Ordinary differential equations for Math

(201.1.0061. Spring 2024. Dmitry Kerner)

Homework 11. Not for submission.

1. Let the operator $D_n x := x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x$ have coefficients with period T.

a. Suppose two solutions, $x(t), \tilde{x}(t)$ of $D_n x = 0$ satisfy: $\tilde{x}^{(i)}(t_o+T) = x^{(i)}(t_o)$ for all $i = 0, \dots, n-1$. Prove: $x(t) = \tilde{x}(t+T)$.

b. Prove: for any basis of solutions $\underline{x}(t) := (x_1(t), \dots, x_n(t))$ one can present $\underline{x}(t) = \underline{y}(t) \cdot e^{Rt}$, where $\underline{y}$ is a row of periodic functions, while $R \in Mat_{n\times n}(\mathbb{C})$.

c. Let $x_1(t), x_2(t)$ be solutions of the equation $D_2 x = 0$. Take the fundamental matrix $\mathbb{X}(t)$ satisfying $\mathbb{X}(0) = 1\!\!1$. Let e^{RT} be the corresponding monodromy matrix. Prove: its eigenvalues are the roots of the polynomial $z^2 - (x_1(T) + x_2'(T))z + e^{-\int_0^T a(t)dt} = 0$.

2. a. Verify: $e^{\underline{a}\cdot\nabla} f(\underline{x}) = f(\underline{x} + \underline{a})$, here $\nabla = (\partial_{x_1}, \dots, \partial_{x_n})$.

b. Why is the expansion $Taylor_{t_0}[f(t)]$ presented as $e^{(t_1-t_0)\frac{d}{dt}} f|_{t=t_0}$ and not just $e^{(t-t_0)\frac{d}{dt}} f|_{t=t_0}$?

c. Find the full Taylor series of the solution of $x'' + t^p \cdot x = 0, \quad p \in \mathbb{N}, \quad x(0) = 0,\ x'(0) = 1$. (A remark: while we have the general formula for $Taylor_0[x(t)]$, it is often worth to start from the expansion $\sum c_j t^j$, and to identify the coefficients.)

d. Let $x(t)$ be the (local) solution of $x' = e^{tx^2}$, $x(0) = 0$. Find $Taylor_0[x(t)]$ up to order 6.

e. Write down the Taylor expansion (at t_o) of a solution $\underline{x}(t)$ of $\underline{x}' = A \cdot \underline{x}$. Verify that you get $\underline{x}(t) = e^{A(t-t_0)} \cdot \underline{x}_0$.

f. Let $\underline{x}(t)$ be the solution of $\underline{x}' = A(t) \cdot \underline{x}$, $\underline{x}(t_0) = \underline{x}_0$. Compute the Taylor expanion of $\underline{x}(t)$ up to order 3. (Attention, the matrices $A(t)$, $A'(t)$ do not necessarily commute.)