Ordinary differential equations for Math

(201.1.0061. Spring 2024. Dmitry Kerner)

Homework 10. Submission date: 16.07.2024

Questions to submit: 1.a. 1.c.ii. 1.d. 2.a. 2.c. 3.b. 4.b. 4.c.

Either typed or in readable handwriting and scanned in readable resolution.

Below we denote: $D_n x := x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x$.

1. a. Find a system $\underline{x}' = A(t)\cdot\underline{x}$ whose solutions are $\underline{x}_1(t) = [e^t cos(t), e^t sin(t)]$ and $\underline{x}_2(t) = [-sin(t), cos(t)]$.

b. Find a linear homogeneous ODE whose space of solutions is spanned by $sin\frac{1}{t}$, $cos\frac{1}{t}$.

c. Suppose the space of solutions of $D_n x=0$ is spanned by the functions $x_1(t)\dots x_n(t)$. Prove:

i. If all $\{x_i(t)\}$ are T-periodic then so are the coefficients $\{a_i(t)\}$.

ii. $a_{n-1}(t) = \frac{W[x_1,\dots,x_n]'}{W[x_1,\dots,x_n]}$ and $a_0(t) = (-1)^n \cdot \frac{W[x_1',\dots,x_n']}{W[x_1,\dots,x_n]}$.

d. Suppose $e^t, sin(t), t^{17}$ are solutions of a linear non-homogeneous equation of 2'nd order. Write the general solution. Find the solution satisfying: $x(0) = a$, $x'(0) = b$.

2. a. Let $\underline{x}(t)=x_1(t),\dots,x_n(t)$ be a basis of the space of solutions of the equation $D_n x=0$. Verify: $\int_{t_o}^t \frac{W(s,t)}{W(s)} b(s)ds$ is a particular solution of $D_n x=b(t)$. Here $W(s):=W(\underline{x}(s))$ is the Wronskian, while $W(s,t):=det[\underline{x}(s)//\underline{x}'(s)//\dots//\underline{x}^{(n-2)}(s)//\underline{x}(t)]$ is the 'mixed Wronskian'.

b. Verify: for $n = 2$ the particular solution of the equation $x^{(2)} + a_1(t)x' + a_0(t)x = b(t)$ is: $\int_{t_o}^t det\begin{bmatrix}\underline{x}(s)\\ \underline{x}(t)\end{bmatrix} \cdot \frac{b(s)}{W(s)} ds$.

c. Find the general solution of Bessel's equation $t^2x'' + tx' + (t^2 - \frac{1}{4})x = 3t^{\frac{3}{2}} sin(t)$, $t > 0$, given a particular solution of the corresponding homogeneous equation, $x(t) = \frac{sin(t)}{\sqrt{t}}$.

d. Take an equation $D_n x = 0$ with constant coefficients. Suppose all the roots of the characteristic polynomial of D_n are distinct, $\{\lambda_\bullet\}$. Prove: a solution of $D_n x = b(t)$ is $\sum_{k=1}^n \int_{t_o}^t \frac{e^{\lambda_k(t-s)}}{\prod_{i\neq k}(\lambda_k-\lambda_i)} b(s)ds$. (hint: Vandermonde)

3. A set $x_1(t),\dots,x_n(t)$ of solutions of $D_n x = 0$ is called 'the standard basis at t_o' if the corresponding fundamental matrix $\mathbb{X}(t)$ satisfies $\mathbb{X}(t_o) = 1\!\!1$.

a. Prove: the standard basis exists, is unique, and satisfies: $x_j(t) = \frac{(t-t_o)^{j-1}}{(j-1)!} + o((t-t_o)^n)$.

b. Let $x_1(t),\dots,x_n(t)$ be the standard basis of $D_n x = 0$, with constant coefficients. Prove: $\int_{t_o}^t x_n(t-s+t_o)b(s)ds$ is a solution of $D_n x = b(t)$.

4. a. Compute the monodromy matrix, e^{RT}, for the system $\underline{x}' = A\cdot\underline{x}$, with constant A.

b. Recall, the fundamental matrix $\mathbb{X}(t)$ of the (periodic) system $\underline{x}' = A(t)\cdot\underline{x}$ is non-unique. Prove: the monodromy matrix e^{RT} is well defined up to conjugation and does not depend on the choice of t_o.

c. Prove: λ is an eigenvalue of the monodromy matrix e^{RT} iff there exists a solution $\underline{x}(t)$ satisfying $\underline{x}(t+T) = \lambda\cdot\underline{x}(t)$.

d. Let $\underline{x}' = A(t)\underline{x}$ where $A(t)$ is periodic with period T. Take the fundamental matrix $\mathbb{X}(t)$ satisfying: $\mathbb{X}(0) = 1\!\!1$. Prove: $\mathbb{X}(d\cdot T) = \mathbb{X}(T)^d$.