Ordinary differential equations for Math

(201.1.0061. Spring 2023. Dmitry Kerner)

Homework 2. Submission date: 19.05.2024

Questions to submit (by mail): 1.ii-iv. 2a. 2c. 2f. 3c. 4c. 4d.
Either typed or in readable handwriting and in readable resolution.

1. In the following cases draw the integral curves. When possible, write down the general solution. In each case extend the solution to the maximal interval. Does a solution explode at a finite time? Which solutions are monotonic/bounded/periodic? When you have a constant solution, is this an (un)stable equilibrium point? For which initital conditions does the solution of the corresponding IVP exist/is unique? Check the C^r-properties for $1 \leq r \leq \infty, \omega$.

i. $x' = \frac{t+|t|}{x+|x|}$ ii. $x' = e^x \cdot sin(x)$ iii. $x' = \frac{1}{\sqrt[3]{sin(x)}}$ iv. $x' = e^{\frac{1}{x}}$

v. $x' = \frac{sin(x)}{sin(t)}$ vi. $x' = \sqrt{|x|} \cdot \sqrt[3]{sin(x)}$

2. Consider the equation $x' = f(x)$, $f \in C^0(a,b)$, $(a,b) \subseteq \mathbb{R}^1$, and its solution $x(t)$.

a. Show that there is a continuum of global solutions, $x(t) \in C^1(\mathbb{R})$, to the IVP $x' = \sqrt{|x|}$, $x(t_0) = -1$. Check that they are all C^1, but none of them is C^2.
Find the largest interval around t_0 on which the solution to this IVP is unique.

b. Give an example of non-differentiable f such that all the solutions of $x' = f(x)$ are real-analytic. (Hint at the end of page)

c. Suppose $f(\pi) = 0$ and $\int_\pi^{\pi+\epsilon} \frac{dx}{f(x)} = \infty$. Find all the solutions that satisfy $x(\sqrt{2}) = \pi$.

d. Suppose $f \in C^0(-\epsilon, \epsilon)$ and $f(x) = O(x \cdot ln|x|)$. (Does this condition imply that f is locally Lipshitz?) Prove: the solution of the IVP $x' = f(x)$, $x(0) = 0$ is locally unique.

e. Prove: any solution $x(t)$ is (weakly) monotonic.

f. Suppose f is locally Lipschitz at each point, and the set of zeros of f has a and b in its closure. Prove: every local solution extends to the unique global solution, $x(t) \in C^1(a,b)$.

3. Suppose $x(t)$ is a solution of the equation $x' = f(t,x)$, here $f \in C^0(\mathcal{U})$ for an open $\mathcal{U} \subseteq \mathbb{R}^2$.

a. Solve the equation $x' = -x + g(t) + g'(t)$, here $g \in C^1(a,b)$.

b. Suppose $f(t,-x) = -f(t,x)$ for all $(t,x) \in \mathbb{R}^2$. Prove: $-x(t)$ is a solution as well.
In the case of uniqueness conclude: either $x(t) \equiv 0$ or $x(t)$ has no zeros.

c. Suppose $f \in C^1(\mathcal{U})$ and $x(t), y(t) \in C^1(a,b)$ are two solutions satisfying $x(t_0) < y(t_0)$. Prove: $x(t) < y(t)$ for any $t \in (a,b)$. Is the assumption $f \in C^1(\mathcal{U})$ necessary here?

4. Consider the equation $x' \cdot sin(t) = x \cdot r \cdot cos(t)$ for $r \in \mathbb{N}$, with $r \geq 2$.

a. Write down the general local solution near $t_0 \in \mathbb{R} \setminus \pi\mathbb{Z}$. Which initial conditions, (t_0, x_0), are allowed?

b. Prove: every local solution at $t_0 \in \mathbb{R} \setminus \pi\mathbb{Z}$ extends (uniquely) to a global solution $x(t) \in C^\infty(\mathbb{R})$.

c. Prove: for any number sequence $\{x_k\}_k$ there exists a C^{r-1}-solution satisfying: $\{x(\frac{\pi}{13} + \pi k) = x_k\}_k$. (Any contradiction to the uniqueness theorem?)

d. Prove: the set of C^{r-1}-solutions is a vector space of uncountable dimension.

e. Prove: any (global) C^r-solution is in fact a C^∞-solution.
Deduce: the set of C^r-solutions is a vector space of dimension 1.

5. As a motivation for systems of ODE's (i.e. "Vector fields integration"), read wiki "Eversion of spheres" and watch youtube "Turning spheres inside-out".

Hint: $x' = |x|$.