Ordinary differential equations for Math

(201.1.0061. Spring 2024. Dmitry Kerner)

Homework 8. Submission date: 2.07.2024

Questions to submit: 1.b. 1.e. 2. 3.b. 3.d. 4.a. 4.b. 4.c.

Either typed or in readable handwriting and scanned in readable resolution.

1. Equations of type $t^n x^{(n)} + a_{n-1}t^{n-1}x^{(n-1)} + \cdots + a_1 t x' + a_0 x = 0$ (here $a_\bullet$ are constants) are called Euler-Cauchy equations. They are used e.g. in physics and in finance.

a. One approach to solve is by time rescaling. Prove: the substitution $t = e^\tau$ transforms the Euler-Cauchy equation into a linear ODE with constant coefficients.

b. Write the general solution for the ODE $t^2x'' + tx' + a_0 x = 0$.

c. In the general case prove: the characteristic polynomial of the obtained ODE with constant coefficient is $L(\lambda) := \lambda(\lambda-1)\cdots(\lambda-n+1) + a_{n-1}\lambda(\lambda-1)\cdots(\lambda-n+2) + \cdots + a_0$.

d. Prove: if one presents $L(\lambda) = \lambda^n + b_{n-1}\lambda^{n-1} + \cdots + b_0$ then the initial equation can be presented as $(t\frac{d}{dt})^n x + b_{b-1}(t\frac{d}{dt})^{n-1}x + \cdots + b_0 x = 0$, for some constants $b_\bullet$.

e. Conclude: for $t \neq 0$ the space of solutions of a Euler-Cauchy equation is spanned by the functions of type $\{ln(t)^{k_j} \cdot t^{\lambda_j}\}$, with $k_j \in \mathbb{N}$. Here the function t^λ for $\lambda \in \mathbb{C}$ is defined by $t^\lambda := e^{\lambda \cdot ln(t)} = |t|^{Re(\lambda)}\big(cos(Im(\lambda)ln(t)) + i \cdot sin(Im(\lambda)ln(t))\big)$.

2. (A bound on the speed of separation of solutions) Let $x_1(t), x_2(t)$ be solutions of the ODE $x^{(n)} = f(t, x, \dots, x^{(n-1)})$. Suppose $|f(t,\underline{y}) - f(t,\underline{\tilde{y}})| \leq g(t) \cdot |\underline{y} - \underline{\tilde{y}}|$ for a function $g(t) \in C^0$.
Prove: $|x_1(t) - x_2(t)| \leq e^{\int_{t_o}^t g(s)ds} \cdot \sqrt{\sum_{j=0}^{n-1} |x_1^{(j)}(t_o) - x_2^{(j)}(t_o)|^2}$.
Hint: use the Grönwall-Bellman inequality for the system $\underline{x}' = \underline{f}(t, \underline{x})$.

3. a. (Variation of constants) Suppose $x(t)$ is a solution of $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = 0$. Then one looks for the solution in the form $x(t) \cdot y$ to pass to an equation $y^{(n-1)} + \tilde{a}_{n-2}(t)y^{(n-2)} + \cdots + \tilde{a}_0(t)y = 0$. Let $y_1(t), \dots, y_{n-1}(t)$ be its independent solutions. Prove: the functions $x(t), x(t) \cdot \int^t y_1(s)ds, \dots, x(t) \cdot \int^t y_{n-1}(s)ds$ form a basis for the space of solutions of the initial equation.

b. Find the general solution of $tx'' - (t+n)x' + nx = 0$, for $n \in \mathbb{N}$, given a solution e^t.

c. Find the general solution of $(t^2-1)x'' + 4tx' + 2x = 6t$, given the particular solutions $x_1(t) = t$, $x_2(t) = \frac{t^2+t+1}{t+1}$.

d. (Factorizing diff.operators) Take an operator $D_n := \frac{d^n}{dt^n} + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_0(t)$. Let $y(t) \neq 0$ be a solution of $D_n x = 0$. Prove: $D_n = D_{n-1} \circ [\frac{d}{dt} - \frac{y'(t)}{t(t)}]$, where D_{n-1} is a differential operator of order$=n-1$. (We did this in the class.)

4. a. Give an example of equation $\underline{x}' = A(t) \cdot \underline{x}$ for which $e^{\int_{t_0}^t A(s)ds} \cdot \underline{x}_0$ is not a solution.

b. Suppose the matrices $A(t)$ and $\int_{t_0}^t A(s)ds$ commute for each $t \in (a,b)$.
Prove: the (unique) solution of $\underline{x}' = A(t) \cdot \underline{x}$, $\underline{x}(t_0) = \underline{x}_0$ is given by $\underline{x}(t) = e^{\int_{t_0}^t A(s)ds}\underline{x}_0$.

c. Let $\{A_j\}$ be some constant pairwise commuting matrices. Let $\{g_j(t)\}$ be $C^0(a,b)$.
Solve the system $\underline{x}' = (\sum g_j(t)A_j)\underline{x}$, $\underline{x}(t_0) = \underline{x}_0$.

d. Prove that the assumption in **b.** implies: the matrices $A(t)'$ and $\int_{t_0}^t A(s)ds$ commute for each $t \in (a,b)$.