

ON PROBLEMS OF DANZER AND GOWERS AND DYNAMICS ON THE SPACE OF CLOSED SUBSETS OF \mathbb{R}^d

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ABSTRACT. Considering the space of closed subsets of \mathbb{R}^d , endowed with the Chabauty-Fell topology, and the affine action of $\mathrm{SL}_d(\mathbb{R}) \times \mathbb{R}^d$, we prove that the only minimal subsystems are the fixed points $\{\emptyset\}$ and $\{\mathbb{R}^d\}$. As a consequence we resolve a question of Gowers concerning the existence of certain Danzer sets: there is no set $Y \subseteq \mathbb{R}^d$ such that for every convex set $C \subseteq \mathbb{R}^d$ of volume one, the cardinality of $C \cap Y$ is bounded above and below by nonzero constants independent of C . We also provide a short independent proof of this fact and deduce a quantitative consequence: for every ε -net N for convex sets in $[0, 1]^d$ there is a convex set of volume ε containing at least $\Omega(\log \log(1/\varepsilon))$ points of N .

1. INTRODUCTION

A set $Y \subseteq \mathbb{R}^d$ is called a *Danzer set* if there exists an $s > 0$ such that Y intersects every convex set of volume s . The following question is due to Danzer, and is open since the sixties, see [F], [CFG, p. 148], [GL, p. 288]: is there a Danzer set $Y \subseteq \mathbb{R}^d$ ($d \geq 2$) such that with a bounded upper density, i.e. with

$$\limsup_{R \rightarrow \infty} \frac{\#(B(0, R) \cap Y)}{R^d} < \infty?$$

Different authors have asked about variants of Danzer's problem. In [Go] Gowers asked whether there exists a set $Y \subseteq \mathbb{R}^d$, and $C > 0$, such that for every convex set K of volume 1 we have

$$1 \leq \#(K \cap Y) \leq C.$$

In this paper we answer Gowers' question negatively, namely:

Theorem 1.1. *Let $Y \subseteq \mathbb{R}^d$ be a Danzer set, then for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there is an ellipsoid E_n with $\mathrm{Vol}(E_n) < \varepsilon$ and $\#(E_n \cap Y) \geq n$.*

In fact we give two different proofs of Theorem 1.1. Our first proof is dynamical. We denote by \mathcal{X} the space of closed subsets of \mathbb{R}^d equipped with the Chabauty-Fell topology, and let $G \stackrel{\mathrm{def}}{=} \mathrm{SL}_d(\mathbb{R}) \times \mathbb{R}^d$ denote the group of affine

transformations of \mathbb{R}^d which preserve Lebesgue measure and orientation. The action of this group on \mathbb{R}^d induces a natural action on \mathcal{X} . Denote by

$$U_0 \stackrel{\text{def}}{=} \{u(\mathbf{a}) : \mathbf{a} \in \mathbb{R}^{d-1}\},$$

where

$$u(\mathbf{a}) \stackrel{\text{def}}{=} \begin{pmatrix} 1 & a_2 & a_3 & \cdots & a_d \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}, \quad \text{for } \mathbf{a} = (a_2, \dots, a_d), \quad (1.1)$$

and let $U = U_0 \times \mathbb{R}^d$. Also let

$$g_t \stackrel{\text{def}}{=} \text{diag}(e^{(d-1)t}, e^{-t}, \dots, e^{-t}), \quad (1.2)$$

let H_0 be the subgroup of $\text{SL}_d(\mathbb{R})$ generated by $\{g_t : t \in \mathbb{R}\}$ and U_0 , and let $H \stackrel{\text{def}}{=} H_0 \times \mathbb{R}^d$.

Our main result about this dynamical system is:

Theorem 1.2. *For every $F \in \mathcal{X}$, either $\emptyset \in \overline{H.F}$ or $\mathbb{R}^d \in \overline{U.F}$.*

As we will show, Theorem 1.1 is a straightforward consequence of Theorem 1.2. Theorem 1.2 also immediately yields a classification of the minimal subsystems of the dynamical system (\mathcal{X}, G) . Recall that a subset $\mathcal{Y} \subseteq \mathcal{X}$ is *minimal* if it is non-empty, closed, G -invariant, and minimal with respect to inclusion with these properties. Fixed points are obvious examples of minimal subsystems. The following is an immediate consequence of Theorem 1.2:

Corollary 1.3. *The only minimal subsystems of (\mathcal{X}, G) are the fixed points $\{\emptyset\}$ and $\{\mathbb{R}^d\}$.*

Note that in ergodic theory, a classification result for minimal subsystems is a topological analogue of a classification of invariant measures. Indeed the measure classification problem for (\mathcal{X}, G) has been raised by Marklof [Mar, §20], who has shown (in a series of works with Strömbergsson, see [Mar, MS] and references therein) that such a classification is of great interest for a variety of problems in mathematical physics. Besides the Dirac measures on the fixed points above, there are additional measures provided by the Poisson point process, and natural probability measures on spaces of grids and cut-and-project sets, embedded in \mathcal{X} .

We also provide an independent direct proof of Theorem 1.1. This second proof can be made quantitative, see Proposition 4.2. We deduce the following result about certain “ ε -nets” in “range spaces” (see [AS, §14.4] and [Mat, §10] for definitions and further reading about these notions):

Theorem 1.4. *For every $\varepsilon > 0$, if $N_\varepsilon \subseteq [0, 1]^d$ intersects every convex set of volume ε in $[0, 1]^d$, then there exist a convex set $K \subseteq [0, 1]^d$ of volume ε with $\#(N_\varepsilon \cap K) = \Omega(\log \log(1/\varepsilon))$.*

Additional results and open questions are discussed in §5. For more results related to Danzer’s questions and weaker formulations, we refer the reader to the papers [BW, SW, Bi, SS, PT].

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2. PRELIMINARIES

For $x \in \mathbb{R}^d$ and $r > 0$ we denote by $B(x, r)$ the ball of radius r centered at x with respect to the Euclidean norm $\|\cdot\|$, by $B_r \stackrel{\text{def}}{=} B(0, r)$, and by \bar{A} the closure of a set $A \subseteq \mathbb{R}^d$. For a set $F \subseteq \mathbb{R}^d$ denote its ε -neighborhood by $U_\varepsilon(F) \stackrel{\text{def}}{=} \bigcup \{B(x, \varepsilon) : x \in F\}$. The notation $\#S$ denotes the cardinality of a set S .

We will need the following theorem (see [Ba, Lecture 3], [J]).

Theorem 2.1 (John’s Theorem). *For every convex set $K \subseteq \mathbb{R}^d$ there exist ellipsoids $E_1 \subseteq K \subseteq E_2$ such that $\text{Vol}(E_2)/\text{Vol}(E_1) \leq C_d$, where C_d is a constant that depends only on d .*

Let \mathcal{X}, G be as in the introduction, and for $F_1, F_2 \in \mathcal{X}$ define

$$D(F_1, F_2) = \inf \left(\left\{ \varepsilon : \begin{array}{l} F_1 \cap B_{1/\varepsilon} \subseteq U_\varepsilon(F_2) \\ F_2 \cap B_{1/\varepsilon} \subseteq U_\varepsilon(F_1) \end{array} \right\} \cup \{1\} \right). \quad (2.1)$$

The following facts are standard, see e.g. [GS, H, LS]:

Proposition 2.2. • *D is a complete metric on \mathcal{X} .*

- *With this metric, \mathcal{X} is homeomorphic to the space of nonempty compact subsets of the one-point compactification $\mathbb{R}^d \cup \{\infty\}$, equipped with the Hausdorff metric, via the map $F \mapsto F \cup \{\infty\}$. In particular \mathcal{X} is compact.*
- *If (F_n) is a convergent sequence in \mathcal{X} then*

$$\lim_{n \rightarrow \infty} F_n = \{x \in \mathbb{R}^d : \exists x_n \in F_n \text{ such that } x_n \rightarrow_{n \rightarrow \infty} x\}.$$

The topology induced by this metric is also called the *Chabauty-Fell topology*. It will be the only topology we consider on \mathcal{X} and thus in the sequel we will omit reference to the metric D .

The standard affine action of G on \mathbb{R}^d induces a G -action on \mathcal{X} . The following lemma connects the properties considered by Danzer and Gowers, to the dynamical system (\mathcal{X}, G) .

Lemma 2.3. *For a set $Y \subseteq \mathbb{R}^d$ we have*

- (a) *Y is a Danzer set $\iff \emptyset \notin \overline{G.Y}$.*
- (b) *If there exist $\varepsilon, C > 0$ such that for every convex set K of volume ε we have $\#(K \cap Y) \leq C$, then $\mathbb{R}^d \notin \overline{G.Y}$.*

Proof. To prove (a), note that G acts transitively on the collections of ellipsoids with the same volume in \mathbb{R}^d , so using Theorem 2.1 we have the following:

$$\begin{aligned}
 Y \text{ is not Danzer} &\iff \text{for any } T \text{ there is an ellipsoid of } \text{Vol} \geq T \text{ disjoint from } Y \\
 &\iff \forall r > 0 \exists g \in G \text{ such that } g^{-1}.B_r \cap Y = \emptyset \\
 &\iff \forall r > 0 \exists g \in G \text{ such that } B_r \cap g.Y = \emptyset \\
 &\iff \emptyset \in \overline{G.Y}.
 \end{aligned}$$

Statement (b) follows from the fact that for sufficiently small δ (depending on ε and C), a set consisting of at most C points cannot be δ -dense in the ball of volume ε . \square

For $s > 0$, we say that $Y \subseteq \mathbb{R}^d$ is *Danzer with volume parameter s* if for any closed convex set K of volume s we have $Y \cap K \neq \emptyset$.

Lemma 2.4. *The set of Danzer sets with volume parameter s is closed in \mathcal{X} .*

Proof. Let (Y_n) be a sequence of Danzer sets with volume parameter s in \mathbb{R}^d such that $Y_n \xrightarrow{n \rightarrow \infty} Y$, and assume by way of contradiction that Y misses a closed convex set A of volume s . Since each Y_n is a Danzer set with parameter s , let $p_n \in A \cap Y_n$. Since A is compact, the sequence (p_n) has a subsequence (p_{n_i}) that converges to a point $p \in A$. This implies that $p \in Y = \lim_{i \rightarrow \infty} Y_{n_i}$, a contradiction. \square

Motivated by Lemmas 2.3 and 2.4 we make the following definitions. For a subgroup $G_0 \subset G$, and $Y \in \mathcal{X}$, we say that Y is *Danzer for G_0* if $\emptyset \notin \overline{G_0.Y}$. Also for $r > 0$, we say that Y is *Danzer for G_0 with parameter r* if for any $g_0 \in G_0$, $Y \cap g_0 \overline{B}_r \neq \emptyset$. The arguments used in the proofs of Lemmas 2.3 and 2.4 show:

Proposition 2.5. *Let G_0, r be as above. Then:*

- (i) *Y is Danzer for G_0 if and only if there is $r > 0$ such that Y is Danzer for G_0 with parameter r .*
- (ii) *The collection of $Y \in \mathcal{X}$ which are Danzer for G_0 is G_0 -invariant.*
- (iii) *The collection of $Y \in \mathcal{X}$ which are Danzer for G_0 with parameter r is closed and G_0 -invariant.*

3. PROOFS OF THE MAIN THEOREMS

We introduce the following notation. We denote by $\mathbf{x} = (x_1, \dots, x_d)$ a vector in \mathbb{R}^d and its coordinates, and by $\mathbf{e}_1, \dots, \mathbf{e}_d \in \mathbb{R}^d$ the standard basis vectors. For $1 \leq k \leq d$ we write

$$V_k \stackrel{\text{def}}{=} \text{span}(\mathbf{e}_i : 2 \leq i \leq k);$$

in particular $V_1 = \{0\}$. We let $P_k : \mathbb{R}^d \rightarrow V_k$ be the orthogonal projection onto V_k . The x_i -axis refers to the set $\text{span}(\mathbf{e}_i)$, and given $\mathbf{y} \in V_d$ by a *horizontal line through \mathbf{y}* we mean the affine line parallel to the x_1 -axis through \mathbf{y} , that is, the set $\{\mathbf{x} \in \mathbb{R}^d : (x_2, \dots, x_d) = \mathbf{y}\}$. Finally let U_0, H_0, H be the subgroups of $\text{SL}_d(\mathbb{R})$ defined in the introduction.

3.1. A dynamical proof. For the proof of Theorem 1.2 we first prove the following proposition.

Proposition 3.1. *For every $S \in \mathcal{X}$, if S is Danzer for H then there exists $Y \in \overline{U_0.S}$ such that Y contains the x_1 -axis.*

Proof. By Proposition 2.5(i), there is $r > 0$ such that S is Danzer for H with parameter r , and by Proposition 2.5(iii), the same is true for any Y in $\overline{H.S}$. By definition, for any $g \in H$, $S \cap g\overline{B_r} \neq \emptyset$. In particular, since $U_0 \subset H$, every $Y \in \overline{U_0.S}$ intersects every translate of $g_t\overline{B_r}$, where g_t is as in (1.2).

It suffices to show that for every $N \in \mathbb{N}$ and $\varepsilon > 0$ there exists a set $Y_{\varepsilon, N} \in \overline{U_0.S}$ such that

$$\{(n\varepsilon, 0, \dots, 0) : n \in \mathbb{Z}, |n| \leq N\} \subseteq Y_{\varepsilon, N}.$$

We fix ε and use induction on N . For $N = 0$, we need to find an element of $\overline{U_0.S} \subset \mathcal{X}$ which contains the origin. For $t > 0$, $g_t\overline{B_r}$ is a closed ellipsoid which is long in the x_1 direction and small in all the other coordinate directions. Therefore, given $\eta > 0$ we can choose $t > 0$ large enough so that the image of $g_t\overline{B_r}$ under the projection $P = P_d : \mathbb{R}^d \rightarrow V_d$ is contained in a ball of diameter less than η . There is therefore a translate B' of $g_t\overline{B_r}$ such that for any $\mathbf{x} \in B'$,

$$0 < \|P(\mathbf{x})\| < \eta. \tag{3.1}$$

Since S intersects every translate of $g_t\overline{B_r}$, there is a point $\mathbf{p} = \mathbf{p}_{0, \eta} \in S \cap B'$. The group U_0 in (1.1) acts on \mathbb{R}^d as follows:

$$u(\mathbf{a}).\mathbf{x} = \left(x_1 + \sum_{i=2}^d a_i x_i, x_2, \dots, x_d \right).$$

In particular it shears along horizontal lines, keeps the x_1 -axis fixed, satisfies $P(u.\mathbf{x}) = P(\mathbf{x})$, and if $P(\mathbf{x}) \neq 0$ then the x_1 -coordinate of $u(\mathbf{a}).\mathbf{x}$ can be made

arbitrary by using suitable \mathbf{a} . Thus we can find $u_\eta \in U_0$ such that the x_1 -coordinate of the point $u_\eta \cdot \mathbf{p}$ is 0, and such that $\|P(u_\eta \cdot \mathbf{p})\| < \eta$. Since η was arbitrary, and \mathcal{X} is compact, passing to a subsequence and taking a convergent subsequence we obtain a set $Y_{\varepsilon,0} \in \overline{U_0 \cdot S}$ that contains the origin.

The induction step is similar. Let $Y_{\varepsilon,N} \in \overline{U_0 \cdot S}$ be the set that is obtained from the induction hypothesis. For an arbitrary $\eta > 0$ choose t so that the diameter of $P(g_t \overline{B_r})$ is less than η . Let B' be a translate of $g_t \overline{B_r}$ so that (3.1) holds for any element of B' . Then $Y_{\varepsilon,N} \cap B' \neq \emptyset$. Let $\mathbf{p} \in Y_{\varepsilon,N} \cap B'$, then there is some $u_\eta \in U_0$ such that the x_1 -coordinate of the point $u_\eta \cdot \mathbf{p}$ is equal to $(N+1)\varepsilon$, and $\|P(u_\eta \cdot \mathbf{p})\| < \eta$. Letting $\eta \rightarrow 0$ and taking subsequences we find $Y'_{\varepsilon,N+1} \in \overline{U_0 \cdot Y_{\varepsilon,N}} \subseteq \overline{U_0 \cdot S}$ that contains the set

$$\{(n\varepsilon, 0) : n = -N, -(N-1), \dots, N, N+1\}.$$

Using the set $Y'_{\varepsilon,N+1}$ instead of $Y_{\varepsilon,N}$ and a similar argument we obtain the required set $Y_{\varepsilon,N+1}$. This completes the proof. \square

Proof of Theorem 1.2. We set $W_k \stackrel{\text{def}}{=} \text{span}(\mathbf{e}_1, \dots, \mathbf{e}_k)$, and prove the following claim by induction on k : for every $F \in \mathcal{X}$ which is Danzer for H , there exists a set $Z_k \in \overline{U_0 \times V_k \cdot F}$ that contains W_k . Note that here we have identified the subspaces V_k with subgroups of the group of translations \mathbb{R}^d .

Proposition 3.1 proves the case $k = 1$. Suppose the statement is valid for $k \geq 1$, and we prove its validity for $k+1$. Let $Z_k^{(0)} = Z_k \in \overline{U_0 \times V_k \cdot F}$ be the set obtained from the induction hypothesis, and let $\varepsilon > 0$. Let $S^{(1)} \stackrel{\text{def}}{=} Z_k^{(0)} + \varepsilon \mathbf{e}_{k+1}$, the set obtained by translating $Z_k^{(0)}$ by $\varepsilon \mathbf{e}_{k+1}$. By Proposition 2.5, $S^{(1)}$ is also Danzer for H . By the induction hypothesis there is a set

$$Z_k^{(1)} \in \overline{U_0 \times V_k \cdot S^{(1)}} \subset \overline{U_0 \times V_{k+1} \cdot F}$$

that contains W_k . Note that all the subspaces W_j and their translates are U_0 -invariant, the subspace W_k and its translates are V_k -invariant, and the action of V_k does not change the x_j -coordinates for $j > k$. Therefore $Z_k^{(1)}$ and any element in its orbit-closure under $U_0 \times V_{k+1}$, contains both W_k and its translate by $\varepsilon \mathbf{e}_{k+1}$. By repeating the above argument for every $\ell \in \mathbb{N}$ we obtain sets $Z_k^{(\ell)} \in \overline{U_0 \times V_{k+1} \cdot F}$ such that $Z_k^{(\ell)}$ contains all the k -dimensional hyperplanes $W_k + i\varepsilon \mathbf{e}_{k+1}$, for $0 \leq i \leq \ell$. Now set $Z_{k+1,\varepsilon} \in \overline{U_0 \times V_{k+1} \cdot F}$ to be a limit point of the sequence $(Z_k^{(2n)} - n\varepsilon \mathbf{e}_{k+1})$, then $Z_{k+1,\varepsilon}$ contains a collection of k -dimensional hyperplanes that is ε -dense in W_{k+1} . Taking $\varepsilon \rightarrow 0$ we obtain the required set $Z_{k+1} \in \overline{U \times V_k \cdot F}$. \square

Proof I of Theorem 1.1. Since Y is Danzer, by Lemma 2.3(a) we have $\emptyset \notin \overline{G \cdot Y}$ and in particular $\emptyset \notin \overline{U \cdot Y}$. By Theorem 1.2, $\mathbb{R}^d \in \overline{H \cdot Y}$, so by Lemma 2.3(b),

for all positive ε, C there is a convex subset of \mathbb{R}^d of volume ε containing more than C points of Y , and the statement follows via John's theorem. \square

3.2. A direct proof. Recall our notation that B_r is the Euclidean ball of radius r centered at the origin. A *closed centered ellipsoid* is the image of the closed unit ball $\overline{B_1}$ under a nonsingular linear map $\mathbb{R}^d \rightarrow \mathbb{R}^d$. Let β_d be the volume of a ball of radius 1 in \mathbb{R}^d . For $r > 0$ and a point $\mathbf{x} \notin B_r$ we define a closed centered ellipsoid $E(r, \mathbf{x})$ containing $B_r \cup \{\mathbf{x}\}$, as follows. If $\mathbf{x} = t\mathbf{e}_1$ for $t > r$, then $E(r, \mathbf{x})$ is the image of $\overline{B_1}$ under the linear transformation whose matrix is $\text{diag}(t/r, 1, \dots, 1)$. For general \mathbf{x} , let Θ be an orthogonal linear transformation with $\Theta(\mathbf{e}_1) = \mathbf{x}/\|\mathbf{x}\|$, and let $E(r, \mathbf{x}) = \Theta(E(r, \|\mathbf{x}\|\mathbf{e}_1))$. Clearly

$$\text{Vol}(E(r, \mathbf{x})) = \beta_d r^{d-1} \|\mathbf{x}\|.$$

Proof II of Theorem 1.1. Let $Y \subseteq \mathbb{R}^d$ be a Danzer set with volume parameter s , and by rescaling we may assume that s is the volume of the ball of diameter $1/2$. Assume with no loss of generality that

$$\varepsilon < \varepsilon_0, \quad \text{where } \varepsilon_0 = \frac{\beta_d}{2^{d-1}}. \quad (3.2)$$

The proof is by induction on n . The $n = 1$ case is true since $Y \neq \emptyset$, so we assume the validity of the statement for n , and prove it for $n + 1$. Let $E_n \subseteq \mathbb{R}^d$ be an ellipsoid with $\#(E_n \cap Y) \geq n$ and

$$\text{Vol}(E_n) < \varepsilon' = \beta_d^{-1/(d-1)} \varepsilon^{d/(d-1)} = \left(\frac{\varepsilon}{\beta_d}\right)^{d/d-1} \beta_d. \quad (3.3)$$

The group G acts transitively on closed ellipsoids of the same volume, so let $g \in G$ be such that $g.E_n$ is a ball centered at the origin, and denote its radius by r . Then $\#(g.E_n \cap g.Y) \geq n$ and the choice of ε' guarantees that

$$r^d \beta_d = \text{Vol}(g.E_n) = \text{Vol}(E_n) \stackrel{(3.3)}{<} \left(\frac{\varepsilon}{\beta_d}\right)^{d/d-1} \beta_d,$$

and hence $r^{d-1} \beta_d < \varepsilon$. From (3.2) we find that $r < 1/2$. Let $D \subseteq B_1$ be a ball of diameter $1/2$ that is disjoint from $g.E_n$. Since $g.Y$ is also a Danzer set with parameter s , D contains a point $\mathbf{p} \in g.Y$. Set $E'_{n+1} \stackrel{\text{def}}{=} E(r, \mathbf{p})$, and $E_{n+1} \stackrel{\text{def}}{=} g^{-1}.E'_{n+1}$. Since $\|\mathbf{p}\| < 1$, we have that $\text{Vol}(E'_{n+1}) \leq r^{d-1} \beta_d$. Thus

$$\text{Vol}(E_{n+1}) = \text{Vol}(E'_{n+1}) \leq r^{d-1} \beta_d < \varepsilon,$$

and

$$\#(E_{n+1} \cap Y) = \#(E'_{n+1} \cap g.Y) \geq n + 1.$$

\square

4. A FINITARY VERSION AND A QUANTITATIVE RESULT

Consider the following finitary version of Gowers' question.

Question 4.1. *Is there a constant $C > 0$ such that for every $\varepsilon > 0$ there exists a set $N_\varepsilon \subseteq [0, 1]^d$ such that for every convex set $K \subseteq [0, 1]^d$ of volume ε we have $1 \leq \#(N_\varepsilon \cap K) \leq C$?*

In this section we prove Theorem 1.4, which implies a negative answer to this question. We will need the following proposition.

Proposition 4.2. *Suppose that $Y \subseteq \mathbb{R}^d$ is a Danzer set with volume parameter s , then for every $n \in \mathbb{N}$ there exist a convex set K_n such that $\text{Vol}(K_n) = s$, $0 \in K_n$, $\#(Y \cap K_n) \geq n$, and*

$$\text{diam}(K_n) \leq C_{d,s} \cdot 4^{\frac{d^n}{(d-1)^{n-1}}}, \quad (4.1)$$

where $C_{d,s}$ is the diameter of a ball of volume s in \mathbb{R}^d .

Proof. We retain the notations as in §3, and repeat the idea of Proof II of Theorem 1.1 using only elements of the group $\text{SL}_d(\mathbb{R})$, and keeping track of the elements that are used. Since both sides of the inequality (4.1) scale by the same amount under dilations, for convenience we assume once more that s is the volume of a ball of diameter $1/2$. It follows that there is some $\mathbf{y}_1 \in Y$ with $\|\mathbf{y}_1\| \leq 1/2$. Set

$$\varepsilon_1^{-1} = 4^{\left(\frac{d}{d-1}\right)^{n-1}} \quad (4.2)$$

(the reason for this choice will become clear below). If $\|\mathbf{y}_1\| \leq \varepsilon_1$ let h_1 be the identity map, and otherwise let $h_1 \in \text{SL}_d(\mathbb{R})$ be the linear transformation which multiplies \mathbf{y}_1 by a scalar to have length ε_1 , and uniformly dilates the perpendicular subspace \mathbf{y}_1^\perp . Note that in both cases the operator norm of h_1^{-1} satisfies $\|h_1^{-1}\|_{\text{op}} \leq \varepsilon_1^{-1}$.

Now let $D_2 \subseteq B_1$ be a ball of diameter $1/2$ which is disjoint from B_{ε_1} . Note that $h_1.Y$ is also a Danzer set with parameter s , hence there is a point $\mathbf{y}_2 \in D_2 \cap h_1.Y$. Let $E_2 = E\left(\varepsilon_1, \frac{\mathbf{y}_2}{\|\mathbf{y}_2\|}\right)$, then $\mathbf{y}_2 \in E_2$ since $\|\mathbf{y}_2\| < 1$, and we have $\text{Vol}(E_2) = \varepsilon_1^{d-1} \beta_d$. Let $h_2 \in \text{SL}_d(\mathbb{R})$ be the element that maps E_2 to a ball, whose radius is $\varepsilon_2 \stackrel{\text{def}}{=} \varepsilon_1^{1-1/d}$. Note that $\|h_2^{-1}\|_{\text{op}} = \varepsilon_2^{-1}$, since h_2^{-1} maps a vector of length ε_2 to a unit vector and uniformly contracts the space orthogonal to this vector. Repeat this procedure to obtain the following data for a positive integer k :

- a ball $D_k \subseteq B_1$ of diameter $1/2$ which is disjoint from $B_{\varepsilon_{k-1}}$,
- a point $\mathbf{y}_k \in D_k \cap (h_{k-1} \cdots h_1.Y)$, with $\|\mathbf{y}_k\| < 1$,
- an ellipsoid $E_k = E\left(\varepsilon_{k-1}, \frac{\mathbf{y}_k}{\|\mathbf{y}_k\|}\right)$ with $\mathbf{y}_k \in E_k$, and $\text{Vol}(E_k) = \varepsilon_{k-1}^{d-1} \beta_d$,

- an element $h_k \in \mathrm{SL}_d(\mathbb{R})$ that maps E_k to a ball,
- and a number ε_k which is the radius of that ball, such that $\|h_k^{-1}\|_{\mathrm{op}} = \varepsilon_k^{-1}$.

We can repeat the procedure to go from step $k - 1$ to step k , as long as $2\varepsilon_{k-1} \leq 1/2$. For every k , since h_k is volume preserving, we have $\varepsilon_{k-1}^{d-1}\beta_d = \mathrm{Vol}(E_k) = \mathrm{Vol}(B_{\varepsilon_k}) = \varepsilon_k^d\beta_d$, and hence

$$\varepsilon_k = \varepsilon_{k-1}^{1-1/d}.$$

Therefore (ε_k) is an increasing sequence that approaches 1, and

$$\varepsilon_k = \varepsilon_1^{(1-1/d)^{k-1}}. \quad (4.3)$$

Since the operator norm is submultiplicative, we have from (4.3) that

$$\|h_1^{-1} \cdots h_k^{-1}\|_{\mathrm{op}} \leq \prod_{i=1}^k \varepsilon_i^{-1} = \varepsilon_1^{-m_k}, \quad (4.4)$$

where

$$m_k = \sum_{i=1}^k \left(1 - \frac{1}{d}\right)^{i-1} = d \left(1 - \left(1 - \frac{1}{d}\right)^k\right). \quad (4.5)$$

Given $n \in \mathbb{N}$, after n steps we obtain a number ε_n and matrices $h_1, \dots, h_n \in \mathrm{SL}_d(\mathbb{R})$ such that $h = h_n \cdots h_1$ satisfies $\#(h^{-1} \cdot \overline{B_{\varepsilon_n}} \cap Y) = \#\overline{B_{\varepsilon_n}} \cap h \cdot Y \geq n$. Set $K_n = h^{-1} \cdot \overline{B_{\varepsilon_n}}$. The choice of ε_1 in (4.2) and (4.3) ensure that

$$2\varepsilon_n = 1/2, \quad \mathrm{Vol}(K_n) = s \quad \text{and} \quad \mathrm{diam}(B_{\varepsilon_n}) = C_{d,s}.$$

Plugging into (4.4) and (4.5), we find

$$\|h^{-1}\|_{\mathrm{op}} \leq 4^{m_n} \left(\frac{d}{d-1}\right)^{n-1} \leq 4^{\frac{d^n}{(d-1)^{n-1}}}$$

This implies (4.1). □

By using $s = 1$ in Proposition 4.2, and noting that $C_{d,1} = 2\beta_d^{-1/d}$, we obtain the following:

Corollary 4.3. *Let*

$$\alpha_{d,n} \stackrel{\mathrm{def}}{=} 2\beta_d^{-1/d} \cdot 4^{\frac{d^n}{(d-1)^{n-1}}},$$

and let $Q_{d,n}$ be the closed cube of edge length $2\alpha_{d,n}$, centered at the origin.

For every $n \in \mathbb{N}$ and $d \geq 2$, if $Y \subseteq Q_{d,n}$ intersects every convex set of volume 1 in $Q_{d,n}$ then there is a convex set $K \subseteq Q_{d,n}$ of volume 1 so that $0 \in K$ and $\#(K \cap Y) \geq n$.

Proof of Theorem 1.4. Without loss of generality we prove the statement for $[-1/2, 1/2]^d$ instead of $[0, 1]^d$. Let $\varepsilon > 0$ and suppose that $N \subseteq [-1/2, 1/2]^d$ intersects every convex set of volume ε in $[-1/2, 1/2]^d$. Let φ be the linear transformation $\varphi(\mathbf{x}) = \varepsilon^{-1/d}\mathbf{x}$. Let $n \in \mathbb{N}$ such that

$$\alpha_{d,n} \leq \varepsilon^{-1/d}/2 < \alpha_{d,n+1}. \quad (4.6)$$

By the left-hand side of (4.6), $Q_{d,n} \subseteq \varphi([-1/2, 1/2]^d)$, and by the right-hand side, $n = \Omega(\log \log(1/\varepsilon))$. Observe that φ maps convex sets of volume ε to convex sets of volume 1, and thus $\varphi(N)$ intersects every convex set of volume 1 in $\varphi([-1/2, 1/2]^d)$, and in particular in $Q_{d,n}$. Hence the claim follows from Corollary 4.3. \square

5. QUESTIONS AND CONCLUDING REMARKS

5.1. Restricting to subgroups of G . The proof of Theorem 1.2 and Corollary 1.3 can be adapted to prove the following:

Theorem 5.1. *Let g_t be a one-parameter diagonal subgroup of $\mathrm{SL}_d(\mathbb{R})$, let U_0 be the expanding horospherical subgroup*

$$U_0 = \{h \in \mathrm{SL}_d(\mathbb{R}) : g_t h g_{-t} \rightarrow_{t \rightarrow -\infty} 1\},$$

let H_0 be the subgroup of $\mathrm{SL}_d(\mathbb{R})$ generated by $\{g_t\}$ and U_0 , and let $H = H_0 \ltimes \mathbb{R}^d$. Then for any $Y \in \mathcal{X}$, $\overline{H.Y}$ contains at least one of the fixed points \emptyset, \mathbb{R}^d .

Recall that $P \subset G$ is called *parabolic* if it is connected and G/P is compact. It is well-known that a parabolic subgroup contains a group H satisfying the conditions of Theorem 5.1. Thus we obtain:

Corollary 5.2. *For any parabolic subgroup P of G , the only minimal sets for the action of P on \mathcal{X} are the fixed points $\{\emptyset\}, \{\mathbb{R}^d\}$.*

Question 5.3. Which subgroups $H \subset G$ satisfy the conclusion of Corollary 5.2 in place of P ?

It is not hard to see that $H = \mathrm{SL}_d(\mathbb{R})$ does not satisfy the conclusion of Corollary 5.2, and in fact the collection of minimal sets in this case contains fixed points, closed orbits which are not fixed points, and minimal sets which are not closed orbits; for an example of the latter, consider the case $d = 2$, and the orbit-closure of $\{0\} \cup A_1 \cup A_2 \in \mathcal{X}$, where

$$A_1 \stackrel{\text{def}}{=} \{(e^m, 0) : m \in \mathbb{Z}\}, \quad A_2 \stackrel{\text{def}}{=} \{(e^{n\sqrt{2}}, 0) : n \in \mathbb{Z}\}.$$

Nevertheless, motivated by [Mar], we ask:

Question 5.4. Classify all minimal sets for the action of $\mathrm{SL}_d(\mathbb{R})$ on \mathcal{X} .

5.2. Other range spaces. Several weakenings of the Danzer question have been proposed, in which the collection (or “range space”) of convex subsets of \mathbb{R}^d is replaced with another collection. Recall that an *aligned box* is a set of the form $[a_1, b_1] \times \cdots \times [a_d, b_d]$, for some choice of $a_i < b_i$, $i = 1, \dots, d$. We note that the conclusion of Theorem 1.1 is not valid for aligned boxes:

Proposition 5.5. *For any d there is $Y \subset \mathbb{R}^d$ and $C > 0$ such that for any aligned box B of volume 1,*

$$1 \leq \#(B \cap Y) \leq C. \tag{5.1}$$

Proof. Let $Y \subset \mathbb{R}^d$ be a lattice arising from the geometric embedding of the ring of integers in a degree d totally real number field. This is a classical source of examples of lattices with interesting properties, see [GL]. In particular it is known that Y has a compact orbit in the space of grids (translated lattices) $G/G(\mathbb{Z})$, under the group $H_1 = A \times \mathbb{R}^d$, where A is the subgroup of diagonal matrices in $SL_d(\mathbb{R})$. The map from the space of grids to \mathcal{X} is continuous and G -equivariant, and therefore Y has a compact H_1 -orbit in \mathcal{X} as well. It is also shown in [SS] that Y is a Danzer set for aligned boxes, i.e. satisfies the left-hand side of (5.1). Suppose by contradiction that the right-hand side of (5.1) fails, that is, for each $n \geq 1$ there is an aligned boxes B_n with $\text{Vol}(B_n) = 1$ and $\#(Y \cap B_n) \geq n$. Since H_1 acts transitively on aligned boxes of volume 1, we can take $h_n \in H_1$ so that $h_n \cdot B_n = [-1/2, 1/2]^n$. Using the compactness of $H_1 \cdot Y$, we can pass to a subsequence and find that the sequence $\{h_n \cdot Y\}$ converges to a translate of a lattice. At the same time, $\#(h_n \cdot Y \cap [-1/2, 1/2]^n) \geq n$ for all n , so that the shortest non-zero vector in the lattice $h_n \cdot (Y - Y)$ has length tending to zero as $n \rightarrow \infty$. This is a contradiction. \square

Note that the group H_1 in the above proof admits a cocompact lattice while the group G does not. This motivates the following:

Question 5.6. For which collections \mathcal{R} of subsets of \mathbb{R}^d of volume 1, is it true that there is $Y \subset \mathbb{R}^d$ and $C > 0$ such that for every $B \in \mathcal{R}$, (5.1) holds? Suppose \mathcal{R} is acted upon transitively by a subgroup $H_{\mathcal{R}} \subset G$, and the collection satisfies the above property. Does it follow that $H_{\mathcal{R}}$ admits a cocompact lattice?

5.3. Improving Theorem 1.4. For fixed d and ε , let $f_d(\varepsilon)$ denote the smallest number k such that for every $N \subset [0, 1]^d$, such that N intersects every convex subset of $[0, 1]^d$ of volume ε , there is a convex $K \subset [0, 1]^d$ of volume ε such that $\#(K \cap Y) \geq k$. From Theorem 1.4 we know that $f_d(\varepsilon) = \Omega(\log \log(1/\varepsilon))$ (where the implicit constant may depend on d).

Question 5.7 (Nati Linial). Is it true that $f_d(\varepsilon) = \Omega(\log(1/\varepsilon))$?

5.4. Other forms of Danzer’s question. Gowers’ question is a weakening of Danzer’s in the sense that Gowers asked about the existence of a set satisfying more stringent conditions than Danzer does. We record a similar weakened version with a dynamical flavor. Recall that $Y \subset \mathbb{R}^d$ is called *uniformly discrete* if

$$\inf\{\|y_1 - y_2\| : y_1, y_2 \in Y, y_1 \neq y_2\} > 0.$$

Note that uniform discreteness implies bounded upper density.

Question 5.8 (Michael Boshernitzan). Is there a uniformly discrete Danzer set $Y \subset \mathbb{R}^d$?

A simple dynamical argument for the translation action of \mathbb{R}^d on \mathcal{X} implies that if the answer is positive, then there exists a uniformly discrete Danzer set which is also *repetitive*, i.e. for any $\varepsilon > 0$ there is $R > 0$ such that for every $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^d$ there is $\mathbf{x} \in B(\mathbf{y}_2, R)$ such that $D(Y - \mathbf{y}_1, Y - \mathbf{x}) < \varepsilon$. Here D is the metric defined in (2.1).

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