Ordinary differential equations for Math

(201.1.0061. Spring 2024. Dmitry Kerner)

Homework 5. Submission date: 11.06.2024

Questions to submit: 1. 2. 3.a. 3.b. 4.b. 4.c. 4.e.

Either typed or in readable handwriting and scanned in readable resolution.

1. Define the function $f(x)$ as $x^2 \cdot sin\frac{1}{x^2}$ for $x<0$, as $\sqrt{x}$ for $0<x<1$ and as $e^{-x^2}\cdot sin(e^{x^3})$ for $x>1$. (Dis)Prove:

a. f is locally Lipschitz at each point where it is defined.

b. f is Lipschitz on $(-\epsilon,\epsilon)\setminus\{0\}$.

c. f is Lipschitz on $(1-\epsilon,1+\epsilon)\setminus\{1\}$.

d. f is Lipschitz on $(-\infty,-1)$ and on $(1,\infty)$.

2. Let $A\in Mat_{2\times 2}(\mathbb{R})$, with eigenvalues $\lambda_\pm = a\pm i\cdot b$, $b\neq 0$. Take the corresponding eigenvectors $v_\pm$. Prove: in the basis of $\mathbb{R}^2$ composed of $Re[v_+], Im[v_+]$ the matrix becomes: $\begin{bmatrix} a & -b\\ b & a\end{bmatrix}$.

3. In the following cases (without solving the equations):

i. Identify the equilibria points. When are these points (un)stable nodes/saddles?

ii. For which λ are there (un)bounded/periodic solutions?

iii. For the cases a. and b. draw the phase portraits.

Now write down the general (real) solutions, and verify the previously obtained properties.

a. $x'=y,\ y'=\lambda\cdot x$. (Distinguish between the cases $\lambda>0$, $\lambda<0$.)

b. $x'=\lambda x+y,\ y'=\lambda y$.

c. $\underline{x}'=A\cdot\underline{x}$ for $A=\begin{bmatrix}1 & 1 & -1\\ 1 & 1 & -1\\ 2 & -1 & 0\end{bmatrix}$.

4. Consider the system of differential equations $\underline{x}'=A\cdot\underline{x}$, $A\in Mat_{n\times n}(\mathbb{R})$. Prove:

a. If $\underline{x}(t)$ is a solution then all its derivatives are solutions.

b. If $A=A^t$ then there are no (non-constant) periodic solutions.

c. If $A=-A^t$ then the space of solutions is spanned by periodic solutions. Does this imply that every solution is periodic?

d. If A is of odd size then there exists an unbounded solution.

e. What is the necessary and sufficient condition on A to ensure $\lim_{t\to\infty}\underline{x}(t)=\underline{0}$ for each solution?

f. If A is $\mathbb{R}$-diagonalizable and the eigenvalues have the same sign then $\underline{x}=0$ is a nodal point. (Attracting or repelling)

g. The solutions are analytic in the initial data, $x(t,t_0,x_0,A)\in C^\omega(\mathbb{R}_t\times\mathbb{R}_{t_0}\times\mathbb{R}^n_{\underline{x}_0}\times Mat_{n\times n}(\mathbb{R}))$.

h. The set of equilibrium points is a vector subspace of $\mathbb{R}^n$. (What is the dimension?)

5. Define the functions $Mat_{n\times n}(\mathbb{C})\overset{sin,cos}{\to} Mat_{n\times n}(\mathbb{C})$ via the Taylor expansion of sin, cos. Prove:

a. These series converge absolutely, the convergence is uniform on bounded subsets of $Mat_{n\times n}(\mathbb{C})$.

b. Prove: $e^{iA}=cos(A)+i\cdot sin(A)$. $cos(A)=\frac{e^{iA}+e^{-iA}}{2}$. $sin(A)=\frac{e^{iA}-e^{-iA}}{2i}$.

c. Prove: $sin^2(A)+cos^2(A)=\mathbb{1}$. If $AB=BA$ then $sin(A+B)=\cdots$, $cos(A+B)=\cdots$.

d. Compute $\frac{d}{dt}cos(At)$ and $\frac{d}{dt}sin(At)$. (Do this in two ways, as we did for $\frac{d}{dt}e^{At}$ in the lecture.)

6. Write the general solution of the system $\underline{x}'=\begin{bmatrix}1 & -1 & 1\\ 1 & 1 & -1\\ -1 & 2 & 3\end{bmatrix}\cdot\underline{x}+\begin{bmatrix}3e^t\\ 0\\ 3e^{-t}\end{bmatrix}$.