Ordinary differential equations

(201.1.0061. Spring 2024. Dmitry Kerner)

Homework 0. Not for submission

Notations/conventions:

- The unit vector in j'th direction is $\hat{x}_j \in \mathbb{R}^n$. A point in the standard coordinates is $\underline{x}=(x_1,\dots,x_n)\in\mathbb{R}^n$.
- An open subset $\mathcal{U} \subseteq \mathbb{R}^n$. The standard sphere $\mathbb{S}^{n-1} := \{\underline{x} \mid ||\underline{x}|| = 1\} \subset \mathbb{R}^n$.
- The partial derivative $\partial_j f$. The (total) k'th order derivative at a point $f^{(k)}|_{x_0}$. Thus $f^{(1)}|_{x_o}$ is a vector, $f^{(2)}|_{x_o}$ is a (symmetric) matrix, and so on.
- Denote by $C^k(\mathcal{U})$ the ring of functions with continuous k'th derivative. (Here $0 \le k \le \infty$)
- Denote by $C^\omega(\mathcal{U})$ the ring of functions analytic on $\mathcal{U}$. Namely, for each $x_o \in \mathcal{U}$ the function equals (locally near x_o) to its Taylor series at x_o.
- For $[a,b] \subset \mathbb{R}^1$ denote by $C^k[a,b] \subset C^k(a,b)$ the ring of functions with finite limits $\lim\limits_{x\to a^+} f^{(k)}|_x$, $\lim\limits_{x\to b^-} f^{(k)}|_x$. The max-norm on $C^0[a,b]$ is defined by $\|f\| = max_{[a,b]}|f(x)|$.
- A function $\mathbb{R}^n \supseteq X \xrightarrow{f} \mathbb{R}$ is called Lipschitz if $|f(x)-f(\tilde{x})| \le C\cdot\|x-\tilde{x}\|$ for a (fixed) constant $C\in\mathbb{R}_{>0}$ and any points $x,\tilde{x}\in X$.
  The function is called locally Lipschitz at $x_o \in X$ if f is Lipschitz on some neighborhood $x \in \mathcal{U} \subset X$. And "locally Lipschitz on X" means "locally Lipschitz at each point of X".

1. a. Fix a function $g \in C^0[a,b]$, with $g(a) \neq 0 \neq g(b)$. For which sub-spaces of $C^0(a,b)$ does the map $f \to \int_a^b g(x)\cdot f(x)\cdot dx$ define an $\mathbb{R}$-linear functional?

b. Take a continuous vector valued function on a compact set $\mathbb{R}^n \supset X \xrightarrow{f} \mathbb{R}^m$.
Prove: $||\int_X f(x)d^n x|| \le \int_X ||f(x)||d^n x$.

c. Prove: all the norms on $\mathbb{R}^n$ are equivalent. (You have seen this proof in the previous courses. Recall that it is enough to consider the restriction onto $\mathbb{S}^{n-1}$.)

2. a. Expand $arctan\frac{x+y}{1+x^2}$ to the Taylor power series at the point $(0,0)$ up to the order 5.

b. Take some real numbers $0<a_1<\dots<a_k$, $0<b_1<\dots<b_l$ and $c_1<\dots<c_r$.
Prove: the functions $\{sin(a_i\cdot x)\}_i$, $\{cos(b_i\cdot x)\}_i$, $\{exp(c_i\cdot x)\}_i$ are $\mathbb{R}$-linearly independent.
(Can you prove this in several different ways?)

c. Define the function $\mathbb{R}_{>0} \xrightarrow{f} \mathbb{R}$ by $f(x) = x\cdot sin\frac{1}{x} + \frac{sin(x^2)}{x^2} + \frac{x\cdot ln(x)}{1+x}$. Is it uniformly continuous?

3. A function $\mathbb{R}^n \supseteq \mathcal{U} \xrightarrow{f} \mathbb{R}^1$ is called *homogeneous of order* $d\in\mathbb{R}$ if it satisfies $f(t\cdot\underline{x}) = t^d\cdot f(\underline{x}), \forall\, t\in\mathbb{R}_{\ge 0}$.

a. Given a (not necessarily continuous) function on the standard sphere, $\mathbb{S}^{n-1} \xrightarrow{g} \mathbb{R}^1$, define $f(\underline{x}) := \|\underline{x}\|^d\cdot g(\frac{\underline{x}}{\|\underline{x}\|})$ for $\underline{x}\neq o$, and $f(o)=0$. Prove: f is homogeneous of order d.
Give a condition (on g and d) to ensure: i. f is a polynomial. ii. $f\in C^k(\mathbb{R}^n)$.

b. Suppose $f\in C^1(\mathbb{R}^n)$ is homogeneous of order d. Prove: $\sum x_i\partial_i f = d\cdot f$.

c. Let f be homogeneous of order 0. Prove: f is a function of $(n-1)$ variables locally at each point of $\mathbb{R}^n\setminus\{o\}$.

4. a. Fix a *continuous* function $\mathbb{R}^n \supseteq X \xrightarrow{f} \mathbb{R}$. (Dis)Prove:

i. If f is locally Lipschitz on X then it is Lipschitz on every compact subset of X.
ii. If f is locally Lipschitz on X then it is uniformly continuous.
iii. If f is uniformly continuous on X then it is locally Lipschitz on X.
iv. If f is C^∞ on X then f is Lipschitz on X.
v. If $f(x) = o(\|x\|^{1001})$, then f is locally Lipschitz at o.

b. Suppose $f\in C^1(\mathcal{U})$ for a convex set $\mathcal{U}\subseteq\mathbb{R}^n$. Prove: f is Lipschitz on $\mathcal{U}$ iff f' is bounded.

c. Define $\mathcal{U}\subset\mathbb{R}^2$ in polar coordinates, by $1<r<2$, $\phi\in(0,2\pi)$. Define the function $\mathcal{U}\xrightarrow{f}(0,2\pi)$ by $f(r,\phi)=\phi$. Prove: $f\in C^1(\mathcal{U})$, with bounded derivative, but f is not Lipschitz on $\mathcal{U}$.

d. The standard confusion is to define: "A function $\mathbb{R}^n\supseteq X\xrightarrow{f}\mathbb{R}$ is called Lipschitz near $x_0\in X$ if $|f(x)-f(x_0)|\le C\cdot||x-x_0||$, for a constant $C\in\mathbb{R}_{>0}$ and any $x\in X$ close to x_0."
In which sense this is wrong?

5. **a.** For which constants $0<\alpha,\beta$ does the series $\sum_n \frac{1}{n^\alpha \cdot ln^\beta(n)}$ converge?

b. Prove: $C^0[a,b]$ is a complete vector space (for the max-norm).

c. Let $\{f_n\}_n \subset C^0[a,b]$ be a Cauchy sequence of functions (for the max-norm on $C^0[a,b]$).
Prove: if x_n is a Cauchy sequence on $[a,b]$ then the sequence $f_n(x_n)$ converges. Moreover, $\exists\, lim_{m,n\to\infty} f_n(x_m)$.

d. Take a sequence of continuous functions $\{f_n\}_n \subset C^0(X)$. Prove: if the series $\sum f_n$ converges uniformly on X then the limit is a continuous function.
(Do not just cite the well-known theorem, write the actual proof.)

e. Let $\{[a,b] \stackrel{f_n}{\to} \mathbb{R}\}$ be a sequence of monotonic functions. Suppose $\{f_n\}$ converges pointwise to a continuous function. Prove: the convergence is uniform. Is the monotonicity necessary here?

f. Given a Rieman-integrable function $[a,b] \stackrel{f_0}{\to} \mathbb{R}$, define the sequence of functions by $f_{k+1}(x) := \int_a^x f_k(t)dt$. Prove: $\{f_k\}$ converges uniformly. And find the limit.
Do the same question for the sequence $f_{k+1}(x) := C + \int_a^x f_k(t)dt$, for a constant $C \in \mathbb{R}$.

6. **a.** Let $A \in Mat_{n\times n}(\mathbb{R})$ and suppose $v_1,\dots,v_k$ are eigenvectors with pairwise distinct eigenvalues $\lambda_1,\dots,\lambda_k$.
Prove: $v_1,\dots,v_k$ are linearly independent.

b. Prove: $\det[\mathbb{1} + t\cdot A] = 1 + t\cdot trace(A) + O(t^2)$.

c. Let $A = \{a_{ij}(t)\} \in Mat_{n\times n}$, here $a_{ij}(t)$ are differentiable functions of one variable. Prove:

$$det[A]' = det\begin{bmatrix} --\ a_{1\bullet}(t)'\ -- \\ --\ a_{2\bullet}(t)\ -- \\ \dots \end{bmatrix} + det\begin{bmatrix} --\ a_{1\bullet}(t)\ -- \\ --\ a_{2\bullet}(t)'\ -- \\ --\ a_{3\bullet}(t)\ -- \\ \dots \end{bmatrix} + \dots + det\begin{bmatrix} --\ a_{1\bullet}(t)\ -- \\ \dots \\ --\ a_{n\bullet}(t)'\ -- \end{bmatrix}$$

7. Define the map $Mat_{n\times n}(\mathbb{R}) \stackrel{\phi}{\to} \mathbb{R}^{n^2}$ by $\phi(A) = \{a_{ij}\}$ (the long vector of all the matrix entries). Define the norm on $Mat_{n\times n}(\mathbb{R})$ by $\|A\| = \sqrt{trace(A\cdot A^t)}$. (This is not the operator norm.) Prove: this norm is induced from the standard norm on $\mathbb{R}^{n^2}$, i.e. $\|A\| = \|\phi(A)\|$. Conclude: ϕ is an isomorphism of normed vector spaces. Thus we can speak of C^k-functions $Mat_{n\times n}(\mathbb{R}) \to \mathbb{R}$, for $0 \le k \le \infty, \omega$.

a. Prove: the following functions are C^ω.
i. $Mat_{n\times n}(\mathbb{R}) \stackrel{trace,det}{\to} \mathbb{R}$. ii. The coefficients $\{c_j(A)\}$ of the characteristic polynomial of A.

b. Prove: the matrix product, $Mat_{n\times n}(\mathbb{R}) \times Mat_{n\times n}(\mathbb{R}) \to Mat_{n\times n}(\mathbb{R})$, $(A,B) \to A\cdot B$, is a C^ω-function.
Prove: the inverse of a matrix, $GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$, $A \to A^{-1}$, is a C^ω-function.

c. Let $\Sigma_{diag} \subset Mat_{n\times n}(\mathbb{R})$ be the subset of all the matrices that are diagonalizable over $\mathbb{C}$. (i.e. $U\cdot A\cdot U^{-1}$ is diagonal for some $U \in GL(n,\mathbb{C})$) Prove: any matrix whose eigenvalues are pairwise distinct complex numbers belongs to the interior $int(\Sigma_{diag})$. (You can use the fact: if all the complex roots of a polynomial are distinct then locally they are C^∞-functions of the coefficients of the polynomial.)

d. Is $Mat_{n\times n}(\mathbb{R}) \setminus \Sigma_{diag}$ a closed subset of $Mat_{n\times n}(\mathbb{R})$? (Hint: look at $Mat_{2\times 2}(\mathbb{R})$)

8. Define the map $Mat_{n\times n}(\mathbb{R}) \stackrel{exp}{\to} Mat_{n\times n}(\mathbb{R})$ by $exp(A) = \sum\limits_{j=0}^{\infty} \frac{A^j}{j!}$. (Convention: $A^0 = \mathbb{1}$)

a.
- Compute $exp(A)$ for a diagonal matrix. (In particular verify that the series converges)
- Compute $exp(A)$ for $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and for $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

b. Prove: the power series of $exp(A)$ converges absolutely, and the convergence is uniform on compact subsets of $Mat_{n\times n}(\mathbb{R})$. You can use $||A\cdot B|| \le ||A||\cdot||B||$ (follows from Cauchy-Schwarz inequality).

c. Consider A as a complex matrix and take its Jordan form, $A = U^{-1}(D_A + C_A)U$, where $U \in GL(n,\mathbb{C})$, D_A is diagonal and C_A is strictly upper-triangular (corresponding to the Jordan cell structure). Verify: $C_A^n = \mathbb{O}$ and $D_A\cdot C_A = C_A \cdot D_A$.
Prove: $exp(A) = U^{-1}\cdot exp(D_A)\cdot(\sum_{k=0}^n \frac{C_A^k}{k!})\cdot U$. (You will have to open the brackets/to change the order of summation in the series. Justify these steps.)

d. Prove: if A,B commute then $exp(A+B) = exp(A)exp(B)$.

e. Fix some $A \in Mat_{n\times n}(\mathbb{R})$ and define the "path" $\mathbb{R}^1 \stackrel{\gamma}{\to} Mat_{n\times n}(\mathbb{R})$, by $\gamma(t) = exp(t\cdot A)$. Compute $\frac{d\gamma}{dt}$.

f. Can you define the function $ln(\mathbb{1} + A)$ and establish its (corresponding) properties?