Ordinary differential equations for Math

(201.1.0061. Spring 2024. Dmitry Kerner)

Homework 4. Submission date: 03.06.2024

Questions to submit: 1.b. 1.d. 2.b. 2.c. 2.g. 3.a. 3.f.

Either typed or in readable handwriting and scanned in readable resolution.

1. a. Prove: the solutions of $x' = e^{x^2} - t$ have no local minima. ($\exists$ at least two different approaches.)

b. Prove: every local solution of $x' = sin^2(t) \cdot e^{t \cdot cos(x)}$ extends (uniquely) to $x(t) \in C^\omega(\mathbb{R})$, this global solution has infinite number of critical points, and all the critical points are flexes (i.e. neither maxima nor minima).

c. Prove: the local solution of $x' = \frac{(x-1)sin(t \cdot x)}{t^2+x^2+1}$, $x(0) = \frac{1}{2}$ extends (uniquely) to the global solution, $x(t) \in C^\omega(\mathbb{R})$. Moreover it satisfies: $0 < x(t) < 1$.

d. Prove: the IVP $x' = \sum_{m=1}^{\infty} \frac{sin(m \cdot x) \cdot cos(m \cdot t)}{m^{\sqrt{5}}}$, $x(t_0) = x_0$ admits the unique local solution for any $(t_0, x_0) \in \mathbb{R}^2$. Moreover, this solution extends (uniquely) to $x(t) \in C^\omega(\mathbb{R})$.

e. (Comparison test) Consider two equations $x' = f_i(t,x)$, $i = 1,2$, and their solutions $x_i(t)$, both defined on $[t_0, t_1)$. Suppose $x_1(t_0) \le x_2(t_0)$. Suppose the bound $f_1(t,x) \le f_2(t,x)$ holds in a neighborhood of the curve $\{(t, x_1(t))\} \subset \mathcal{U}$. Prove: $x_1(t) \le x_2(t)$ on $[t_0, t_1)$.

2. a. Prove: the functions $A = \sqrt{trace(\bar{A}^t \cdot A)}$, $||A||_{op} := \sup_{||v|| \neq 0} \frac{||Av||}{||v||}$ define norms on $Mat_{n \times n}(\mathbb{R})$ and $Mat_{n \times n}(\mathbb{C})$. Moreover, $||A \cdot B||_{op} \le ||A||_{op} \cdot ||B||_{op}$.

b. Disprove: i. $\|A\|_{op}$ equals the largest eigenvalue of A.
ii. The norm $\| * \|_{op}$ is conjugation-invariant (i.e. $\|A\|_{op} = \|UAU^{-1}\|_{op}$.)

c. Prove: the norms $|| * ||$, $|| * ||_{op}$ are equivalent. Prove: these normed spaces are complete.

d. Review questions 7,8 of homework.0

e. Prove: if $e^{At} e^{Bt} = e^{(A+B)t}$ holds for all $t \in (-\epsilon, \epsilon)$ then $AB = BA$.

f. Prove: if $e^A = \mathbb{1}$ then A is $\mathbb{C}$-diagonalizable. What are the possible eigenvalues?

g. Prove: if $A \in Mat_{n \times n}(\mathbb{R})$ is $\mathbb{C}$-diagonalizable then A is $\mathbb{R}$-conjugate to a (real) block-diagonal matrix, with blocks of size ≤ 2. Moreover, each 2×2 block can be brought to the form: $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.
(Hint: the non-real eigenvectors of A come in conjugate pairs.
But we want a real basis, ...)

3. a. Take the unit ball $Ball_1(\mathbb{O})_{op} := \{A | \ ||A||_{op} < 1\} \subset Mat_{n \times n}(\mathbb{C})$.
Prove: if $A \in Ball_1(\mathbb{O})_{op}$ then $\mathbb{1} + A \in GL(n, \mathbb{C})$.

b. For a marix $A \in Ball_1(\mathbb{O})_{op}$ define $ln(\mathbb{1} + A) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1} A^k}{k}$. Prove: the series converges absolutely, and the convergence is uniform on compact subsets of $Ball_1(\mathbb{O})_{op}$.

c. Prove: $exp(ln(\mathbb{1} + A)) = \mathbb{1} + A = ln(e^{(\mathbb{1}+A)})$ for every $A \in Ball_1(\mathbb{O})$. (No long computations are needed here.)

d. Take the subset $Mat_{2 \times 2}(\mathbb{C}) \supset \Sigma := \{A | \ e^A = \mathbb{1}, \ det(t\mathbb{1} - A) = t^2 + 4\pi^2\}$. Identifying $Mat_{2 \times 2}(\mathbb{C}) \cong \mathbb{C}^4$ prove: Σ is defined by one linear and one quadratic equation. (Hint: observe that any matrix in Σ must be diagonalizable.)
Conclude: Σ contains a two-parametric family of matrices. (Therefore, while the map exp is locally invertible near 0, it is highly non-injective globally.)

e. Prove: if $AB = BA$ and $A, B, A + B + AB \in Ball_1(\mathbb{O})_{op}$ then $ln\big[(\mathbb{1} + A)(\mathbb{1} + B)\big] = ln(\mathbb{1} + A) + ln(\mathbb{1} + B)$. In particular, $ln[(\mathbb{1} + A)^k] = k \cdot ln(\mathbb{1} + A)$ for every $k \in \mathbb{Z}$.

f. Compute $\frac{d}{dt} ln(\mathbb{1} + At)$. (Do this in two ways, as we did for $\frac{d}{dt} e^{At}$ in the lecture.)