Ordinary differential equations for Math

(201.1.0061. Spring 2024. Dmitry Kerner)

Homework 9. Submission date: 9.07.2024

Questions to submit: 1.b. 1.d. 2.a. 2.c. 3.e. 3.f. 4.b. 4.d.

Either typed or in readable handwriting and scanned in readable resolution.

1. **a.** Let $A(t) \in Mat_{n\times n}(C^r(\mathbb{R}))$, for $1 \le r \le \infty, \omega$. Prove: any local solution of $\underline{x}' = A(t)\cdot\underline{x}$ extends (uniquely) to a global solution $\underline{x}(t) \in C^{r+1}(\mathbb{R})$.
   b. Prove: the dimension of the space of solutions of $x^{(n)} + a_{n-1}(t)x^{(n-1)} \cdots + a_0(t)x = 0$ is n.
   c. Let $\underline{x}(t)$, $\underline{y}(t)$ be solutions of $\underline{x}' = A(t)\cdot\underline{x}$. Prove Grönwall's bound on the separation of solutions: $||\underline{x}(t) - \underline{y}(t)|| \le ||\underline{x}(t_0) - \underline{y}(t_0)|| \cdot e^{\int_{t_0}^t ||A(s)||_{op} ds}$.
   In particular, any solution of $\underline{x}' = A(t)\underline{x}$ satisfies: $\|\underline{x}(t)\| \le \|x(t_0)\| \cdot e^{\int_{t_0}^t ||A(s)||_{op} ds}$.
   d. Obtain a similar bound on $\|x(t)\|$ for any solution of $\underline{x}' = A(t)\underline{x} + \underline{b}(t)$.

2. **a.** Suppose a solution $x(t)$ of equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} \cdots + a_0(t)x = 0$, with $a_j(t) \in C^0$ has infinitely many zeros on a compact interval. Prove: $x(t) = 0$ on this interval.
   Can the compactness be weakened to boundedness here?
   b. Prove: the function $sin(t^p)$, $p \in \mathbb{N}$, cannot be a solution of equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = 0$ with C^0-coefficients, for $n < p$.
   c. Prove: the function $e^{-\frac{1}{t^2}}$, extended to $(-1,1)$, cannot be a solution of equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = 0$ with C^0-coefficients, for any n.

3. Let $\mathbb{X}(t)$ be a fundamental matrix of $\underline{x}' = A(t)\underline{x}$ or $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = 0$. Prove:
   a. There exists a fundamental matrix $\mathbb{X}(t)$ satisfying $\mathbb{X}(t_0) = \mathbb{1}$.
   b. Any two fundamental matrices are related by $\tilde{\mathbb{X}}(t) = \mathbb{X}(t)\cdot U$ for $U \in GL(n,\mathbb{R})$ a constant matrix.
   c. $\mathbb{X}(t)$ is non-degenerate for each t. In the case $\underline{x}' = A(t)\underline{x}$ it satisfies: $\mathbb{X}'(t) = A(t)\cdot\mathbb{X}(t)$.
   d. The (unique) solution for the initial condition $\underline{x}(t_0) = \underline{x}_0$ is: $\mathbb{X}(t)\cdot\mathbb{X}^{-1}(t_0)\cdot\underline{x}_0$.
   e. Let $\mathbb{X}(t), \tilde{\mathbb{X}}(t)$ be two fundamental matrices of the system $\underline{x}' = A(t)\underline{x}$. Suppose $\mathbb{X}(t_1) = \mathbb{1}$ and $\tilde{\mathbb{X}}(t_2) = \mathbb{1}$. Prove: $\mathbb{X}(t_2)\cdot\tilde{\mathbb{X}}(t_1) = \mathbb{1}$.
   f. Let $\mathbb{X}_A(t)$ be a fundamental matrix for $\underline{x}' = A(t)\cdot\underline{x}$, let $\mathbb{X}_B(t)$ be a fundamental matrix for $\underline{x}' = B(t)\cdot\underline{x}$. Prove: if $\mathbb{X}_A(t)B(t) = B(t)\mathbb{X}_A(t)$ then $\mathbb{X}_A(t)\mathbb{X}_B(t)$ is a fundamental matrix of $\underline{x}' = (A(t)+B(t))\underline{x}$.

4. **a.** Given two functions $x_1(t), x_2(t) \in C^1(a,b)$ (not necessarily solutions of some ODE), suppose $W(x_1(t), x_2(t)) = 0$ on (a,b). Does this imply the $\mathbb{R}$-linear dependence of $x_1(t)$, $x_2(t)$?
   (Hint at the end of page)
   b. Prove: if $W(x_1(t),\dots,x_n(t)) = 0$ on (a,b) for some analytic functions, then these functions are $\mathbb{R}$-linearly dependent on (a,b).
   c. Verify: the functions $sin(t^2)$, $cos(t^2)$ are (linearly independent) solutions of $tx'' - x' + 4t^3x = 0$, but the Wronskian of these functions vanishes at a point. Any contradiction to **4.b**?
   d. Prove: if $\lim_{t\to\infty}\int^t trace[A(s)]ds = \infty$, then at least one solution of $\underline{x}' = A(t)\underline{x}$ is unbounded.
   Show by an example that the conclusion "$\|\underline{x}(t)\| \to \infty$ for at least one solutions" fails.
   e. Prove: the rescaling $\underline{x} \to e^{\pm\int^t \frac{trace[A(s)]}{n}ds}\underline{x}$ transforms $\underline{x}' = A(t)\cdot\underline{x}$ into a system $\underline{x}' = \tilde{A}(t)\cdot\underline{x}$ with $trace[\tilde{A}(t)] = 0$. (Should one choose here $+$ or $-$?)
   f. Prove: the rescaling $x \to e^{\pm\int^t \frac{a_{n-1}(s)}{n}ds}x$ transforms the equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = 0$ into an equation with $\tilde{a}_{n-1}(t) = 0$. (Should one choose here $+$ or $-$?)

Hint: x^2, $x\cdot|x|$.