Ordinary differential equations for Math

(201.1.0061. Spring 2021. Dmitry Kerner)

Homework 9. Submission date: 28.05.2021

Questions to submit: 1.i. 1.ii. 1.v. 2.b. 3.c. 4.b. 4.c.

Homeworks must be either typed (e.g. in Latex) or written in readable handwriting and scanned in readable resolution.

1. Prove: **i.** $det(e^A) = e^{trace(A)}$. **ii.** $\det[\mathbb{I} + \epsilon A] = 1 + \epsilon \cdot trace(A) + O(\epsilon^2)$. **iii.** $||e^A||_{op} \le e^{||A||_{op}}$.
iv. $e^A = \lim\limits_{k\to\infty}(\mathbb{I} + \frac{A}{k})^k$. **v.** If $A(t) \in GL(n, C^1(a,b))$ then $(A(t)^{-1})' = -A(t)^{-1}A'(t)A(t)^{-1}$.

2. Equations of type $t^n x^{(n)} + a_{n-1}t^{n-1}x^{(n-1)} + \cdots + a_1 t x' + a_0 x = 0$ are called Euler-Cauchy equations. They are used e.g. in physics and in finance.

- **a.** One approach to solve is by time rescaling. Prove: the substitution $t = e^\tau$ transforms the Euler-Cauchy equation into a linear ODE with constant coefficients.
- **b.** Write the general solution for the ODE $t^2x'' + tx' + a_0 x = 0$.
- **c.** In the general case prove: the characteristic polynomial of the obtained ODE with constant coefficient is $L(\lambda) := \lambda(\lambda-1)\cdots(\lambda-n+1) + a_{n-1}\lambda(\lambda-1)\cdots(\lambda-n+2) + \cdots + a_0$.
- **d.** Prove: if one presents $L(\lambda) = \lambda^n + b_{n-1}\lambda^{n-1} + \cdots + b_0$ then the initial equation can be presented as $(t\frac{d}{dt})^n x + b_{b-1}(t\frac{d}{dt})^{n-1}x + \cdots + b_0 x = 0$.
- **e.** Conclude: for $t \neq 0$ the space of solutions of a Euler-Cauchy equation is spanned by the functions of type $\{ln(t)^{k_j} \cdot t^{\lambda_j}\}$, with $k_j \in \mathbb{N}$. Here the function t^λ for $\lambda \in \mathbb{C}$ is defined by $t^\lambda := e^{\lambda \cdot ln(t)} = |t|^{Re(\lambda)}\big(cos(Im(\lambda)ln(t)) + i \cdot sin(Im(\lambda)ln(t))\big)$.

3. Let $\mathbb{X}(t)$ be a fundamental matrix of solutions of the ODE $\underline{x}' = A(t)\underline{x}$. Prove:

- **a.** Any two fundamental matrices are related by $\tilde{\mathbb{X}}(t) = \mathbb{X}(t) \cdot U$ for $U \in GL(n, \mathbb{R})$ (a constant matrix).
- **b.** $\mathbb{X}(t)$ is non-degenerate for each t and satisfies: $\mathbb{X}'(t) = A(t) \cdot \mathbb{X}(t)$.
- **c.** Given the initial condition $\underline{x}(t_0) = \underline{x}_0$, the solution is: $\mathbb{X}(t) \cdot \mathbb{X}^{-1}(t_0) \cdot \underline{x}_0$.
- **d.** Given an equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} \cdots + a_0(t)x = 0$ and the corresponding system $\underline{x}' = A(t)\underline{x}$. Prove: the fundamental matrix of the equation is exactly the fundamental matrix of this system. And the same for Wronskians.

4.
- **a.** Given two functions $x_1(t), x_2(t) \in C^1(a,b)$ (not necessarily solutions of some ODE), suppose $W(x_1(t), x_2(t)) = 0$ on (a,b). Does this imply the $\mathbb{R}$-linear dependence of $x_1(t)$, $x_2(t)$? (Hint at the end of page)
- **b.** Prove: if $W(x_1(t), \ldots, x_n(t)) = 0$ on (a,b) for some analytic functions, then these functions are $\mathbb{R}$-linearly dependent on (a,b).
- **c.** Suppose a solution $x(t)$ of equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} \cdots + a_0(t)x = 0$, with $a_j \in C^0$ has infinitely many zeros on a compact interval. Prove: $x(t) = 0$ on this interval.
  Can the compactness be weakened to boundedness here?
- **d.** Prove: the function $sin(t^p)$, $p \in \mathbb{N}$, cannot be a solution of equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = 0$ with C^0-coefficients, for $n < p$.
- **e.** Prove: the function $e^{-\frac{1}{t^2}}$, extended to $(-1,1)$, cannot be a solution of equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = 0$ with C^0-coefficients, for any n.

Hint: x^2, $x \cdot |x|$.