A Polynomial Approach to Linear Algebra

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Preface

Linear Algebra is a well entrenched mathematical subject that is taught in virtually every undergraduate program both in the sciences and engineering. Over the years many texts have been written on linear algebra, therefore it is up to the author to justify the presentation of another book in this area to the public.

I feel that my jusification for the writing of this book is based on a different choice of material and a different approach to the classical core of linear algebra. The main innovation in it is the emphasis placed on functional models and polynomial algebra as the best vehicle for the analysis of linear transformations and quadratic forms. In pursuing this innovation, a long lasting trend in mathematics is being reversed. Modern algebra went from the specific to the general, abstracting the underlying unifying concepts and structures. The epitomy of this trend was represented by the Bourbaki school. No doubt this was an important part in the development of modern mathematics, but it had its faults too. It led to several generations of students who could not compute nor could they give interesting examples of theorems they proved. Even worse, it increased the gap between pure mathematics and the general user of mathematics. It is the last group, which is made up of engineers and applied mathematicians, which is interested not only in understanding a problem but also in its computational aspects. A very similar development occured in functional analysis and operator theory. Initially the axiomatization of Banach and Hilbert spaces led to a search for general methods and results. While there were some significant successes in these direction, it soon became apparent, especially when trying to understand the structure of bounded operators, that one has to be much more specific. In particular, the introduction of functional models, through the work of Livsic, De Branges and Sz.-Nagy and Foias, provided a new approach to structure theory. It is these ideas that we have taken as our motivation in the writing of this book.

In the present book, at least where the structure theory is concerned, we look at a special class of shift operators. These are defined by using polynomial modular arithmetic. The interesting fact about this class is its property of universality, in the sense that every cyclic operator is similar to a shift and every linear operator on a finite dimensional vector space is similar to a direct sum of shifts. Thus the shifts are the building blocks of an arbitrary linear operator.

Basically, the approach taken in this book is a variation on the study of a linear transformation via the study of the module structure induced by it over the ring of polynomials. While module theory provides great elegance, it is also difficult to grasp by students. Furthermore, it seems too far removed from computation. Matrix theory seems to be at the other extreme, it is too much concerned with computation and not enough with structure. Functional models, especially the polynomial models, lie on an intermediate level of absraction between module theory and matrix theory.

The book includes specific chapters devoted to quadratic forms and the establishments of algebraic stability criteria. The emphasis is shared between the general theory and the specific examples, which are in this case the study of the Hankel and Bezout forms. This general area, via the work of Hermite, is one of the roots of the theory of Hilbert spaces. I feel that it is most illuminating to see the Euclidean algorithm and the associated Bezout identity not as isolated results, but as an extremely effective tool in the development of fast inversion algorithms for structured matrices.

Another innovation in this book is the inclusion of basic system theoretic ideas. It is my conviction that it is no longer possible to separate in a natural way the study of linear algebra from the study of linear systems. The two topics have benefitted greatly from cross fertilization. In particular the theory of finite dimensional linear systems seems to provide an unending flow of problems, ideas and concepts which are quickly assimilated in linear algebra. Realization theory is as much a part of linear algebra as is the long familiar companion matrix.

The inclusion of a whole chapter on Hankel norm approximation theory, or AAK theory as it is commonly known, is also a new addition as far as linear algebra books are concerned. This part requires very little mathematical knowledge not covered in the book, but a certain mathematical maturity is assumed. In believe it is very much within the grasp of a well motivated undergraduate. In this part several results from early chapters are reconstructed in a context where stability is central. Thus the rational Hardy spaces enter, and we have analytic models and shifts. Lagrange and Hermite interpolation are replaced by Nevanlinna-Pick interpolation. Finally coprimeness and the Bezout identity reappear, but over a different ring. I believe the study of these analogies goes a long way towards demonsrating to the student the underlying unity of mathematics.

Let me explain the philosophy which underlies the writing of this book. In a way I share the aim of Halmos [1958] in trying to treat linear transformations on finite dimensional vector spaces by methods of more general theories. These theories were functional analysis and operator theory in Hilbert space. This is still the case in this book. However, in the intervening years operator theory has changed remarkably. The emphasis has moved from the study of selfadjoint and normal operators to the study of nonselfadjoint operators. The hope that a general structure theory for linear operators might be developed seems to be too naive. The methods utilizing Riesz-Dunford integrals proved to be too restrictive. On the other hand, a whole new area centering around the theory of invariant subspaces and the construction and study of functional models, was developed. This new development had its roots not only in pure mathematics but also in many applied areas, notably scattering, network, control theories, and some areas of stochastic processes as estimation and prediction theories.

I hope that this book will show how linear algebra is related to other, more advanced, areas of mathematics. Polynomial models have their root in operator theory, especially that part of operator theory which centered around invariant subspace theory and Hardy spaces. Thus the point of view adopted here provides a natural link with that area of mathematics, as well as those application areas I have already mentioned.

In writing this book I chose to work almost exclusively with scalar polynomials. The one exception in this project being the invariant factor algorithm and its application to structure theory. My choice was influenced by the desire to have the book accessible to most undergraduates. Virtually all results about scalar polynomial models have polynomial matrix generalizations, and some of the appropriate references are pointed out in the notes and remarks sections.

The exercises at the end of chapters have been partly chosen to indicate directions not covered in the book. I have refrained from including routine computational problems. This does not indicate a negative attitude towards computation. Quite to the contrary, I am a great believer in the exercise of computation and I suggest that readers choose, and work out, their own problems. This is the best way to get a better grasp of the presented material.

I usually use the first seven chapters for a one year course on linear algebra at BGU. If the group is a bit more advanced, one can supplement this by more material on qudratic forms. The material on qudratic forms and stability can be used as a one semester course of special topics in linear algebra. Also the material on linear systems and Hankel norm approximations can be used as a basis for either a one term course or a seminar.

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