Ordinary Differential Equations for Math

(201.1.0061. Spring 2024. Dmitry Kerner)

Homework 1. Submission date: 12.05.2024

Questions to submit (by mail): 1 b. 1 d. 2 c. 3 a. 3 b. 4 b.
(Either typed or in readable handwriting and in readable resolution.)

1. a. The earth is defined by the condition $\{\underline{x}|\ x_n \leq 0\} \subset \mathbb{R}^n$. Suppose every freely falling particle (with $x_n > 0$) experiences the constant acceleration, $\vec{a} = -g \cdot \hat{x}_n$. (The law of Gallileo, $g = 9.8m/sec^2$.) A frog jumps at the moment t_0 from the point $0 \in \mathbb{R}^n$, with the initial velocity $\vec{v}$ (here $v_n > 0$). Find the trajectory $\underline{x}(t)$, the total time lapse of the jump (i.e. $t_1 - t_0$) and the total displacement (i.e. $\underline{x}(t_1) - \underline{x}(t_0)$).

b. A particle moves on the standard sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Prove: $\vec{v}(t) \perp \underline{x}(t)$ for each t.
Suppose $n = 2$ and $||\vec{v}|| = const$. Prove: $\vec{a}(t)||\underline{x}(t)$ for each t. (wiki: centrifugal force)

c. How does a differentiable coordinate change, $\mathbb{R}^n \xrightarrow{\phi} \mathbb{R}^n$, $\underline{x} \to \phi(\underline{x})$, affect $\vec{v}$, $\vec{a}$?

d. Take the second law of Newton in dimension 1, i.e. $m \cdot \vec{a} = \vec{F}(x)$, for a continuous vector field $\mathbb{R}^1 \xrightarrow{\vec{F}} \mathbb{R}^1$. Multiply this 2'nd order ODE by $\vec{v}$ and integrate. You get a 1'st order ODE, called the "Energy conservation law". Here $\frac{m \cdot \vec{v}^2}{2}$ is the kinetic energy, while $-\int F(x)dx$ is the potential energy.
(To extend this to $n > 1$ one needs some ingredients from Geometric Calculus 2.)

2. Consider the equation $x' + f(t) \cdot x = g(t)$, for $f, g \in C^0(a, b)$.

a. We have obtained the complete solution in the class. Rederive the general formula. What is the dimensionality of the family of solutions? (It seems to depend on two constants?)

b. Impose the initial condition $x(t_0) = x_0$. Prove: the solution is unique and is globally defined.

c. For which pairs $(t_0, x_0) \in \mathbb{R}^2$ does the IVP $t \cdot x' + x = sin(t)$, $x(t_0) = x_0$ posses a solution? When is it unique? Draw the field of directions and the integral curves.

3. The hare runs over $\mathbb{R}^2_{x,y}$ with constant velocity $(0, v_h)$, starting at $t = 0$ from the origin. The dog starts to run from the point $(x_0, 0)$ at $t = 0$ with velocity $\vec{v}$ satisfying: $||\vec{v}| = v_d$, and at each moment $\vec{v}$ is directed towards the hare. Present the trajectory of the dog as $y = y(x)$.

a. Prove: the function $y(x)$ satisfies the differential equation $x \cdot y'' = \frac{v_h}{v_d} \cdot \sqrt{1 + (y')^2}$.
Hint: write all the equations on the functions $x(t), y(t)$. You will get, e.g. $y - x\frac{\dot{y}}{\dot{x}} = v_h t$ and $\|(\dot{x}, \dot{y})\| = v_d$. Observe that the functions $x(t), y(t)$ are invertible. Eliminate t from this system of equations, and pass from $\dot{y}$ to $y' = \frac{dy(x)}{dx}$.

b. Substitute $z = y'$ to convert this 2'nd order equation to a first order ODE. Solve it.

c. Give the necessary and sufficient condition (on v_h, v_d) for the dog to catch the hare in finite time. (Verify that the natural guess means the converges of some improper integral.)

d. Wiki: *The curve of pursuit.* (The french version is better than the english one, 13.03.2023.)

4. Suppose $x(t)$ is a solution of the equation $x' = f(t, x)$, for some $f \in C^\infty(\mathcal{U})$.

a. Prove: if t_0 is an extremum of the function $x(t)$ then $f(t_0, x(t_0)) = 0$.
Is this also a sufficient condition for an extremum of the solution?

b. Suppose $f(t_0, x(t_0)) = 0$ and $\partial_t f|_{(t_0, x(t_0))} > 0$. Prove: $x(t)$ has a local minimum at t_0.

c. Suppose $\partial_t^j f|_{(t_0, x(t_0))} = 0$ for $j = 0, \ldots, r-1$ and $\partial_t^r f|_{(t_0, x(t_0))} < 0$. Is t_0 a local min/max of $x(t)$?