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A study of (A, B)-invariant subspaces via polynomial models

PAUL A. FUHRMANN† and J. C. WILLEMS†

This paper describes an application of the theory of polynomial models to the study of some natural objects in geometric control theory. In particular, it utilizes the correspondence between factorization of polynomial matrices and invariant subspaces to obtain, by the use of Toeplitz operators, a polynomial characterization of (A, B)-invariant subspaces as well as those included in ker C. A geometric characterization of feedback irreducibility is rederived.

1. Introduction

Over the last decade work in system theory was fragmented between groups following a variety of approaches. Among the various techniques currently in use one finds classical transfer function techniques (Wonham 1974), polynomial system matrices (Rosenbrock 1970), an algebraic theory based on modules (Kalman *et al.* 1969) and state space theory including geometric control theory (Wonham 1974). To make communication feasible one needs an easy way of translating results from one context to the other. So far what seems to be the best unifying tool is the theory of polynomial models developed by one of the authors. In Fuhrmann (1976) state space notions, coprime factorizations of rational functions were related and a short route to realization theory described. This has been pushed further in Fuhrmann (1977) with the results of showing a natural connection with the theory of polynomial system matrices developed by Rosenbrock (1970). Further results along these lines were done in Fuhrmann (1978) which introduced also models of rational functions and Fuhrmann (1979) where feedback was studied by use of Toeplitz operators. The time seemed ripe for further progress towards understanding geometric control theory in terms of polynomial methods. Initial results were obtained by Emre (1978), Emre and Hautus (1978) and Münzner and Prätzel-Wolters (1978, 1979). This paper tackles essentially the same problems as Emre and Hautus (1978) but with somewhat different techniques and more complete results giving a closer relation of the geometric concepts to problems of factorization of polynomial matrices. Some of the results were first presented in Fuhrmann (1979).

2. Preliminaries

We collect in this section some basic information about functional, polynomial and rational, models for linear transformations and linear systems. The whole development will be over an arbitrary field F.

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Define the $F^n((\lambda^{-1}))$ to be the set of all truncated Laurent series with coefficients in F^n, that is the set of all formal series of the form

$$f(\lambda)=\sum_{-\infty<j\leq k} f_j\lambda^j, \quad f_j\in F^n, \quad k\in Z \tag{2.1}$$

$F^n((\lambda^{-1}))$ contains two important subsets, namely $F^n[\lambda]$ the set of all vector polynomials with coefficients in F^n and $\lambda^{-1}F^n[[\lambda^{-1}]]$ the set of all formal power series in λ^{-1} with coefficients in F^n and vanishing constant term. We define two projections π_+ and π_- acting in $F^n(\lambda^{-1}))$, suppressing in the notation the dependence on n, by

$$\pi_+ \sum_{-\infty<j\leq k} f_j\lambda^j = \sum_{j=0}^{k} f_j\lambda^j \tag{2.2}$$

and

$$\pi_- \sum_{-\infty<j\leq k} f_j\lambda^j = \sum_{j=-\infty}^{-1} f_j\lambda^j \tag{2.3}$$

The ranges of the two projections coincide with $F^n[\lambda]$ and $\lambda^{-1}F^n[[\lambda^{-1}]]$ and in view of the direct sum representation

$$F^n((\lambda^{-1}))=F^n[\lambda]\oplus\lambda^{-1}F^n[[\lambda^{-1}]] \tag{2.4}$$

we obtain

$$I=\pi_+ + \pi_- \tag{2.5}$$

$F^n((\lambda^{-1}))$ is a module over $F[\lambda]$ and $F^n[\lambda]$ a submodule. Since there is a natural isomorphism between $F^n((\lambda^{-1}))/F^n[\lambda]$ and $\lambda^{-1}F^n[[\lambda^{-1}]]$ the later has a natural induced $F[\lambda]$-module structure given by

$$p\cdot y=\pi_-(py), \quad p\in F(\lambda), \quad y\in\lambda^{-1}F^n[[\lambda^{-1}]] \tag{2.6}$$

We introduce the shift operators in $F^n((\lambda^{-1}))$, $F^n[\lambda]$ and $\lambda^{-1}F^n[[\lambda^{-1}]]$ and denote them by S, S_+ and S_- respectively. The shift operators are the representation of the identity polynomial in the various modules. Thus

$$(Sf)(\lambda)=\lambda f(\lambda) \quad \text{for } f\in F^n((\lambda^{-1})) \tag{2.7}$$

$$S_+f=Sf \quad \text{for } f\in F^n[\lambda], \quad \text{i.e. } S_+=S|F^n[\lambda] \tag{2.8}$$

Finally $S_-: \lambda^{-1}F^n[[\lambda^{-1}]]\to\lambda^{-1}F^n[[\lambda^{-1}]]$ is defined by

$$S_-g=\pi_-Sg, \quad \forall g\in\lambda^{-1}F^n[[\lambda^{-1}]] \tag{2.9}$$

We recall that given any linear transformation A in a finite dimensional vector space V over F then there is an induced $F[\lambda]$-module structure in V given by

$$p\cdot v=p(A)v \quad \text{for } p\in F[\lambda], \quad v\in V \tag{2.10}$$

With this definition V becomes a finitely generated torsion module over $F[\lambda]$. Thus from our point of view it is of interest to study the set of finitely generated torsion modules which can be derived from $F^n[\lambda]$ or $\lambda^{-1}F^n[[\lambda^{-1}]]$. Since $F^n[\lambda]$ is a free $F[\lambda]$-module all its submodules are also free [14], but certain quotient modules are torsion modules. In the case of $\lambda^{-1}F^n[[\lambda^{-1}]]$

certain of its submodules are finitely generated torsion modules and we will give a characterization of the two. To this end we recall the following.

Theorem 2.1 (Fuhrmann 1976)

(i) A subset M of $F^n[\lambda]$ is a submodule of $F^n[\lambda]$ if and only if for some polynomial matrix D, or matrix polynomial as we do not distinguish between the two, in $F^{n\times n}[\lambda]$ we have $M = DF^n[\lambda]$.

Since $F^n[\lambda]$ is finitely generated so is any of its quotient modules.

Theorem 2.2 (Fuhrmann 1976)

The quotient module $F^n[\lambda]/DF^n[\lambda]$ is a finitely generated torsion module if and only if D is non-singular.

The inclusion relation between *full submodules*, that is submodules of $F^n[\lambda]$ that correspond to non-singular D, is reflected in a division relation between the corresponding polynomial matrices.

Theorem 2.3

(i) Let $M_i = D_i F^{n\times n}[\lambda]$ be two full submodules. Then $M_2 \subset M_1$ if and only if

$$D_2 = D_1 E_1 \tag{2.11}$$

for some $E_1 \in F^{n\times n}[\lambda]$.

(ii) Two full submodules are equal, i.e. $D_1 F^{n\times n}[\lambda] = D_2 F^{n\times n}[\lambda]$ if and only if

$$D_2 = D_1 U \tag{2.12}$$

for a unimodular $U \in F^{n\times n}[\lambda]$.

From now we will assume D is non-singular. We define a map $\pi_D : F^n[\lambda] \to F^n[\lambda]$ by

$$\pi_D f = D\pi_- D^{-1} f, \quad \forall f \in F^n[\lambda] \tag{2.13}$$

Theorem 2.4 (Fuhrmann 1976)

π_D defined by (2.13) is a projection and $\ker \pi_D = DF^n[\lambda]$.

Let $K_D = \text{range } \pi_D$ and, as K_D is isomorphic to $F^n[\lambda]/DF^n[\lambda]$, we can give K_D an induced module structure by defining

$$p \cdot f = \pi_D(pf) \quad \text{for } p \in F[\lambda], \quad f \in K_D \tag{2.14}$$

We define the restriction of the shift to K_D, compression of the shift may be a better terminology, by $S_D : K_D \to K_D$ where

$$S_D f = \pi_D \lambda \cdot f = \pi_D S_+ f, \quad \forall f \in K_D \tag{2.15}$$

We have the following characterizations of K_D

Theorem 2.5

(i) $K_D = \{f \in F^n[\lambda] \mid D^{-1} f \in \lambda^{-1} F^n[[\lambda^{-1}]]\}$

(ii) $K_D = \{f \in F^n[\lambda] \mid f = Dh,\ h \in \lambda^{-1} F^n[[\lambda^{-1}]]\}$.

While the first characterization requires D to be non-singular the second one does not. This allows one to define (Emre and Hautus 1978) for any polynomial matrix $U \in F^{n \times m}[\lambda]$

$$K_U = \{f \in F^n[\lambda] \mid f = Uh,\ h \in \lambda^{-1} F^m[[\lambda^{-1}]]\} \tag{2.16}$$

Since for every $h \in \lambda^{-1} F^n[[\lambda^{-1}]]$ we have $\lambda \cdot h = \pi_+ \lambda \cdot h + \pi_- \lambda \cdot h = \pi_+ \lambda \cdot h + S_- h$ it is clear that $\pi_+ \lambda \cdot h = \xi \in F^n$. Thus the next lemma follows from the definition of S_D.

Lemma 2.6

For each $f \in K_D$ there exists a unique vector $\xi \in F^n$ s.t.

$$(S_D f)(\lambda) = \lambda \cdot f(\lambda) - D(\lambda)\xi \tag{2.17}$$

If we take as our setting $\lambda^{-1} F^n[[\lambda^{-1}]]$ we can obtain similar related results. Given a non-singular polynomial matrix $D \in F^{n \times n}[\lambda]$ we define the map $\pi^D :\ \lambda^{-1} F^n[[\lambda^{-1}]] \to \lambda^{-1} F^n[[\lambda^{-1}]]$ by

$$\pi^D g = \pi_- D^{-1} \pi + Dg, \quad \forall g \in \lambda^{-1} F^n[[\lambda^{-1}]] \tag{2.18}$$

Theorem 2.7 (Fuhrmann 1978)

π^D defined by (2.18) is a projection map in $\lambda^{-1} F^n[[\lambda^{-1}]]$ and $L_D = \text{range } \pi^D$ is a finitely generated torsion submodule of $\lambda^{-1} F^n[[\lambda^{-1}]]$. Conversely if $L \subset \lambda^{-1} F^n[[\lambda^{-1}]]$ is a finitely generated torsion submodule then $L = L_D$ for some non-singular $D \in F^{n \times n}[\lambda]$.

Since L_D is a submodule of $\lambda^{-1} F^n[[\lambda^{-1}]]$ we can define the restricted shift in L_D, denoted by S_D, which is defined by

$$S^D g = S_- g = \pi_- \lambda \cdot g, \quad \forall g \in L_D \tag{2.19}$$

or equivalently

$$S^D = S_- | L_D \tag{2.20}$$

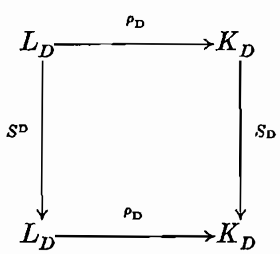
K_D and L_D are isomorphic modules. Define a map $\rho_D :\ L_D \to K_D$ by

$$\rho_D g = Dg \tag{2.21}$$

then its inverse $\rho_D^{-1} :\ K_D \to L_D$ is given by

$$\rho_D^{-1} f = \pi_- D^{-1} f \tag{2.22}$$

In terms of ρ_D we have the following commutative diagram which exhibits the isomorphism of K_D and L_D.



(2.23)

K_D and L_D, with the actions of S_D and S^D respectively, are the *functional models associated with D*. The first one is a *polynomial model* whereas the latter is a *rational model*. In spite of the isomorphism of the two classes of models it is extremely important to carry along both of them as in some cases the use of one is easier and both will be needed for the final classification of questions of duality.

Given two polynomial matrices $D \in F^{n\times n}[\lambda]$ and $D_1 \in F^{m\times m}[\lambda]$ we want to characterize the similarity of S_D and S_{D_1}. This amounts to the isomorphism of K_D and K_{D_1}. Similarly S^D and S^{D_1} are similar if and only if L_D and L_{D_1} are isomorphic as $F[\lambda]$-modules. The following theorem sums up the situation.

Theorem 2.8 (Fuhrmann 1976)

(i) A map $X: K_D \to K_{D_1}$ is an $F[\lambda]$-module homomorphism if and only if it is of the form

$$Xf = \pi_{D_1}\Xi f, \quad \forall f \in K_D \tag{2.24}$$

where Ξ and Ξ_1 are polynomial matrices in $F^{m\times n}[\lambda]$ satisfying

$$\Xi D = D_1 \Xi_1 \tag{2.25}$$

X is injective if and only if D and Ξ_1 are right coprime and surjective if and only if Ξ and D_1 are left coprime.

(ii) *A* map $Y: L_D \to L_{D_1}$ is an $F[\lambda]$-module homomorphism if and only if it is of the form

$$Yg = \pi_{-}\Xi_1 g, \quad \forall g \in L_D \tag{2.26}$$

where Ξ and Ξ_1 are as before.

Y is injective if and only if D and Ξ_1 are right coprime and surjective if and only if D_1 and Ξ are left coprime.

An extremely useful property of polynomial models is the correspondence between invariant subspaces and factorization of polynomial matrices.

Theorem 2.9

(i) A subset M of K_D is a submodule, or equivalently an S_D-invariant subspace, if and only if

$$M = D_1 K_{D_2} \tag{2.27}$$

for some factorization

$$D = D_1 D_2 \tag{2.28}$$

with $D_i \in F^{n\times n}[\lambda]$.

(ii) A subset N of L_D is a submodule, or equivalently an S^D-invariant subspace, if and only if

$$N = L_{D_2} \tag{2.29}$$

for some factorization (2.28).

Proof

(i) Let $D=D_1D_2$ and $M=D_1K_{D_2}$. Clearly $M\subset K_D$. If $f\in M$ then $f=D_1g$ with $g\in K_{D_2}$. We compute

$$S_Df=\pi_D\lambda\cdot f=\pi_D\lambda\cdot D_1g=D_1D_2\pi_-D_2^{-1}\,D_1^{-1}\,\lambda\cdot D_1g=D_1\pi_{D_2}\lambda\cdot g$$
$$=D_1S_{D_2}g\in D_1K_{D_2}$$

Thus M is S_D invariant. In fact we proved that $D_1K_{D_2}$ and K_{D_2} are isomorphic and $S_D|D_1K_{D_2}=D_1S_{D_2}D_1^{-1}$.

Conversely assume $M\subset K_D$ is S_D-invariant. Clearly $M+DF^n[\lambda]$ is a submodule of $F^n[\lambda]$. Hence $M+DF^n[\lambda]=D_1F^n[\lambda]$ for some $D_1\in F^{n\times n}[\lambda]$. Since $DF^n[\lambda]\subset D_1F^n[\lambda]$ we have, Theorem 2.3, that $D=D_1D_2$ for some $D_2\in F^{n\times n}[\lambda]$. Thus

$$DF^n[\lambda]=D_1D_2F^n[\lambda]=D_1D_2(K_{D_2}\oplus D_2F^n[\lambda])=D_1K_{D_2}\oplus DF^n[\lambda]$$

From this it follows that $M=D_1K_{D_2}$.

Part (ii) follows from (i) by application of the isomorphism between K_D and L_D.

Under certain extra conditions we can get a more complete decomposition of K_D.

Theorem 2.10

Given a factorization of a non-singular $D\in F^{n\times n}[\lambda]$ of the form

$$D=D_1D_2 \tag{2.30}$$

with $D_i\in F^{n\times n}[\lambda]$ and $D_2(\lambda)^{-1}$ proper then

$$K_D=K_{D_1}\oplus D_1K_{D_2} \tag{2.31}$$

The following lemma, which partially generalizes the previous theorem to the non-square case will be useful later on.

Lemma 2.11

Let $U\in F^{n\times m}[\lambda]$ for every $E\in F^{p\times n}[\lambda]$ we have

$$EK_U\subset K_{EU} \tag{2.32}$$

If E is left invertible, i.e. there exists an $E_1\in F^{n\times p}[\lambda]$ such that $E_1(\lambda)E(\lambda)=F$, then

$$EK_U=K_{EU} \tag{2.33}$$

Proof

If $h\in\lambda^{-1}F^m[[\lambda^{-1}]]$ is such that $f=Uh$ then $f=EUh\in K_{EU}$ which proves (2.32). If $E_1E=I$ and $f\in K_{EU}$ then $f=EUh$ which implies that $Uh=E_1f$ is a vector polynomial. Thus $f=E(Uh)\in EK_U$.

The simplest submodules of K_D are the 1-dimensional. A 1-dimensional submodule is an S_D-invariant subspace generated by an eigenfunction of S_D. The next lemma characterizes these.

Lemma 2.12

A polynomial vector $f \in K_D$ is an eigenfunction of S_D, corresponding to an eigenvalue $\alpha \in F$, if and only if

$$f(\lambda) = \frac{D(\lambda)\xi}{\lambda - \alpha} \tag{2.34}$$

for the same vector $\xi \in F^n$ that satisfies $D(\alpha)\xi = 0$.

Proof

$f \in K_D$ is an eigenfunction if and only if $(S_D - \alpha I)f = 0$. Since, by Lemma 2.6, $(S_D f)(\lambda) = \lambda f(\lambda) - D(\lambda)\xi$ for some ξ we have $\lambda f(\lambda) - D(\lambda)\xi - \alpha f(\lambda) = 0$ or $f(\lambda) = D(\lambda)\xi/(\lambda - \alpha)$. Since f is a polynomial we must have $D(\alpha)\xi = 0$. This condition is clearly sufficient, for then $D(\lambda)\xi/(\lambda - \alpha) \in K_D$ as $D^{-1}D\xi/(\lambda - \alpha) = \xi(\lambda - \alpha) \in \lambda^{-1}F^n[[\lambda^{-1}]]$ and

$$(S_D - \alpha I)\,\frac{D\xi}{\lambda - \alpha} = \pi_D(\lambda - \alpha) \cdot \frac{D(\lambda)\xi}{\lambda - \alpha} = \pi_D D\xi = 0$$

Similarly all eigenfunctions of S_- are given by $g(\lambda) = \xi/(\lambda - \alpha)$ for same $\xi \in F^n$ and $\alpha \in F$.

To see the relation between this lemma and Theorem 2.9 let

$$f(\lambda) = \frac{D(\lambda)\xi}{\lambda - \alpha}$$

and choose a basis in F^n for which ξ is the first element. Let $(d_{ij}(\lambda))$ be the matrix representation of $D(\lambda)$. Since $D(\alpha)\xi = 0$ it follows that $d_{i1}(\alpha) = 0$, $i = 1, \ldots, n$. But then the diagonal matrix

$$\begin{pmatrix} \lambda - \alpha & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

is clearly a right factor of D.

In the analysis of linear transformations in finite dimensional vector spaces it is of interest to decompose the space into a direct sum of invariant subspaces.

This analysis can be carried out effectively in the case of polynomial models. The following result, related to the resultant theorem, extends those of Fuhrmann (1976).

Theorem 2.13

Let $D = D_1E_1 = D_2E_2$ be two factorizations of $D \in F^{n \times n}[\lambda]$ with $D_i E_i \in F^{n \times n}[\lambda]$.

(i) We have the representation

$$K_D = D_1 K_{E_1} + D_2 K_{E_2} \tag{2.35}$$

if and only if D_1 and D_2 are left coprime.

(ii) The representation (2.35) is a direct sum representation, i.e.

$$D_1K_{E_1}\cap D_2K_{E_2}=\{0\} \tag{2.36}$$

if and only if E_1 and E_2 are right coprime.

This result can be immediately dualized.

Theorem 2.14

Let $D=D_1E_1=D_2E_2$ be two factorizations with D, D_i, $E_i\in F^{n\times n}[\lambda]$. Then

$$L_D=L_{E_1}+L_{E_2} \tag{2.37}$$

if and only if D_1 and D_2 are left coprime, and

$$L_{E_1}\cap L_{E_2}=\{0\} \tag{2.38}$$

if and only if E_1 and E_2 are right coprime.

Corollary 2.15

Let $M_i=D_iK_{E_i}$, $i=1, \ldots, k$, be submodules of K_D. Let $M=M_1+\ \ldots\ +M_k$, and $M_0=\cap M_i$ then $M=DK_E$ and $M_0=D_0K_{E_0}$ where

(i) D is the g.c.l.d. of the D_i, equivalently E is the l.c.l.m. of the E_i, and
(ii) D_0 is the l.c.r.m. of the D_i, equivalently E_0 is the g.c.r.d. of the E_i.

We note that $L_{E_1}+\ \ldots\ +L_{E_k}=L_E$ and $\cap L_{E_i}=L_{E_0}$.

If D_1, $D_2\in F^{n\times n}[\lambda]$ then they may be left coprime or right coprime with $d_1=\det D_1$ and $D_2=\det D_2$ having a common factor. However, if $d_1\wedge d_2=1$, i.e. if the g.c.d. of d_1 and d_2 is one, then D_1 and D_2 are necessarily left coprime.

However if $D\in F^{n\times n}[\lambda]$ and $d=\det D$ has a factorization $d=d_1d_2$ into coprime factors then this factorization induces related factorizations of D into spectrally different factors.

Theorem 2.16

Let $D\in F^{n\times n}[\lambda]$, $d(\lambda)=\det D(\lambda)$ and let d have a factorization

$$d=d_1\cdot d_2 \tag{2.39}$$

into coprime factors. Then D admits two factorizations

$$D=D_1E_1=D_2E_2 \tag{2.40}$$

with $\det E_i=d_i$ (and hence with $\det D_1=d_2$ and $\det D_2=d_1$). As a consequence we have

$$K_D=D_1K_{E_1}\oplus D_2K_{E_2} \tag{2.41}$$

Proof

Define $M_i=\{f\in K_D|\pi_Dd_if=0\}$. Clearly M_i are submodules of K_D and the coprimeness condition on d_1 and d_2 implies that $M_1\cap M_2=\{0\}$. Since clearly $K_D=M_1+M_2$ it follows that $K_D=M_1\oplus M_2$. Now M_i have, by Theorem 2.9, the representations $M_i=D_iK_{E_i}$ and we obtain (2.41).

Since $S_D|D_iK_{E_i}$ has, by Theorem 2.9 the characteristic polynomial det $E_i=e_i$ and this by construction divides d_i. Since $d=d_1d_2$ and on the other hand from the representations (2.40) $d=e_1e_2$ it follows necessarily that $e_i=d_i$ up to a constant factor.

We conclude this section by relating factorizations of transfer functions and realization theory. If $G(\lambda)$ is a $p\times m$ strictly proper rational matrix function then there are associated with it two coprime factorizations (Wolovich 1974, Rosenbrock 1970, Fuhrmann 1976)

$$G(\lambda)=N(\lambda)D(\lambda)^{-1}=T(\lambda)^{-1}U(\lambda) \tag{2.42}$$

where $D\in F^{m\times m}[\lambda]$, $T\in F^{p\times p}[\lambda]$ are non-singular and N, $U\in F^{p\times m}[\lambda]$. These factorizations are unique up to right unimodular factor in one and a left unimodular factor in the other. It is useful however not to restrict the generality and assume a factorization of G of the form

$$G(\lambda)=N(\lambda)D(\lambda)^{-1}M(\lambda)+P(\lambda) \tag{2.43}$$

with no coprimeness conditions imposed. We may however assume without loss of generality that $D^{-1}M$ is strictly proper. We associate now with the factorization (2.43) a realization by the following procedure. We take K_D as a state space and define a triple (A, B, C) by

$$A=S_D \tag{2.44}$$

$$B\xi=\pi_D M\xi \quad \text{for } \xi\in p^m \tag{2.45}$$

and

$$Cf=(ND^{-1}f)_{-1}=\pi_f(\lambda ND^{-1}f) \tag{2.46}$$

for all $f\in K_D$.

It has been proved in Fuhrmann (1977) that this is indeed a realizaton of G which is reachable if and only if M and D are left coprime and observable if and only N and D are right coprime. We call this the realization associated with the factorization (2.43).

3. Toeplitz operators and feedback

It is generally known what important role the Kronecker indices, i.e. the reachability and observability indices, play in system theoretic problems in particular those related to feedback invariance, pole shifting and canonical forms (Rosenbrock 1970). Preceding the study of these problems certain factorizations and factorization indices have been for a long time the object of study in the mathematical literature, mainly in the study of Wiener–Hopf systems of equations, Toeplitz operators, etc. (Gohberg and Krein 1960, Gohberg and Feldman 1971). Lately there has been some effort in clarifying some of the connections between the various theories.

In Fuhrmann (1979) the role of Toeplitz operators in the study of feedback equivalence has been elucidated whereas Fuhrmann and Willems (1979) is an attempt to clarify the relations between Wiener–Hopf factorizations of rational functions, coprime factorizations and the connection between factorization indices on the one hand and Kronecker indices on the other. Other recent work in this direction is Gohberg *et al.* (1978).

Here we establish the facts relevant to this paper.

Given $A \in F^{p\times m}((\lambda^{-1}))$ then the *Toeplitz operator* induced by A, which will be denoted by T_A, is the map $T_A: F^m[\lambda] \to F^p[\lambda]$ defined by

$$T_A f = \pi_+ A f \quad \text{for all } f \in F^m[\lambda] \tag{3.1}$$

A Toeplitz operator T_A is called *rational* if A is rational, *causal* if $A \in F^{p\times m}[[\lambda^{-1}]]$ and *strictly causal* if A is an invertible element of the ring $F^{m\times m}[[\lambda^{-1}]]$, in which case it is called a *bicausal isomorphism*. If $A(\lambda) = \sum_{j=0}^{\infty} A_j \lambda^j$ then A is a bicausal isomorphism if and only if A_0 is an invertible element of $F^{m\times m}$.

Let $A \in F^{p\times m}((\lambda^{-1}))$. We say a factorization of the form

$$A(\lambda) = \Gamma(\lambda) D(\lambda) U(\lambda) \tag{3.2}$$

is a *left Wiener–Hopf factorization at infinity* of A if Γ is a bicausal isomorphism in $F^{p\times p}[[\lambda^{-1}]]$, U is unimodular in $F^{m\times m}[\lambda]$ and

$$D(\lambda) = \begin{pmatrix} \Delta(\lambda) & 0 \\ 0 & 0 \end{pmatrix}$$

where $\Delta(\lambda) = \operatorname{diag}(\lambda^{\kappa_1}, \ldots, \lambda^{\kappa_r}) \in F^{r\times r}((\lambda^{-1}))$. The indices $\kappa_1, \ldots, \kappa_r$ are called the left factorization indices. We assume them decreasingly ordered that is $\kappa_1 \geqslant \kappa_2 \geqslant \ldots \geqslant \kappa_r$. In an analogous way we define the right factorizations and the right factorization indices. It is quite easy to prove that left and right factorizations of infinity of a rational function $A \in F^{p\times m}((\lambda^{-1}))$ exist. For a proof one reference is Fuhrmann and Willems (1979).

Feedback is going to play an important role in the sequel. For a reachable pair (A, B) we characterize feedback equivalent pairs in terms of coprime factorizations.

Let (A, B) be reachable and let $H(\lambda)D(\lambda)^{-1}$ be a right coprime factorization of $(\lambda I - A)^{-1}B$. Then the pair (S_D, π_D) is isomorphic to (A, B). Thus all the information relative to (A, B), including the reachability indices, are derivable from D. The following theorem, due to Hautus and Heyman (1978) (Fuhrmann 1979) is useful in the analysis of feedback.

Theorem 3.1

Let (A, B) be a reachable pair, with $A \in F^{n\times n}$, $B \in F^{m\times m}$, and let $H(\lambda)D(\lambda)^{-1}$ be a coprime factorization of $(\lambda I - A)^{-1}B$. Then a necessary and sufficient condition for a pair (A_1, B_1) to be feedback equivalent to (A, B) is that

$$(\lambda I - A_1)^{-1} B_1 = R H(\lambda)(P(\lambda) + Q(\lambda))^{-1} P^{-1} \tag{3.3}$$

where R and P are non-singular matrices and $Q \in F^{m\times m}[\lambda]$ is such that QD^{-1} is strictly proper.

If we let $D_1 = D + Q$ and associate with the pairs (A, B) and (A_1, B_1) the equivalent pairs (S_D, π_D) and (S_{D_1}, π_{D_1}) then the invertible map $Y: K_{D_1} \to K_D$ for which

$$S_D Y - Y S_{D_1} = BK = \pi_D K \tag{3.4}$$

is given by a Toeplitz operator induced by DD_1^{-1}, i.e. by $T_{DD_1^{-1}}$ where

$$T_{DD_1^{-1}}f = \pi_+ DD_1^{-1} f \tag{3.5}$$

Thus D and D_1 are related to feedback equivalent pairs if and only if $D(\lambda)D_1(\lambda)^{-1} = \Gamma(\lambda)$ is an invertible element of $F^{m\times m}[[\lambda^{-1}]]$, i.e. by a bicausal isomorphism.

The connection between the Kronecker indices, coprime factorizations and Wiener–Hopf factorizations is summarized in the following theorem (Fuhrmann and Willems 1979).

Theorem 3.2

Let $G(\lambda)$ be a strictly proper transfer function and let

$$G(\lambda) = N_r(\lambda)D_r(\lambda)^{-1} = D_l(\lambda)^{-1}N_l(\lambda) \tag{3.6}$$

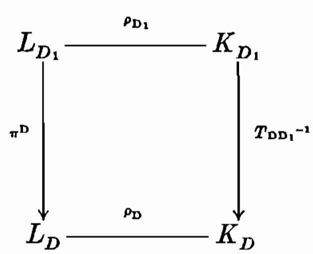
be coprime factorizations of G. Let (A, B, C) be any canonical realization of G. Then

(i) The reachability indices of (A, B) coincide with the left factorization indices of D_r.
(ii) The observability indices of (A, C) coincide with the right factorization indices of D_l.

There is a close connection between Toeplitz operators and projections. We state it as a theorem.

Theorem 3.3

Let $D, D_1 \in F^{m\times m}[\lambda]$ such that DD_1^{-1} is a bicausal isomorphism. Then the following diagram is commutative



(3.7)

Thus $\pi^D | L_{D_1}$ is isomorphic to $T_{DD_1^{-1}} | K_{D_1}$.

Proof

Let $h \in L_{D_1}$ then

$$\rho_D \pi^D h = D\pi_- D^{-1}\pi_+ Dh = D\pi_- D^{-1}\pi_+ DD_1^{-1} D_1 h$$
$$= \pi_D T_{DD_1^{-1}}(D_1 h) = T_{DD_1^{-1}}\rho_{D_1} h$$

as $D_1 h \in K_{D_1}$ and $T_{DD_1^{-1}}$ maps K_{D_1} onto K_D.

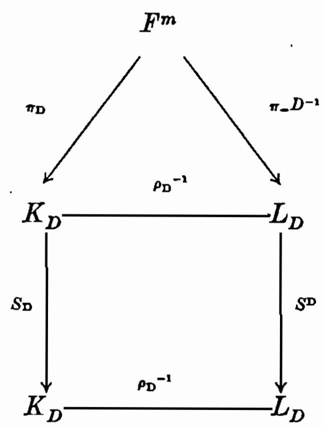
Corollary 3.4

$D, D_1 \in F^{m\times m}[\lambda]$. Then DD_1^{-1} is a bicausal isomorphism if and only if $\pi^D | L_{D_1}$ is an invertible map from L_{D_1} onto L_D. In that case its inverse is given by $\pi^{D_1} | L_D$.

Proof

Follows from the commutative diagram (3.7).

The preceding results allow us to study feedback also in the setting of rational models. Consider the following diagram



(3.8)

which is clearly commutative as for $\xi \in F^m \phi_D^{-1}\,\pi_D\xi = D^{-1}D\pi_-D^{-1}\xi = \pi D^{-1}\xi$. Thus the pair (S_D, π_D) acting in the state space K_D is isomorphic to the pair (S^D, π_-D^{-1}) acting in L_D.

The question of feedback equivalence of two such pairs is resolved by the following.

Theorem 3.5

The pairs (S^D, π_-D^{-1}) and $(S^{D_1}, \pi_-D_1^{-1})$ acting in the state spaces L_D and L_{D_1} respectively are feedback equivalent if and only if DD_1^{-1} is a bicausal isomorphism. In that case $\pi^D|L_{D_1}$ is an invertible map of L_{D_1} and L_D and we have

$$S^D\pi^D - \pi^D S^{D_1} = \pi_-D^{-1}K_1 \tag{3.9}$$

for some $K_1 : L_{D_1} \to F^m$.

Proof

The result follows easily, by isomorphism, from Theorem 3.1. To check (3.9) we start with (3.4) and obtain

$$\rho_D^{-1}\{S_D T_{DD_1^{-1}} - T_{DD_1^{-1}} S_{D_1}\}\rho_{D_1} = \rho_D^{-1}\,\pi_D K \rho_{D_1}$$

$$S^D \rho_D^{-1}\, T_{DD_1^{-1}}\rho_{D_1} - \pi^D \rho_{D_1^{-1}} S_{D_1}\rho_{D_1} = \rho_- D^{-1} K \rho_{D_1}$$

$$S^D\pi^D - \pi^D S^{D_1} = \pi D^{-1} K_1$$

with $K_1 = K\rho_{D_1}$.

We conclude this section with some easy factorization results which are useful in the sequel.

Theorem 3.6

Let $A \in F^{p\times m}[\lambda]$ with $p \leqslant m$. Then the following conditions are equivalent.

(i) $AF^m[\lambda] = F^p[\lambda]$.
(ii) A is right invertible, i.e. there exists a $B \in F^{m\times p}[\lambda]$ for which $AB = I$.
(iii) The g.c.d. of the determinants of all $p \times p$ minors is equal to one.
(iv) The left factorization indices at infinity are all zero.

Proof

Let $e_1, \ldots, e_p$ be the standard basis for F^p. If we assume (i) then there exist $b_i \in F^m[\lambda]$ for which $e_i = Ab_i$. If $B(\lambda)$ is the $m \times p$ matrix whose columns are $b_1, \ldots, b_p$ then $AB = I$ and (i) implies (ii). Conversely assuming (ii) we have $BF^p[\lambda] \subset F^m[\lambda]$ and $F^p[\lambda] = ABF^p[\lambda] \subset AF^m[\lambda] \subset F^p[\lambda]$ and necessarily we have equality throughout so (i) follows.

Let $A^{(p)}$ be the $1 \times \begin{pmatrix} m \\ p \end{pmatrix}$ compound matrix induced by A consisting of all $p \times p$ minors ordered lexicographically. $B^{(p)}$ is similarly defined and we have the general relation

$$(AB)^{(p)} = A^{(p)}B^{(p)}$$

and it follows that $A^{(p)}B^{(p)} = 1$ which shows that (ii) implies (iii).

If (iii) is assumed then the Smith form of A is $(I, 0)$ and so for some unimodular matrices U and V we have $U(\lambda)A(\lambda)V(\lambda) = (I, 0)$ or

$$A(\lambda) = (U(\lambda)^{-1} \;\; 0)\, V(\lambda)^{-1} = (I \;\; 0) \begin{pmatrix} U(\lambda)^{-1} & 0 \\ 0 & I \end{pmatrix} V(\lambda)^{-1}$$

and (iv) follows. Finally assume (iv) then choose a unimodular matrix U so that $A(\lambda)U(\lambda)$ is column proper. This means that $A(\lambda) = (D, 0)U(\lambda)$ with D constant square non-singular. But then

$$B(\lambda) = U(\lambda)^{-1} \begin{pmatrix} D^{-1} \\ 0 \end{pmatrix}$$

is a right inverse.

The preceding theorem is useful in proving a factorization result for singular polynomial matrices.

Theorem 3.7

Let $A \in F^{p\times m}[\lambda]$ be of full row rank. Then there exists a factorization

$$A(\lambda) = A_0(\lambda)A_1(\lambda) \tag{3.10}$$

such that $A_0 \in F^{p\times p}[\lambda]$ is non-singular and A_1 is right invertible. A_0 is uniquely determined up to a right unimodular factor.

Proof

Let $M = AF^m[\lambda]$ then M is a submodule, in fact a full submodule, of $F^p[\lambda]$ and hence has, by Theorem 2.1, a representation as $M = A_0F^p[\lambda]$. Since $A(\lambda)\xi \in M$ for each $\xi \in F^m[\lambda]$ (3.10) follows.

4. Polynomial models and (A, B)-invariant subspaces.

Let us consider the pair (A, B) acting in the state space X. Denote by $\mathscr{B}$ the range of B. A subspace $V \subset X$ is called an (A, B)-*invariant subspace* (Wonham 1974) of

$$AV \subset V + \mathscr{B} \tag{4.1}$$

A subspace V is an (A, B)-invariant subspace if and only if for some feedback map F we have

$$(A + BF)V \subset V \tag{4.2}$$

Thus by feedback we may make V into an invariant subspace of a feedback equivalent operator. A family $\{V_\alpha\}$ of (A, B)-invariant subspaces is called *compatible* if for some F

$$AV_\alpha \subset V_\alpha + \mathscr{B} \quad \text{for all } \alpha \tag{4.5}$$

The following simple lemma will be of use later on.

Lemma 4.1

Given the pair (A, B) and two (A, B)-invariant subspaces V_1 and V_2. If $V_1 \subset V_2$ then V_1 and V_2 are compatible.

Proof

Choose a basis $v_1, \ldots, v_{\mu_1}, v_{\mu_1+1}, \ldots, v_{\mu_2}$ for V_2 such that $v_1, \ldots, v_{\mu_1}$ is a basis of V_1. Since $Av_i = w_i - Bu_i$ with $w_i \in V_2$ and $w_i \in V_1$ if $1 \leqslant i \leqslant \mu_i$, we let $Fv_i = u_i$ and extend the definition of F arbitrarily to the whole space.

By induction the lemma can be extended to cover the case of chains of (A, B)-invariant subspaces.

Let now $H(\lambda)D(\lambda)^{-1}$ be any right coprime factorization of $(\lambda I - A)^{-1}B$. Then if (A_1, B_1) is a pair feedback equivalent to (A, B) and isomorphic to (S_{D_1}, π_{D_1}) this, by Theorem 3.1, occurs if and only if DD_1^{-1} is a bicausal isomorphism. This is the key to the characterization of (A, B)-invariant subspaces of K_D. In the following the reference to (A, B)-invariant subspaces is always relative to a realization associated with a given factorization of the transfer function by the procedure outlined in § 2.

Theorem 4.2

Let $D(\lambda) \in F^{m \times m}[\lambda]$. Then a subspace M of K_D is an (A, B)-invariant subspace, i.e. (S_D, π_D)-invariant subspace, if and only if there exist $D_1, E_1, F_1 \in F^{m \times m}[\lambda]$ such that

(i) $M = T_{DD_1^{-1}}(E_1 K_{F_1})$

(ii) $D_1 = E_1 F_1$

(iii) DD_1^{-1} is a bicausal isomorphism.

Proof

Assume there exist D_1, E_1 and $F_1 \in F^{m\times n}[\lambda]$ such that (i)–(iii) are satisfied, then (ii) implies that $E_1 K_{F_1}$ is an S_{D_1}-invariant subspace of K_{D_1}. From (iii) it follows that the pairs (S_D, π_D) and (S_{D_1}, π_{D_1}) are feedback equivalent with $T_{DD_1^{-1}}$, the Toeplitz operator induced by DD_1^{-1} being an invertible map from K_{D_1} onto K_D satisfying

$$S_D T_{DD_1^{-1}} - T_{DD_1^{-1}} S_{D_1} = \pi_D \cdot K \tag{4.4}$$

for some $K : K_{D_1} \to F^m$. This implies that M is an (A, B)-invariant subspace.

Conversely assume $M \subset K_D$ is an (A, B)-invariant subspace of K_D. By the definition of (A, B)-invariant subspaces M, or its isomorphic image, is an invariant subspaces of a feedback equivalent pair, a pair which without loss of generality we identify with (S_{D_1}, π_{D_1}). This implies that DD_1^{-1} is a bicausal isomorphism. The map from K_{D_1} onto K_D that exhibits the feedback, i.e. that satisfies (4.4) is the corresponding Toeplitz map. Since S_{D_1}-invariant subspace are of the form $E_1 K_{F_1}$ for some factorization (ii) it follows that M has a representation (i).

In the representation of (A, B)-invariant subspaces we do not expect uniqueness but we do expect uniqueness modulo feedback equivalence and indeed this is the case.

Theorem 4.3

Let $M = T_{DD_i^{-1}}(E_i K_{F_i})$, $i = 1, 2$ be two representations of an (A, B)-invariant subspace of K_D satisfying the conditions of Theorem 4.2 then (S_{F_1}, π_{F_1}) and (S_{F_2}, π_{F_2}) are feedback equivalent or stated differently $F_2 F_1^{-1}$ is a bicausal isomorphism.

Proof

If $M = T_{DD_i^{-1}}(E_i K_{F_i})$ then $T_{D_2 D_1^{-1}} = T_{D_2 D^{-1}}(T_{D_1 D^{-1}})^{-1}$ is an invertible map of K_{D_1} onto K_{D_2} which satisfies $T_{D_2 D_1^{-1}}(E_1 K_{F_1}) = E_2 K_{F_2}$. Now for $f \in K_{F_1}$ we have $T_{D_2 D_1^{-1}} E_1 f = \pi_+ D_2 D_1^{-1} E_1 f = \pi_+ E_2 F_2 F_1^{-1} f = \pi_+ E_2 \pi_+ F_2 F_1^{-1} f + \pi_+ E_2 \pi_- F_2 F_1^{-1} f = E_2 g$ for some $g \in K_{F_2}$. This implies $g = \pi_+ F_2 F_1^{-1} f$. By symmetry we have the invertibility of $T_{F_2 F_1^{-1}}$.

It is of considerable interest to characterize the right factors $F_1 \in F^{m\times m}[\lambda]$ which can be left multiplied to yield a feedback equivalent system to (S_D, π_D). The key to this are Wiener–Hopf factorizations.

Theorem 4.4

Given non-singular D and F_1 in $F^{m\times m}[\lambda]$. Then there exist $E_1 \in F^{m\times m}[\lambda]$ such that

(i) $D_1 = E_1 F_1$, and
(ii) DD_1^{-1} is a bicausal isomorphism if and only if all the left Wiener–Hopf factorization indices (at infinity) of DF_1^{-1} are non-negative.

Proof

If $D_1 = E_1 F_1$ and $\Gamma = DD_1^{-1}$ is a bicausal isomorphism then $DF_1^{-1} = \Gamma E_1$. Now let $E_1 = \Omega \Delta U$ be a left factorization of E_1 then necessarily the factorization indices of E_1 are non-negative, being the reachability indices of the pair (S_{E_1}, π_{E_1}). It follows that

$$DF_1^{-1} = (\Gamma\Omega)AU$$

i.e. DF_1^{-1} has non-negative left factorization indices.

Conversely if $DF_1^{-1} = \Gamma\Delta U$ with $\Delta(\lambda) = \text{diag}\,(\lambda\kappa_1, \ldots, \lambda\kappa_m)$ with $\kappa_1 \geqslant \ldots \geqslant \kappa_m \geqslant 0$ then define $E_1 = \Delta U$. It follows that $D = \Omega D_1$ with $D_1 = E_1 F_1$ and Ω a bicausal isomorphism.

In terms of the rational models we can state Theorem 4.2 as follows.

Theorem 4.5

A subspace M of L_D is an (A, B)-invariant subspace if and only if there exist $D_1, E_1, F_1 \in F^{m\times m}[\lambda]$ such that

(i) $M = \pi^D L_{F_1}$

(ii) $D_1 = E_1 F_1$

(iii) DD_1^{-1} is a bicausal isomorphism.

Actually we can strengthen the previous result a bit.

Theorem 4.6

A subspace M of L_D is an (A, B)-invariant subspace if and only if

$$M = \pi^D L \tag{4.5}$$

for some submodule L of $\lambda^{-1}F^m[[\lambda^{-1}]]$.

Proof

In view of the previous theorem all we have to prove is that the image under the projection π^D of any submodule of $\lambda^{-1}F^m[[\lambda^{-1}]]$ is (A, B)-invariant subspace of L_D. Equivalently we have to show that if $h \in L$ then there exists $h_1 \in L$ and $\xi \in F^m$ such that

$$S^D \pi^D h = \pi^D h_1 + \pi_- D^{-1}\xi \tag{4.6}$$

We will prove (4.6) with $h_1 = Sh$ and $\xi = (Dh)_{-1}$. In this case

$$\begin{aligned} S^D\pi^D h - \pi^D S_- h &= \pi_-\lambda\pi_- D^{-1}\pi_+ Dh - \pi_- D^{-1}\pi_+ D\pi_-\lambda h \\ &= \pi_-\lambda D^{-1}\pi_+ D\lambda h - \pi_- D^{-1}\pi_+ D\lambda h \\ &= \pi_- D^{-1}\{\lambda\pi_+ Dh - \pi_+\lambda Dh\} = \pi_- D^{-1}\xi \end{aligned}$$

Corollary 4.7

A subspace M of K_D is an (A, B)-invariant subspace if and only if it has the representation

$$M = \pi_D \pi_+ DL \tag{4.7}$$

for any submodule L of $\lambda^{-1}F^m[[\lambda^{-1}]]$.

We point out that if $D(\lambda)^{-1}$ is proper then (4.7) can be replaced by the simpler form

$$M = \pi_+ DL \tag{4.8}$$

Corollary 4.8

If M_1 and M_2 are (A, B)-invariant subspaces of L_D then so is $M_1 + M_2$.

Proof

Let $M_i = \pi^D L_i$ then $M = \pi^D L$ with $L = L_1 + L_2$ independently of course of the setting the result holds.

As an example we compute the 1-dimensional (A, B)-invariant subspaces of K_D.

Lemma 4.9

A subspace M of K_D is a 1-dimensional (A, B)-invariant subspace if and only if it is spanned by a vector polynomial of the form

$$f(\lambda) = \pi_D \cdot \left(\frac{D(\lambda) - D(\alpha)}{\lambda - \alpha} \right) \xi \tag{4.9}$$

for some $\xi \in F$ and $\xi \in F^m$.

Proof

The 1-dimensional submodules of $\lambda^{-1} F^m[[\lambda^{-1}]]$ are spanned by one function of the form $\xi/(\lambda - \alpha)$. The result now follows from Corollary 4.7.

Again we remark that if $D(\lambda)^{-1}$ is proper (4.9) simplifies to

$$f(\lambda) = \frac{(D(\lambda) - D(\alpha))\xi}{\lambda - \alpha} \tag{4.10}$$

The structure of (A, B)-invariant subspaces reflects the structure of invariant subspaces of a linear transformation. In that case a 2-dimensional invariant subspace may be one of two kinds, either spanned by two linearly independent eigenfunctions or by an eigenfunction and a related generalized eigenfunction. Thus a 2-dimensional (A, B)-invariant subspace of K_D, assuming for simplicity that $D(\lambda)^{-1}$ is proper, is either of the form

$$\text{span} \left\{ \frac{(D(\lambda) - D(\alpha))\xi}{\lambda - \alpha}, \frac{(D(\lambda) - D(\beta))\xi}{\lambda - \beta} \right\}$$

or

$$\text{span} \left\{ \frac{(D(\lambda) - D(\alpha))\xi}{\lambda - \alpha}, \frac{(D(\lambda) - D(\alpha) - D^1(\alpha)(\lambda - \alpha))\xi}{(\lambda - \alpha)^2} \right\}$$

Obviously this analysis can be pushed further but we prefer to leave it at that.

As an example we consider the case of $d(\lambda) = \lambda^n$, i.e. a single input system given in Brunovsky canonical form. The pair (A, b) is represented in this case by (S_d, π_d) which relative to the basis $\{1, \lambda, \ldots, \lambda^{n-1}\}$ of K_{λ^n} has the matrix representation

$$A = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}_{n \times n}, \quad b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1}$$

The 1-dimensional (A, b)-invariant subspaces are given by multiples of

$$\frac{d(\lambda)-d(\alpha)}{\lambda-\alpha}=\frac{\lambda^n-\alpha^n}{\lambda-\alpha}=\lambda^{n-1}+\alpha\lambda^{n-2}+\ \dots\ +\alpha^{n-1}$$

or in a vector representation by multiples of

$$\begin{pmatrix}\alpha^{n-1}\\ \vdots\\ \alpha\\ \\ 1\end{pmatrix}.$$

The 2-dimensional subspaces are either spanned by vectors of the form

$$\begin{pmatrix}\alpha^{n-1}\\ \vdots\\ \alpha\\ \\ 1\end{pmatrix},\quad \begin{pmatrix}\beta^{n-1}\\ \vdots\\ \beta\\ \\ 1\end{pmatrix}$$

with $\alpha\neq\beta$ or alternatively by the span of

$$\frac{\lambda^n-\alpha^n}{\lambda-\alpha}$$

and

$$\frac{\lambda^n-\alpha^n-n\alpha^{n-1}(\lambda\alpha)}{\lambda-\alpha}=\lambda^{n-2}+2\alpha\lambda^{n-3}+\ \dots\ +(n-\alpha)\alpha^{n-2}$$

or equivalently by

$$\operatorname{span}\begin{bmatrix}\alpha^{n-1}\\ \vdots\\ \alpha\\ \\ 1\end{bmatrix},\quad \begin{bmatrix}(n-1)\alpha^{n-2}\\ \vdots\\ 2\alpha\\ \\ 1\\ \\ 0\end{bmatrix}$$

Starting with the transfer function $G(\lambda)$ and the coprime factorizations

$$G(\lambda)=N(\lambda)D(\lambda)^{-1}=T(\lambda)^{-1}U(\lambda) \tag{4.11}$$

then the pair (A, B) induced by T and U through $A=S_T$ and $B\xi=\pi_T U\xi$ is isomorphic to (S_D, π_D) and hence a characterization of (A, B)-invariant subspaces can be given in these terms.

Theorem 4.10

A subspace $V\subset K_T$ is an (A, B)-invariant subspace if and only if there exist D_1, E_1, $F_1\in F^{m\times m}[\lambda]$ such that

$$V=T\pi_{-}N\pi^{D}L_{F_1} \tag{4.12}$$

$$D_1=E_1F_1 \tag{4.13}$$

and

$$DD_1^{-1} \text{ is a bicausal isomorphism} \tag{4.14}$$

Proof

From $T^{-1}U = ND^{-1}$ we obtain

$$UD = TN \tag{4.15}$$

and the assumption of the coprimeness of the factorizations (4.11) and Theorem 2.8 imply that the map $X: K_D \to K_T$ defined by

$$Xf = \pi_T Uf, \quad \forall f \in K_D \tag{4.16}$$

is an isomorphism. Thus a subspace $V \subset K_T$ is an (A, B)-invariant subspace if and only if it is image under X of an (A, B)-invariant subspace of K_D and these were characterized by Theorem 4.2. Let therefore $XM = V$ with $M = T_{DD_1^{-1}} E_1 K_{F_1}$ then

$$\begin{aligned} V = \pi_T U \pi_+ DD_1^{-1} E_1 K_{F_1} &= T\pi_- T^{-1} U \pi_+ DF_1^{-1} E_1^{-1} E_1 K_{F_1} \\ &= T\pi_- ND^{-1}\pi_+ DL_{F_1} = T\pi_- N\pi_- D^{-1}\pi_+ DL_{F_1} = T\pi_- N \cdot \pi^D L_{F_1} \end{aligned}$$

Note that L_{F_1} is a submodule of L_{D_1} but $\pi^D L_{F_1}$ is just a subspace of L_D. The map $h \to \pi_- Nh$ is an isomorphism of L_D into L_T and so $\pi_- N\pi^D L_{F_1}$ is a subspace of L_T which under T is mapped into an (A, B)-invariant subspace of K_T.

We use the previous theorem to characterize all 1-dimensional (A, B)-invariant subspaces of K_T. Assume without loss of generality that D is column proper. A 1-dimensional subspace of K_D is spanned by

$$f(\lambda) = \frac{(D(\lambda) - D(\alpha)\xi)}{(\lambda - \alpha)}$$

and so we compute Xf.

$$\begin{aligned} Xf = \pi_T Uf = \pi_T U(\lambda)\left(\frac{D(\lambda) - D(\alpha)}{\lambda - \alpha}\right)\xi &= \pi_T \frac{(U(\lambda)D(\lambda) - U(\lambda)D(\alpha))\xi}{\lambda - \alpha} \\ = \pi_T \frac{(T(\lambda)N(\lambda) - U(\lambda)D(\alpha))\xi}{\lambda - \alpha} &= \pi_T \left\{T(\lambda)\left(\frac{N(\lambda) - N(\alpha)}{\lambda - \alpha}\right)\xi \right. \\ &\qquad \left. + \frac{T(\lambda)N(\alpha) - U(\lambda)D(\alpha)\xi}{\lambda - \alpha}\right\} \\ = \pi_T \left\{\frac{T(\lambda)N(\alpha) - U(\lambda)D(\alpha)}{\lambda - \alpha}\right\}\xi & \end{aligned}$$

as

$$T(\lambda)\frac{N(\lambda) - N(\alpha)}{\lambda - \alpha}\xi \in \ker \pi_T$$

On the other hand

$$f_1(\lambda) = \left(\frac{T(\lambda)N(\alpha) - V(\lambda)D(\alpha)}{\lambda - \alpha}\right)\xi$$

is in K_T as, since $T(\alpha)N(\alpha)-U(\alpha)D(\alpha)=0$, it is in $F^p[\lambda]$ and

$$\pi_+T^{-1}f_1=\pi_+\left\{\frac{N(\alpha)}{\lambda-\alpha}+\frac{T(\lambda)^{-1}U(\lambda)D(\alpha)\xi}{\lambda-\alpha}\right\}=0$$

and so the subspace is spanned by

$$f_1(\lambda)=\left(\frac{T(\lambda)N(\alpha)-U(\lambda)D(\alpha)}{\lambda-\alpha}\right)\xi$$

If $U(\lambda)=I$ then $T(\lambda)N(\lambda)=D(\lambda)$ and we can rewrite f_1 in the form

$$f_1(\lambda)=\left(\frac{T(\lambda)-T(\alpha)N(\alpha)}{\lambda-\alpha}\right)\xi$$

which is in agreement with the previously obtained characterization.

Using Theorem 4.6 we can remove the restriction on F_1 in Theorem 4.10 and state it as follows.

Theorem 4.11

A subspace $V\subset K_T$ is an (A, B)-invariant subspace of the pair $(S_T, \pi_T U)$, where

$$G=T^{-1}U=ND^{-1} \tag{4.17}$$

are coprime factorizations of G, if and only if

$$V=T\pi_-N\pi^D L \tag{4.18}$$

for some submodule L of $\lambda^{-1}F^m[[\lambda^{-1}]]$.

5. Polynomial characterization of (A, B)-invariant subspace in ker C

In this section we study (A, B)-invariant subspaces which lie in the kernel of C. This study is in terms of the numerator polynomials in the left or right coprime factorization of the transfer function $G(\lambda)=C(\lambda I-A)^{-1}B$ of the given system. Previous work along these lines has been that of Emre (1978) and Emre and Hautus (1978) where some applications are given. Of course the best source for the usefulness of this concept in system design is Wonham (1974).

In order to obtain some feeling for the problem and the way zeros come into the picture at all we consider first the scalar case. Under an element of the feedback group an (A, B)-invariant subspace is mapped onto another (A, B)-invariant subspace. Thus we may without loss of generality assume that the transfer function is of the form $g(\lambda)=p(\lambda)/\lambda^n$ with $\deg p\leqslant n-1$. This is equivalent to the assumption that we have a realization in Brunovsky canonical form. In fact if $p(\lambda)=p_{n-1}\lambda^{n-1}+\ldots+p_0$

$$\left(\begin{pmatrix}0 & \cdots\cdots & 0\\ 1 & & \vdots\\ & \ddots & \vdots\\ & 1 & 0\end{pmatrix}, \begin{pmatrix}1\\ 0\\ \vdots\\ 0\end{pmatrix}, (p_{n-1}\cdots p_0)\right) \tag{5.1}$$

is a reachable realization of $p(\lambda)/\lambda^n$. Any 1-dimensional (A, B)-invariant subspace of this realization is spanned by a vector of the form

$$x=\begin{pmatrix}\alpha^{n-1}\\ \vdots\\ \alpha\\ \\ 1\end{pmatrix}.$$

Thus $x\in\ker C$ if and only if $p_{n-1}\alpha^{n-1}+p_{n-2}\alpha^{n-2}+\ \dots\ +p_0=0$, i.e. if and only if $p(\alpha)=0$. Of course in this case $\lambda-\alpha$ is a factor of $p(\lambda)$.

We use next our characterization of one-dimensional (A, B)-invariant subspaces in the vectorial case. Let $G(\lambda)$ be a $p\times m$ strictly proper rational function having the coprime factorizations

$$G(\lambda)=N(\lambda)D(\lambda)^{-1}=T(\lambda)^{-1}U(\lambda) \tag{5.2}$$

Using the right coprime factorization as a basis for a realization in K_D we characterized the one dimensional (A, B)-invariant subspaces of K_D as those spanned by vector polynomials of the form

$$f(\lambda)=\pi_D\,\frac{(D(\lambda)-D(\alpha))}{\lambda-\alpha}\,\xi$$

for same $\alpha\in F$ and $\xi\in F^m$. Now for any $f\in K_D$, $Cf=(ND^{-1}f)_{-1}$. In our case

$$Cf=\left(ND^{-1}\pi_D\,\frac{(D(\lambda)-D(\lambda))}{\lambda-\alpha}\,\xi\right)_{-1}=\left(N\pi_{-}D^{-1}\,\frac{(D(\lambda)-D(\alpha))}{\lambda-\alpha}\,\xi\right)_{-1}$$

$$=\left(ND^{-1}\left(\frac{D(\lambda)-D(\alpha)}{\lambda-\alpha}\right)\xi\right)_{-1}=\left(\frac{N(\lambda)\xi}{\lambda-\alpha}\right)_{-1}-\left(\frac{N(\lambda)D(\lambda)^{-1}D(\alpha)}{\lambda-\alpha}\,\xi\right)_{-1}$$

The right term vanishes as ND^{-1} is strictly proper and so is $D(\alpha)\xi/(\lambda-\alpha)$, and so there is no term in λ^{-1}. Also since

$$\frac{N(\lambda)\xi}{\lambda-\alpha}=\frac{(N(\lambda)-N(\alpha))}{\lambda-\alpha}\,\xi+\frac{N(\alpha)\xi}{\lambda-\alpha}$$

we have

$$Cf=\left(\frac{N(\lambda)\xi}{\lambda-\alpha}\right)_{-1}=\left(\frac{N(\alpha)\xi}{\lambda-\alpha}\right)_{-1}=N(\alpha)\xi$$

Thus $f\in\ker C$ if and only if $\xi\in\ker N(\alpha)$. But $N(\alpha)\xi=0$ corresponds to a right factor of N. In fact if we complete ξ arbitrarily to a basis $e_1, \dots, e_m$ of F^m with $e_1=\xi$ then relative to this basis the first column of $N(\alpha)$ is zero. Hence $N(\lambda)$ is divisible on the right by

$$\begin{pmatrix}\lambda-\alpha & 0 \ldots\ldots & 0\\ & & \\ 0 & 1 & \\ \vdots & & 0\\ \vdots & & & \ddots\\ 0 & & & 1\end{pmatrix}$$

Next we start from the coprime factorization $G = T^{-1}U$. One-dimensional (A, B)-invariant subspaces have been characterized previously and are of the form

$$f(\lambda) = \frac{T(\lambda)N(\alpha) - U(\lambda)D(\alpha)}{\lambda - \alpha}\,\xi$$

We have seen, and this is easily checked directly, that $f \in \ker C$ if and only if $N(\alpha)\xi = 0$. Thus $f(\lambda) = U(\lambda)\eta/(\lambda - \alpha)$ where $\eta = D(\alpha)\xi$. Since $f \in K_T$ we must have $U(\alpha)\eta = 0$. Again this corresponds to a right factor of U.

Before attacking the general problem we characterize the A-invariant subspaces in ker C via polynomial models. Let $N(\lambda)D_1(\lambda)^{-1}$ be a factorization of $G_1(\lambda)$ which is assumed to be strictly proper. In K_{D_1} A_1 is defined by $A_1 = S_{D_1}$ and

$$Cf = (ND_1^{-1} f)_{-1} \quad \text{for } f \in K_{D_1} \tag{5.3}$$

Theorem 5.1

A subspace M of K_{D_1} is an S_{D_1}-invariant subspace included in ker C if and only if

$$M = E_1 K_{F_1} \tag{5.4}$$

and F_1 is a non-singular common right factor of N and D_1, with $D_1 = E_1F_1$.

Proof

Let M be an S_{D_1}-invariant subspace. Then $D_1 = E_1F_1$ and $M = E_1K_{F_1}$. If $M \subset \ker C$ and $f \in M$, i.e. $f = E_1 g$ with $g \in K_{F_1}$ then

$$Cf = (ND_1^{-1}\, E_1 g)_{-1} = (NF_1^{-1}\, g)_{-1} = 0$$

Actually more is true. Since $S_{F_1}{}^j\, K_{F_1} \subset K_{F_1}$ we have for $f = E_1 g$, $g \in K_{F_1}$ that

$$\begin{aligned} 0 &= (ND_1^{-1}\, E_1 S_{F_1}{}^j\, g)_{-1} = (ND_1^{-1}\, E_1 \pi_{F_1} \lambda^j g)_{-1} \\ &= (NF_1^{-1}\, E_1^{-1}\, E_1 F_1 \pi_- F_1^{-1}\, \lambda^j g)_{-1} = (N\pi_- F_1^{-1}\, \lambda^j g)_{-1} \\ &= (\pi_- N \pi_- F_1^{-1}\, \lambda^j g)_{-1} = (\pi_- N F_1^{-1}\, \lambda^j g)_{-1} = (NF_1^{-1}\, g)_{-j-1} \end{aligned}$$

and this implies that $\pi_- NF_1^{-1}\, g = 0$ for all $g \in K_{F_1}$.

On the other hand it is clear that for all $g \in F_1 \cdot F^m[\lambda]$ we have, with $g = F_1 h$ and $h \in F^m[\lambda]$, that

$$\pi_- NF_1^{-1}\, g = \pi_- NF_1^{-1}\, F_1 h = \pi_- Nh = 0$$

and as $F^m[\lambda] = K_{F_1} \oplus F_1 F^m[\lambda]$ it follows that

$$\pi_- NF_1^{-1}\, g = 0 \quad \text{for all } g \in F^m[\lambda]$$

This implies that NF_1^{-1} is a polynomial matrix, say N_1 and so

$$N = N_1 F_1 \tag{5.5}$$

Conversely assume $N = N_1F_1$ and $D_1 = E_1F_1$ with F_1 non-trivial. Then $M = E_1K_{F_1}$ is an S_{D_1}-invariant subspace. Let $f \in M$, i.e. $f = E_1 g$ with $g \in K_{F_1}$. We show that $f \in \ker C$.

$$Cf = (ND_1^{-1}\, f)_{-1} = (NF_1^{-1}\, E_1^{-1}\, E_1 g)_{-1} = (N_1 F_1 F_1^{-1}\, g)_{-1} = (N_1 g)_{-1} = 0$$

as $N_1 g$ is a vector polynomial.

Under the linear invertible map Y the A-invariant subspaces are mapped onto the YAY^{-1}-invariant subspaces in a bijective way. Under the feedback group (A, B)-invariant subspaces are mapped into subspaces of the same kind. Analogously this holds also for (A, B)-invariant subspaces in the kernel of C under a larger group. This has been pointed out by Emre (1978).

Lemma 5.2

Let (A, B, C) be a system with a state space X, let $Y: X \to \bar{X}$ be an invertible map and let $V \subset X$ be an (A, B)-invariant subspace included in $\ker C$. If $(\bar{A}, \bar{B}, \bar{C})$ and $\bar{V}$ are defined by

$$\bar{C} = CY^{-1}, \quad \bar{A} = Y(A + BK + HC)Y^{-1}, \quad \bar{B} = YB \tag{5.6}$$

and

$$\bar{V} = YV \tag{5.7}$$

then $\bar{V}$ is an $(\bar{A}, \bar{B})$-invariant subspace in $\ker \bar{C}$.

The previous characterizations of (A, B)-invariant subspaces and invariant subspaces in $\ker C$ yield the following.

Theorem 5.3

Let $N(\lambda)D(\lambda)^{-1}$ be a right coprime factorization $C(\lambda I - A)^{-1}B$. A subspace $M \subset K_D$ is an (A, B)-invariant subspace in $\ker C$ if and only if

$$M = T_{DD_1^{-1}}(E_1 K_{F_1}) \tag{5.8}$$

where

$$DD_1^{-1} \text{ is a bicausal isomorphism} \tag{5.9}$$

$$D_1 = E_1 F_1 \tag{5.10}$$

and

$$N = N_1 F_1 \tag{5.11}$$

Proof

By the previous lemma (A, B)-invariant subspaces included in $\ker C$ correspond to A-invariant subspaces included in $\ker C$ of a feedback equivalent system. These arise, up to similarity, out of factorizations $N(\lambda)D_1(\lambda)^{-1}$ with (5.9) holding. Thus $T_{D_1 D^{-1}}(M)$ is an S_{D_1}-invariant subspace of K_{D_1} and it lies in $\ker C$, by Theorem 5.1, if and only if D_1 and N have a non-trivial common right factor. Since D_1 is non-singular so must be F_1.

The above theorem relates (A, B)-invariant subspaces in $\ker C$ to some non-singular right factors of N in the coprime factorization ND^{-1} of $C(\lambda I - A)^{-1}B$ Not every non-singular right factor of N can give rise to such a subspace. The permissible factors were characterized in Theorem 4.4 in terms of Wiener–Hopf factorizations.

That not all right factors of N produce (A, B)-invariant subspaces included in $\ker C$ can be seen from the following example. Let

$$N(\lambda) = (n(\lambda)\ 0), \quad D(\lambda) = \begin{pmatrix} d(\lambda) & 0 \\ 0 & 1 \end{pmatrix}$$

Then

$$F(\lambda)=\begin{pmatrix}1 & 0\\ 0 & f(\lambda)\end{pmatrix}$$

is a non-singular right factor of N for every $f\in F[\lambda]$ but it does not correspond to such a subspace.

Theorem 5.3 is closely related to the notion of feedback irreducibility (Morse 1976, Fuhrmann and Willems 1979). A strictly proper rational function $G(\lambda)$ is *feedback irreducible* if its McMillan degree is minimal in the set of transfer functions of systems feedback equivalent to a canonical realization of G. It is an immediate corollary from Theorem 5.3 that a strictly proper rational function $G(\lambda)$ is feedback irreducible if and only if in any canonical realization of G there is no non-trivial (A, B)-invariant subspace in ker C. This actually serves as the definition of feedback irreducibility in Morse (1976). For a connection to Wiener–Hopf factorizations we refer to Fuhrmann and Willems (1979).

Next we study the order relation among (A, B)-invariant subspaces in ker C in terms of the right factors of N.

Theorem 5.4

Let M_1, M_2 be two (A, B)-invariant subspaces of K_D which are contained in ker C which are associated with the non-singular right factors F_1, F_2 respectively of N. Then $M_1 \subset M_2$ if and only if F_1 is a right factor of F_2.

Proof

Assume F_1 is a right factor of F_2, i.e. $F_2 = HF_1$ for some, necessarily non-singular H. By Theorem 5.3 there exists E_2 such that $D_1 = E_2F_2$, DD_1^{-1} is a bicausal isomorphism, and $M_2 = T_{DD_1^{-1}}(E_2K_{F_2})$. Let $E_1 = E_2H$ then $D_1 = E_1F_1$. Also from $F_2 = HF_1$ it follows that $HK_{F_1} \subset K_{F_2}$ and hence that $E_1K_{F_1} = E_2HK_{F_1} \subset E_2K_{F_2}$ and from this, since $M_1 = T_{DD_1^{-1}}(E_1K_{F_1})$ the inclusion follows.

Conversely assume $M_1 \subset M_2$ are two (A, B)-invariant subspaces in ker C. In view of Lemma 4.1, M_1 and M_2 are compatible. Thus there exists D_1, with DD_1^{-1} a bicausal isomorphism, having the two factorizations $D_1 = E_1F_1 = E_2F_2$ and $E_1K_{F_1} \subset E_2K_{F_2}$. This implies that $F_2 = HF_1$ for some H. Since M_1, M_2 are in ker C necessarily F_1, F_2 are right factors of N.

Given a rational transfer function $G(\lambda)$ we were able to characterize the (A, B)-invariant subspaces in ker C in terms of non-singular right factors of the numerator polynomial in a right coprime factorization of G. Given a left coprime factorization $G = T^{-1}U$ we expect, purely on grounds of symmetry that the (A, B)-invariant subspaces in ker C of the system associated with this factorization will be related to non-singular factors of U and this in fact turns out to be true.

We will use freely results on feedback dualized to the case of the *output injection group*. Thus if T and T_1 are such that $T_1^{-1}T$ is a bicausal isomorphism then the systems associated with the factorizations $T^{-1}U$ and $T_1^{-1}U$ are output injective.

The following lemma will be used in the study of (A, B)-invariant subspaces in ker C.

Lemma 5.5

If T, $T_1 \in F^{p\times p}[\lambda]$ and $T_1^{-1}T$ is a bicausal isomorphism then K_T and K_{T_1} contain the same elements, that is, they are equal as sets.

Assume $\Gamma = T_1^{-1}T$ is a bicausal isomorphism.

Proof

Let $f \in K_T$ then $\pi_T f = f$. Consider

$$\begin{aligned}\pi_{T_1} f &= \pi_{T_1}\pi_T f = T_1\pi_- T_1^{-1}\, T\pi_- T^{-1} f = T_1\pi_-\Gamma\pi_- T^{-1} f = T_1\Gamma\pi_- T^{-1} f \\ &= T_1 T_1^{-1}\, T\pi_- T^{-1} f = \pi_T f = f\end{aligned}$$

i.e. $f \in K_{T_1}$ and the result follows by symmetry.

Next we pass on to the characterization of (A, B)-invariant subspaces in $\ker C$ in terms of the left coprime factorization $G = T^{-1}U$. The triple (A, B, C) is the one associated with this factorization of G.

Theorem 5.6

Let $G = T^{-1}U$ be a left coprime factorization with $T \in F^{p\times p}[\lambda]$ non-singular and $U \in F^{p\times m}[\lambda]$. Let

$$U = E_1 U_1 \tag{5.12}$$

be a factorization of U with $E_1 \in F^{p\times p}[\lambda]$ non-singular and $U_1 \in F^{p\times m}[\lambda]$. Then V defined by

$$V = E_1 K_{U_1} \tag{5.13}$$

is an (A, B)-invariant subspace in $\ker C$.

Proof

First we show that V defined by (5.13) is (A, B)-invariant. Note that if $h \in \lambda^{-1}F^m[[\lambda^{-1}]]$ satisfies $U_1 h \in F^p[\lambda]$ then also $U_1(S_- h) \in F^p[\lambda]$ where S_- is the shift in $\lambda^{-1}F^m[[\lambda^{-1}]]$ defined by (2.9). Indeed for $h \in \lambda^{-1}F^m[[\lambda^{-1}]]$ we have

$$\lambda h = (\pi_+ + \pi_-)\lambda h) = \pi_+(\lambda h) + \pi_-(\lambda h) = \xi + S_- h$$

with $\xi = h_{-1}$. Hence

$$U_1(S_- h) = U_1(\lambda h) - U_1\xi = \lambda U_1 h - U_1 \xi \in F^p[\lambda]$$

Next we compute $S_T f$ for $f \in E_1 K_{U_1}$. Thus $f = E_1 U_1 h$ with $U_1 h \in K_{U_1}$.

$$\begin{aligned}S_T f &= \pi_T \lambda f = T\pi_- T^{-1}\lambda f = T\pi_- T^{-1}\lambda E_1 U_1 h = T\pi_- T^{-1} U(\xi + S_- h) \\ &= T\pi_- T^{-1}U\xi + T\pi_- T^{-1} U S_- h = U\xi + US_- h = U\xi + E_1 U_1(S_- h)\end{aligned}$$

Now $U\xi \in \beta = \{U\eta \mid \eta \in F^m\}$ and $U_1(S_- h) \in K_{U_1}$ so $E_1 U_1(S_- h) \in E_1 K_{U_1}$. This shows that V is (A, B)-invariant.

That $E_1 K_{U_1}$ is in $\ker C$ follows from the following. Let $f \in E_1 K_{U_1}$ be represented as before, i.e. $f = E_1 U_1 h$. Then

$$Cf = (T^{-1}f)_{-1} = (T^{-1}E_1 U_1 h)_{-1} = (T^{-1}Uh)_{-1} = 0$$

as both $T^{-1}U$ and h are strictly proper and so the coefficient of λ^{-1} in the formal power series of $T^{-1}Uh$ is necessarily zero.

Corollary 5.7

K_U is an (A, B)-invariant subspace in $\ker C$.

Note that the condition in Theorem 5.7 is sufficient. We will see later that it is not necessary, i.e. not every (A, B)-invariant subspace in $\ker C$ has the form (5.13).

A complete characterization is given by the following. The proof is close to an argument used by Emre and Hautus (1978).

Theorem 5.8

Let $G = T^{-1}U$ be as in Theorem 5.6. A necessary and sufficient condition for a subspace $V \subset K_T$ to be an (A, B)-invariant subspace in $\ker C$ is that V has the representation

$$V = U_0 K_{E_0} \tag{5.14}$$

where

$$U = U_0 E_0 \tag{5.15}$$

is a factorization of U with $E_0 \in F^{m\times m}[\lambda]$ non-singular and $U_0 \in F^{p\times m}[\lambda]$.

Proof

The sufficiency of the condition follows the lines of the proof of Theorem 5.6.

To prove necessity assume $V \subset K_T$ is a k-dimensional (A, B)-invariant subspace in $\ker C$. For some state feedback map K we have $(A+BK)V \subset V$ and we let $\bar{A} = (A+BK)|V$. Without loss of generality we may assume $\bar{A}$ to act in $F^k \cdot K_{\lambda I - \bar{A}}$ is a model for $\bar{A}$ and the elements of $K_{\lambda I - \bar{A}}$ coincide with all vectors in F^k.

Let $X: K_{\lambda I - A} \to K_T$ be the invertible map for which

$$X\bar{A} = (A+BK)X \tag{5.16}$$

By the linearity of X it follows that there exists a polynomial matrix Ψ in $F^{p\times k}[\lambda]$ such that

$$(X\xi)(\lambda) = \Psi(\lambda)\xi \quad \text{for every } \xi \in F^k \tag{5.17}$$

Since

$$(S_T f)(\lambda) = f(\lambda) - T(\lambda)\eta_f$$

we have

$$\Psi(\lambda)\bar{A}\xi = X\bar{A}\xi = (A+BK)X\xi = \lambda\Psi(\lambda)\xi - T(\lambda)\eta_1 - U(\lambda)\eta_2$$

where $\eta_1 \in F^p$ and $\eta_2 \in F^m$ depend linearly on ξ. Therefore there exist constant matrices $E \in F^{p\times k}$ and $F \in F^{m\times k}$ for which

$$\Psi(\lambda)\bar{A} = \lambda\Psi(\lambda) - T(\lambda)E - U(\lambda)F \tag{5.18}$$

or

$$\Psi(\lambda)(\lambda I - \bar{A}) = T(\lambda)E + U(\lambda)F$$

and equivalently

$$\Psi(\lambda) = T(\lambda)E(\lambda I - \bar{A})^{-1} + U(\lambda)F(\lambda I - \bar{A})^{-1} \tag{5.19}$$

Since $V \subset \ker C$ we must have for all $\xi \in F^k$ that $(T^{-1}\Psi\xi)_{-1} = 0$. Hence

$$(E(\lambda I - \bar{A})^{-1}\xi)_{-1} + (T(\lambda)^{-1}U(\lambda)F(\lambda I - \bar{A})^{-1}\xi)_{-1} = 0$$

Since $T^{-1}U$ and $F(\lambda I-\bar{A})^{-1}$ are both strictly proper the second term vanishes identically. Thus $E(\lambda I-\bar{A})^{-1}\xi=0$ for all ξ and so necessarily $E=0$. Thus (5.19) reduces to

$$\Psi(\lambda)=U(\lambda)F(\lambda I-\bar{A})^{-1} \tag{5.20}$$

We claim that $(\bar{A}, F)$ is an observable pair, or equivalently that F and $(\lambda I-\bar{A})$ are right coprime. Otherwise for some $\xi\neq 0$ we would have $F(\lambda I-\bar{A})^{-1}\xi=0$ and in that case (5.20) implies that $\Psi(\lambda)\xi=0$ contrary to the assumption that X is injective. Let $E_0^{-1}\,V_0$ be a left coprime factorization of $F(\lambda I-\bar{A})^{-1}$ then

$$\Psi(\lambda)=U(\lambda)E_0(\lambda)^{-1}V_0(\lambda) \tag{5.21}$$

and so E_0 is necessary a right factor of U, i.e.

$$U=U_0E_0 \tag{5.22}$$

and

$$\Psi(\lambda)=U_0(\lambda)V_0(\lambda) \tag{5.23}$$

Now $F(\lambda I-\bar{A})^{-1}$ is a state-to-output transfer function. Dualizing results of Hautus and Heyman (1978) it follows that the columns of V_0 span K_{E_0}, in fact are a basis of K_{E_0}. Combining this with (5.23) and the fact that $V=\{\Psi(\lambda)\xi\mid\xi\in F^k\}$ it follows that $V=U_0K_{E_0}$ and this completes the proof.

Corollary 5.9 [2]

K_U is the maximal (A, B)-invariant subspace in $\ker C$.

Proof

From the preceding theorem and Lemma 2.11 it follows that every (A, B)-invariant subspace of K_T which is contained in $\ker C$ is contained in K_U. But K_U is itself such a subspace, so maximality follows.

One might be tempted to speculate that the converse to Theorem is also true, i.e. that every (A, B)-invariant subspace of K_T which is contained in $\ker C$ has a representation of the form $E_1K_{U_1}$. This is false as the following counterexample shows.

Let $U(\lambda)=(1, \lambda^3)$ and $T(\lambda)$ be any scalar polynomial of degree $\geqslant 4$. Clearly K_U is the set of all polynomial of degree $\leqslant 2$. Since the g.c.d. of 1 and λ^3 is 1 there is no non-trivial factorization of the form $U=E_1U_1$ with E_1 non-singular. However there exist other (A, B)-invariant subspaces in $\ker C$. Consider the following factorization of $(1, \lambda^3)$.

$$(1, \lambda^3)=(1, \lambda^2)\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}=U_0(\lambda)E_0(\lambda)$$

then

$$K_{E_0}=\left\{\begin{pmatrix}0\\ \alpha\end{pmatrix}\middle|\ \alpha\in F\right\}\quad\text{and}\quad U_0K_{E_0}=\{\alpha\lambda^2\mid\alpha\in F\}$$

The (A, B)-invariant subspaces of K_T which are contained in $\ker C$ are ordered by inclusion and this can be immediately related to some division relations as in Theorem 5.4.

Theorem 5.10

Let V_1, V_2 be two (A, B)-invariant subspaces of K_T contained in ker C and assume

$$V_i = U_i K_{E_i}, \quad i = 1, 2 \tag{5.24}$$

with $E_i \in F^{m \times m}[\lambda]$ non-singular. Then $V_1 \subset V_2$ if and only if E_1 is a right factor of E_2, i.e. if and only if $E_2 = HE_1$.

Proof

If $E_2 = HE_1$ then $U = U_2 E_2 = U_2 H E_1 = U_1 E_1$. As E_1 is non-singular then $U_1 = U_2 H$. From the factorization $E_2 = HE_1$ it follows, by Theorem 2.9 or Lemma 2.11, that $HK_{E_1} \subset K_{E_2}$ and hence that

$$U_1 K_{E_1} = U_2 H K_{E_1} \subset U_2 K_{E_2}$$

which proves $V_1 \subset V_2$.

Conversely assume $V_1 \subset V_2$ are (A, B)-invariant subspaces of K_T contained in ker C. Thus if $f \in V_1$, $f = U_1 E_1 h = U_2 E_2 h$. Since $E_i h \in K_{E_i}$ and E_i are non-singular it follows that $h \in L_{E_1} \subset L_{E_2}$. From Theorem 2.9 (ii) it follows that E_1 is a right factor of E_2.

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