dim(W) - 1$. We need to show that for any $J \subseteq \{1, \dots, m+1\}$ and $W \in \mathcal{W}$ we have $\dim(V_W(T, g, J)) = \min\{d - \dim(W), |J|\}$. Fix some $J \subseteq \{1, \dots, m+1\}$ and $W \in \mathcal{W}$. The assertion follows from the induction hypothesis in case $(m+1) \notin J$, so suppose $(m+1) \in J$. By the induction hypothesis, $\dim(V_W(T, g, J \setminus \{m+1\})) = \min\{d - \dim(W), |J \setminus \{m+1\}|\}$. If $|J \setminus \{m+1\}| \geq d - \dim(W)$, then $\dim(V_W(T, g, J \setminus \{m+1\})) = d - \dim(W)$, as required. Otherwise, we have that

$$\dim(V_W(T, g, J \setminus \{m+1\})) = |J \setminus \{m+1\}| = |J| - 1.$$

By our choice of $g(m+1)$, we have that

$$v_{m+1} + g(m+1) \notin \text{span}\{v_j + g(v_j) : j \in J \setminus \{m+1\}\} + W,$$

so

$$\dim(V_W(T, g, J)) = \dim(V_W(T, g, J \setminus \{m+1\})) + 1 = |J|.$$

This completes the induction step, hence the proof. \square

Proof of Theorem 1.9. Suppose $F \oplus A = \mathbb{Z}^d$ where $L \leq \mathbb{Z}^d$ is a finite index subgroup satisfying $A + L = A$. Write $F^* = \{w_1, \dots, w_k\}$. We apply Lemma 7.4 with $m = (d-1)k$, the m -tuple $T = (v_1, \dots, v_m)$, where $v_{kj+i} = w_i$ for $0 \leq j \leq d-2$, and $1 \leq i \leq k$, and $\mathcal{W} = \{\text{span}\{v\} : v \in F\}$ to obtain a function $g : \{1, \dots, m\} \rightarrow L$ so that for every $J \subseteq \{1, \dots, m\}$ and every $W \in \mathcal{W}$ we have

$$\dim(V_W(T, g, J)) = \min\{d - \dim(W), |J|\}.$$

In other words, the set $\tilde{T} := \{v_k + g(k) : 1 \leq k \leq m\} \subset \mathbb{Z}^d$ has the property that any d elements of \tilde{T} are independent, and the union of any $d-1$ elements of \tilde{T} together with any element of F^* are also independent.

For $0 \leq j \leq d-2$ we set

$$F_{j+1} = \{0\} \cup \{v_{kj+i} + g(kj+i) : 1 \leq i \leq k\} = \{0\} \cup \{w_i + g(kj+i) : 1 \leq i \leq k\}.$$

By Lemma 7.2 we indeed have $F_j \oplus A = \mathbb{Z}^d$ for every $1 \leq j \leq d-1$.

To see that (F_1, \dots, F_{d-1}, F) is a d -tuple of independent tiles, note that for any choice of $(u_1, \dots, u_{d-1}, v) \in F_1^* \times \dots \times F_{d-1}^* \times F$, the set $\{u_1, \dots, u_{d-1}, v\}$ is a union of $d-1$ elements of \tilde{T} together with an element of F^* , hence it is an independent set.

Let us check that $\{F_1, \dots, F_{d-2}, F\}$ has the property (\star) . Since property (\star) is vacuous in $d = 2$, we assume that $d \geq 3$. Choose two distinct $(d-2)$ -tuples

$$(u_1, \dots, u_{d-2}), (\tilde{u}_1, \dots, \tilde{u}_{d-2}) \in F_1^* \times \dots \times F_{d-2}^*,$$

and $v, \tilde{v} \in F^*$. Since the $(d-2)$ -tuples are distinct, there exists $1 \leq i \leq d-2$ such that $u_i \neq \tilde{u}_i$. So $\{u_1, \dots, u_{d-2}\} \cup \{\tilde{u}_i\} \cup \{v\}$ is a subset of \mathbb{Z}^d that consists of $d-1$ elements of \tilde{T} together with an element of F^* , hence it is an independent set. In particular,

$$\text{span}(\{\tilde{u}_1, \dots, \tilde{u}_{d-2}\}) \not\subseteq \text{span}(\{u_1, \dots, u_{d-2}, v\}).$$

This shows that

$$\text{span}(\{\tilde{u}_1, \dots, \tilde{u}_{d-2}, \tilde{v}\}) \neq \text{span}(\{u_1, \dots, u_{d-2}, v\}),$$

which proves that (F_1, \dots, F_{d-2}, F) has property (\star) . \square

8. FURTHER COMMENTS AND QUESTIONS

8.1. Integer-valued co-tiles. Given $F \in \Gamma$, we say that a bounded function $f : \Gamma \rightarrow \mathbb{Z}$ is an *integer-valued co-tile* for F if $\mathbf{1}_F * f = 1$. Observe that our proof of [Theorem 1.3](#) holds for integer-valued co-tile as well, thus we have:

Proposition 8.1. *Let k and d be positive integers and let $F_1, \dots, F_k \in \mathbb{Z}^d$. Suppose that F_1, \dots, F_k admit an integer-valued joint co-tile f and that $f = \sum_{i=1}^k f_i$, where each $f_i : \mathbb{Z}^d \rightarrow \mathbb{Z}$ is bounded and $(d-1)$ -periodic. Then F_1, \dots, F_k admit a d -period integer-valued joint co-tile.*

It is natural to ask whether the existence of an integer-valued co-tile for $F \in \Gamma$ implies the existence of a set $A \subseteq \Gamma$ for which $\mathbf{1}_F * \mathbf{1}_A = 1$? The simple example below shows that this is not true even for $\Gamma = \mathbb{Z}$ (or for Γ a finite cyclic group, here $\mathbb{Z}/18\mathbb{Z}$). Let $F_1 = \{0, 1\}$, $F_2 = \{0, 3, 6\}$ and $F = F_1 \oplus F_2 = \{0, 1, 3, 4, 6, 7\}$.

$$\begin{array}{cccccccc} 0 & 1 & & 3 & 4 & & 6 & 7 \\ \bullet & \bullet & \circ & \bullet & \bullet & \circ & \bullet & \bullet \end{array}$$

We claim that F does not tile \mathbb{Z} , but it does admit an integer-valued co-tile. Note that for $A_1 = 2\mathbb{Z}$ and $A_2 = \{0, 1, 2\} \oplus 9\mathbb{Z}$ we have

$$F_1 \oplus A_1 = F_2 \oplus A_2 = \mathbb{Z}.$$

Furthermore, if \tilde{A}_1 is a co-tile for F_1 then \tilde{A}_1 must be a translate of A_1 . To see that F does not tile \mathbb{Z} , suppose by contradiction that $F \oplus A = \mathbb{Z}$ then $F_1 \oplus (F_2 \oplus A) = \mathbb{Z}$, so we must have that $F_2 \oplus A$ is a coset of $2\mathbb{Z}$, but this is clearly impossible since F_2 is not contained in a coset of $2\mathbb{Z}$. Now take

$$f = \mathbf{1}_{A_1} - \mathbf{1}_{A_2}.$$

Then using $\mathbf{1}_F = \mathbf{1}_{F_1} * \mathbf{1}_{F_2}$ and $\mathbf{1}_{F_i} * 1 = |F_i|$ we get:

$$\mathbf{1}_F * f = \mathbf{1}_{F_2} * (\mathbf{1}_{F_1} * \mathbf{1}_{A_1}) - \mathbf{1}_{F_1} * (\mathbf{1}_{F_2} * \mathbf{1}_{A_2}) = |F_2| - |F_1| = 1.$$

8.2. Conditions for joint tilings for d independent tiles in \mathbb{Z}^d . In view of [Theorem 1.2](#), the classical Wang argument (see [\[Ber66\]](#), [\[Rob71\]](#)) implies that it is algorithmically decidable whether a set of d independent tiles in \mathbb{Z}^d admit a joint co-tile: Indeed, any such tiling must be periodic so we can exhaust the possible periodic co-tiles. As in [\[GT21\]](#), from an upper bound for the period of a co-tile one can directly deduce an upper bound for the computational complexity of this tiling problem. It is of interest to find explicit necessary and sufficient conditions for a d -tuple of independent subsets of \mathbb{Z}^d to admit a joint co-tile. In view of [Theorem 1.9](#), the previous problem is closely related to the more basic question of finding explicit necessary and sufficient conditions for a finite set of \mathbb{Z}^d to tile periodically.

Conversely, one can ask about necessary and sufficient conditions for an infinite subset of \mathbb{Z}^d to be a joint co-tile for d -independent tiles. In view of [Theorem 1.2](#) and [Theorem 1.9](#), this is equivalent to the question of finding necessary and sufficient conditions for a periodic subset of \mathbb{Z}^d to be a co-tile for a finite tile.

A complete solution to the above questions involves the factorization of finite abelian groups, namely understanding solutions for $A \oplus B = G$, where G is a finite abelian group. This is a difficult problem even in the cyclic case $G = \mathbb{Z}/M\mathbb{Z}$, which comes up in tilings of \mathbb{Z} .

Coven and Meyerowitz [\[CM99\]](#) found explicit and efficiently verifiable sufficient conditions for tiling the integers by a finite set. It has been conjectured that these conditions are also necessary. This conjecture has been verified in some specific cases recently [\[LL22a, LL22b\]](#). The necessity of the Coven-Meyerowitz conditions would imply an efficient algorithm for determining if a given finite subset $F \subseteq \mathbb{Z}$ can tile \mathbb{Z} , see [\[KM09\]](#).

8.3. Higher level tilings. A level ℓ co-tile of \mathbb{Z}^d by a finite set $F \subseteq \mathbb{Z}^d$ is a set $A \subseteq \mathbb{Z}^d$ such that $\mathbf{1}_F * \mathbf{1}_A = \ell$. An obvious generalization of [Proposition 2.5](#) implies that if $\mathbf{1}_F * f = \ell$ then f has mean $\frac{\ell}{|F|}$. A fairly routine modification of the proof of [Theorem 3.1](#) gives the following generalization:

Theorem 8.2. *Let $\ell_1, \dots, \ell_k \in \mathbb{N}$, $F_1, \dots, F_k \subseteq \mathbb{Z}^d$, with $0 \in F_i$ for all $1 \leq i \leq k$, and let $f : \mathbb{Z}^d \rightarrow \mathbb{Z}$ be a bounded function that satisfies $\mathbf{1}_{F_i} * f = \ell_i$ for all $1 \leq i \leq k$. Then for every $1 \leq i \leq k$ and every $(v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*$ there exists a function $\phi_{v_1, \dots, v_i} : \mathbb{Z}^d \rightarrow [\min f, \max f]$ with the following properties:*

(a) *For $i < k$ we have*

$$\phi_{v_1, \dots, v_i} = \ell_{i+1} - \sum_{v_{i+1} \in F_{i+1}} \phi_{v_1, \dots, v_i, v_{i+1}}.$$

(b)

$$f = (-1)^i \sum_{(v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*} \phi_{v_1, \dots, v_i} + \sum_{j=1}^i (-1)^{j-1} \prod_{t=1}^j \ell_t \prod_{s=1}^{j-1} |F_s|.$$

(c) *Let q denote the product of all primes less than or equal to $\max_{1 \leq i \leq k} \ell_k (\max f - \min f) \max_{1 \leq i \leq k} |F_i|$, then*

$$(\mathbb{Z}q v_1 + \dots + \mathbb{Z}q v_i) \leq \text{stab}(\phi_{v_1, \dots, v_i}),$$

(d) $\mathbf{1}_{F_i} * \phi_{v_1, \dots, v_i} = \ell_i$ for every $1 \leq i \leq k$. In particular, it has mean $\ell_i / |F_i|$.

From [Theorem 8.2](#) it is easy to deduce that both [Theorem 1.1](#) and [Theorem 1.2](#) generalize to level ℓ tilings, in the sense that any “joint higher-level-co-tile” for d -independent tiles in \mathbb{Z}^d is periodic.

8.4. Piecewise 1-periodicity of co-tiles in $\mathbb{Z}^2 \times (\mathbb{Z}/p\mathbb{Z})$. By applying the arguments of Section 4, the methods of [GT21] directly give:

Theorem 8.3. *Let p be a prime number, $\Gamma = \mathbb{Z}^2 \times (\mathbb{Z}/p\mathbb{Z})$ and $F \Subset \Gamma$ be a finite set. Then one of the following holds:*

- (1) *Any $A \subset \Gamma$ satisfying $F \oplus A = \Gamma$ is piecewise 1-periodic.*
- (2) *There exists a finite set $\tilde{F} \subset \mathbb{Z}^2$ such that $F = \tilde{F} \times (\mathbb{Z}/p\mathbb{Z})$.*

Note that according to our definition a subset of A of a finitely generated abelian group Γ is 1-periodic if and only if there an element $\gamma \in \Gamma$ of infinite order such that $\gamma + A = A$.

In fact, using Theorem 4.7 and the results of Section 5, we can deduce the following: For any rank 2 abelian group Γ and any $F \Subset \Gamma$, if $F \oplus A = \Gamma$ then the set $F^{\text{Tor}} \oplus A$ is piecewise 1-periodic, whereas in Section 4, F^{Tor} is the intersection of F with the torsion subgroup of Γ . Then in the case $\Gamma = \mathbb{Z}^2 \times (\mathbb{Z}/p\mathbb{Z})$ with p prime, Lemma 4.8 implies Theorem 8.3.

Corollary 8.4. *Let p be a prime number, $\Gamma = \mathbb{Z}^2 \times (\mathbb{Z}/p\mathbb{Z})$ and $F \Subset \Gamma$ be a finite set. If F tiles Γ , then F tiles Γ periodically.*

Rachel Greenfeld and Terence Tao have informed us in private communication that they also obtained Corollary 8.4.

8.5. A Fourier-analytic and algebraic-geometric approach to the study of joint co-tiles. Fourier analytic methods are a natural approach to translational tiling problems, see [Kol04]. Here we outline a specific application related to [GT21, Remark 1.8]. Let $g_1, \dots, g_d : \mathbb{Z}^d \rightarrow \mathbb{C}$ be finitely supported functions, by which we mean that $g_i(v) = 0$ for all but finitely many $v \in \mathbb{Z}^d$. Suppose $f : \mathbb{Z}^d \rightarrow \mathbb{C}$ is a bounded function that satisfies $g_i * f = 1$ for all $1 \leq i \leq d$. Taking distributional Fourier transform on both sides yields

$$\hat{g}_i \cdot \hat{f} = \delta_0.$$

Thus, the distributional Fourier transform of f is supported on 0 and the intersection of the zeros of \hat{g}_i . In particular, if $\hat{g}_1, \dots, \hat{g}_d$ have finitely many common zeros, f must be the Fourier transform of a multivariate trigonometric polynomial. From the multidimensional version of Weyl’s equidistribution theorem, it follows that f must be either periodic or equidistributed with respect to a measure whose support is an infinite compact subset of \mathbb{C} . So if we further assume f takes finitely many values (as is the case when f is the indicator function of a joint co-tile), f must be periodic.

The set of common zeros for d polynomials in d variables is “generically” a finite set. Given $v = (n_1, \dots, n_d) \in \mathbb{Z}_+^d$ let $X^v := x_1^{n_1} \dots x_d^{n_d}$ denote the corresponding monomial in d variables x_1, \dots, x_d . Given a finite set $F \Subset \mathbb{Z}_+^d$, let $P_F := \sum_{v \in F} X^v$ denote the corresponding multivariate polynomial. We conclude that whenever $F_1, \dots, F_d \Subset \mathbb{Z}_+^d$ are subsets such that the algebraic variety

$$V(P_{F_1}, \dots, P_{F_d}) := \bigcap_{i=1}^d \{(x_1, \dots, x_d) \in \mathbb{C}^d : P_{F_i}(x_1, \dots, x_d) = 0\}$$

has a finite intersection with the d -sphere, then any joint co-tile for F_1, \dots, F_d is periodic.

This raises the question: Is it true that for an independent d -tuple (F_1, \dots, F_d) in \mathbb{Z}^d the algebraic variety $V(P_{F_1}, \dots, P_{F_d})$ is finite?

We note that it can be shown that $V(P_{F_1}, \dots, P_{F_d})$ is finite if we impose the somewhat stronger condition that $(F_1 - F_1, \dots, F_d - F_d)$ is an independent d -tuple in \mathbb{Z}^d . This follows

from the equality of the tropical variety with the Bieri-Groves set of the variety (see Theorem 2.2.5 and Corollary 2.2.6 in [EKL06]), combined with [EKL06, Theorem 2.2.3] and an explicit direct computation. This connection was kindly explained to us by Ilya Tyomkin. This argument gives an alternative derivation of the conclusion of Theorem 1.2, under the slightly stronger assumption that $(F_1 - F_1, \dots, F_d - F_d)$ is an independent tuple of tiles in \mathbb{Z}^d .

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