Ordinary differential equations for Math

(201.1.0061. Spring 2024. Dmitry Kerner)

Homework 6. Submission date: 18.06.2024

Questions to submit: 1.a. 2.b. 2.c. 2.d. 3.a. 3.b. 5.c.

Either typed or in readable handwriting and scanned in readable resolution.

1. Consider the equation $\underline{x}' = A \cdot \underline{x} + e^{\mu t} t^k \cdot \underline{b}$, here $\underline{b}$ is a constant vector. Prove:

a. If μ is not an eigenvalue of A then there exists a solution of the form $e^{\mu t} \cdot \underline{g}(t)$, where the entries of $\underline{g}$ are polynomials of degree$\leq k$. (Hint: the presentation $\underline{g}(t) = (\mu \mathbb{1} - A)^{-1} \cdot [t^k \cdot \underline{b} - \underline{g}'(t)]$ determines the terms of order k in $\underline{g}(t)$. How to determine the terms of order$< k$?)

b. If μ is an eigenvalue of A then there exists a solution of the form $e^{\mu t} \underline{g}(t)$, where the entries of $\underline{g}$ are polynomials of degree$\leq k + Jord.Size.(A)$, here $Jord.Size.(A)$ is the size of the maximal Jordan cell of A.

2. a. Write down the general (real) solutions of the equations

i. $x^{(n+2)} \pm b^2 x^{(n)} = 0$. ii. $x^{(4)} + 2x^{(3)} + 2x^{(2)} = 0$.

b. Find the condition on a, b to ensure: 0 is a stable equilibrium point of $x^{(2)} + a \cdot x' + b \cdot x = 0$.

c. Find an equation $x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_0 x = 0$ of minimal possible order whose general solution contains the functions $sin(2t) \cdot e^t$, $cos(2t) \cdot e^{2t}$, t^2. Explain why is this order minimal. Prove: the coefficients $\{a_j\}$ are uniquely determined.

d. Fix a finite set of pairwise distinct complex numbers $\{\lambda_k\}$ and some natural numbers $\{p_k\}$. Prove: the functions $\{t^j \cdot e^{\lambda_k t}\}_{\substack{1 \leq k \leq n \\ 0 \leq j \leq p_k}}$ are $\mathbb{C}$-linearly independent.

(Can you do this in several different ways?)

e. Prove the $\mathbb{R}$-version of part d., about the $\mathbb{R}$-linear independence of the functions $\{t^j e^{a_k t} \cdot cos(b_k \cdot t)\}$, $\{t^j e^{a_k t} \cdot sin(c_k \cdot t)\}$.

3. Consider the ODE $x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_0 x = 0$ and the equation $\lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_0 = 0$.

a. (Dis)prove: if all the λ-roots are imaginary, simple then every solution of this ODE is periodic.

b. (Dis)prove: if all the λ-roots are real-positive then every solution of this ODE is monotonic.

c. Prove: for any tuple $(a_{n-1}, \ldots, a_1)$ there exists a finite subset $S \subset \mathbb{R}$ such that for $a_0 \in \mathbb{R} \setminus S$ the space of solutions is spanned by exponents. (Hint: when does a polynomial have a multiple root?)

This phenomenon causes statements like "The solutions $t^j e^{\lambda t}$ never appear in laboratory".

4. Prove: any (finite) system of ODE's, $\underline{f}(t, \underline{x}, \underline{x}', \ldots, \underline{x}^{(k+1)}) = \underline{0}$, is equivalent to a system of 1'st order ODE's, $\underline{F}(t, \underline{y}, \underline{y}') = \underline{0}$. (Namely, every solution of $\underline{f}$ leads to a solution of $\underline{F}$ and vice versa.)

Moreover, if the initial system $\underline{f}(\ldots)$ is in the normal form/autonomic/linear/polynomial then the resulting system $\underline{F}(\ldots)$ is of this type as well.

5. a. Prove: if the function $g(\underline{x}) > 0$ is continuous then the systems $\underline{x}' = \underline{f}(\underline{x})$ and $\underline{x}' = g(\underline{x}) \cdot \underline{f}(\underline{x})$ have the same phase potraits. (What happens for $g < 0$?)

b. Prove: the phase curves of the system $\underline{x}' = \underline{f}(\underline{x})$, $\underline{f} \in C^1(\mathcal{U})$ for $\mathcal{U} \subseteq \mathbb{R}^n$, cover the whole $\mathcal{U}$ and either coincide or do not intersect.

c. Consider the system $x' = sin(x) \cdot (e^{y^2} + x^4)$, $y' = sin(cos(y)) \cdot (e^{x^2} + y^3)$.

i. Find the equilibria points.

ii. Prove: there exist infinity of phase curves that are parallel to $\hat{y}$-axis. Moreover, each of these curves is an open interval of length$< \pi$. (And the same for $\hat{x}$-axis.)

iii. Prove: any local solution extends (uniquely) to the global solution $x(t), y(t) \in C^\omega(\mathbb{R})$.