Ordinary differential equations for Math

(201.1.0061. Spring 2024. Dmitry Kerner)

Homework 7. Submission date: 25.06.2024

Questions to submit: 1.a. 1.b. 1.d. 2.b. 2.c. 3.a. 3.b. 3.d. 4.ii. 4.v.

Either typed or in readable handwriting and scanned in readable resolution.

1. Consider the equation $D_n(x) = g(t)$, where $D_n = \frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_0$, with $a_i \in \mathbb{R}$.

a. Write the general solution of $x^{(4)} + 4x = \sum b_j e^{\omega_j t}$, here $\omega_j \in \mathbb{C}$, with $\omega_j = 0$ or $\omega_j^3 = -4$.

b. Suppose $\mu \in \mathbb{C}$ is not a root of the characteristic polynomial of D_n. Prove: the equation $D_n(x) = t^k \cdot e^{\mu t}$, with $k \in \mathbb{N}$, has a solution of the form $g_k(t) \cdot e^{\mu t}$ for a polynomial $g_k(t) \in \mathbb{C}[t]_{\le k}$ of degree k. (Hint. It is enough to show: the operator $D_n \circlearrowright \mathbb{C}[t]_{\le k} \cdot e^{\mu t}$ acts surjectively. And for this it is enough to verify: D_n acts injectively.)

c. Suppose $\mu \in \mathbb{C}$ is a root of the characteristic polynomial of D_n, of multiplicity p. Prove: the equation $D_n(x) = t^k \cdot e^{\mu t}$ has a solution of the form $t^p \cdot g(t) \cdot e^{\mu t}$ for a polynomial $g_k(t) \in \mathbb{C}[t]_{\le k}$ of degree k. Wiki: "Resonance".

d. Write the general solution of $x^{(4)} + 4x = b \cdot t \cdot e^{\mu t}$. (Here $b, \mu \neq 0$ are parameters.)

e. Consider the equation $D_n x = p(t) \cdot e^{\mu t}$, here $p(t) \in \mathbb{C}[t]$. What is the necessary and sufficient condition to ensure that the equation has a periodic solution? A bounded solution?

2. Consider the system $\underline{x}' = \underline{f}(t, \underline{x})$, with $\underline{f} \in C^r((a,b) \times \mathbb{R}^n)$. We have proved: If $|\underline{x} \cdot \underline{f}(t, \underline{x})| \le g(t) \cdot (1 + ||\underline{x}||^2)$ then any solution extends to $C^{r+1}(a,b)$.

a. Instead of the condition $|\underline{x} \cdot \underline{f}(t, \underline{x})| \le g(t) \cdot (1 + ||\underline{x}||^2)$ one could impose $|\underline{x} \cdot \underline{f}(t, \underline{x})| \le g_0(t) + g_1(t) \cdot ||\underline{x}|| + g_2(t) \cdot ||\underline{x}||^2$, for some g_0, g_1, g_2. Prove: this condition is not weaker. Namely, this condition holds for some g_0, g_1, g_2 iff the previous condition holds for some g.

b. Suppose the bound $|\underline{x} \cdot \underline{f}(t, \underline{x})| \le g(t) \cdot (1 + \phi(\|\underline{x}\|^2))$ holds for a function $g(t) \in C^1(a,b)$ and a function $\phi(y) \ge 0$ satisfying: $\int_0^\infty \frac{dy}{1+\phi(y)} = \infty$.

i. Prove: any solution extends to $C^{r+1}(a,b)$. (See the hint downstairs.)

ii. For which function ϕ do we get the criterion proved in the class? For which functions ϕ we get a stronger criterion?

c. Consider the equation $x^{(n)} = f(t, x, \dots, x^{(n-1)})$, where $f \in C^r((a,b) \times \mathbb{R}^n)$. Denote $\underline{y} = (y_0, \dots, y_{n-1})$. Suppose the bound $|y_{n-1} \cdot f(t, \underline{y})| \le g(t) \cdot (1 + |\underline{y}|^2)$ holds in $(a,b) \times \mathbb{R}^n$. Prove: any local solution extends to a global one, $x(t) \in C^{r+1}(a,b)$.

3. a. (A comparison test) Consider an ODE $x' = f(t, x)$, where $f \in C^0((a,b) \times \mathbb{R}^1)$ is locally Lipschitz in x. Suppose there exist functions $x_{min}(t), x_{max}(t) \in C^1(a,b)$ satsifying: $x'_{min}(t) \le f(t, x_{min}(t))$ and $x'_{max}(t) \ge f(t, x_{max}(t))$ for $t \in (a,b)$. Prove: any local solution with $x_{min}(t_o) < x(t_o) < x_{max}(t_o)$ extends to a global solution $x(t) \in C^1(t_o - \epsilon, b)$.

b. (Speed of separation of solutions) Consider the system $\underline{x}' = f(t, \underline{x})$ for $f \in C^0(\mathcal{U})$. Suppose $|(\underline{x} - \underline{y}) \cdot (\underline{f}(t, \underline{x}) - \underline{f}(t, \underline{y}))| \le g(t) \cdot e^{||\underline{x} - \underline{y}||^2}$ in $\mathcal{U}$. Prove: any solutions $\underline{x}(t)$, $\underline{y}(t) \in C^1(a,b)$ satisfy $\|\underline{x}(t) - \underline{y}(t)\|^2 \le \|\underline{x}(t_0) - \underline{y}(t_0)\|^2 - ln[1 - e^{\|\underline{x}(t_0) - \underline{y}(t_0)\|^2} \cdot \int_{t_0}^t 2g(s)ds]$.
(We assume here: $e^{\|\underline{x}(t_0) - \underline{y}(t_0)\|^2} \cdot \int_{t_0}^t g(s)ds < 1$.)

c. Write the general solution of the system $x' = \frac{x}{1+t^2} + y \cdot sin(2t)$, $y' = y \cdot cos(t)$.

d. Write the general solution of the equation $(\frac{d}{dt} - a_1(t)) \circ (\frac{d}{dt} - a_2(t))x = 0$, $a_1(t), a_2(t) \in C^1(a,b)$.

4. Prove: **i.** $det(e^A) = e^{trace(A)}$. **ii.** $\det[\mathbb{I} + \epsilon A] = 1 + \epsilon \cdot trace(A) + O(\epsilon^2)$. **iii.** $\|e^A\|_{op} \le e^{\|A\|_{op}}$.
iv. $e^A = \lim\limits_{k \to \infty} (\mathbb{I} + \frac{A}{k})^k$. **v.** If $A(t) \in GL(n, C^1(a,b))$ then $(A(t)^{-1})' = -A(t)^{-1} A'(t) A(t)^{-1}$.

Hint: denote $y = \|x\|^2$, then one has $y' \le g(t) \cdot (1 + \phi(y))$.