Ordinary differential equations for Math

(201.1.0061. Spring 2021. Dmitry Kerner)

Homework 8. Submission date: 21.05.2021

Questions to submit: 2.a. 2.c. 3.b. 3.c. 4.b. 5.b.

Homeworks must be either typed (e.g. in Latex) or written in readable handwriting and scanned in readable resolution.

1. Prove: any (finite) system of ODE's, $\underline{f}(t,\underline{x},\underline{x}',\dots,\underline{x}^{(k+1)})=\underline{0}$, is equivalent to a system of 1'st order ODE's, $\underline{F}(t,\underline{y},\underline{y}')=\underline{0}$. (Namely, every solution of $\underline{f}$ leads to a solution of $\underline{F}$ and vice versa.)
Moreover, if the initial system $\underline{f}(\dots)$ is in the normal form/autonomic/linear/polynomial then the resulting system $\underline{F}(\dots)$ is of this type as well.

2. **a.** Prove: if the function $g(\underline{x})>0$ is continuous then the systems $\underline{x}'=\underline{f}(\underline{x})$ and $\underline{x}'=g(\underline{x})\cdot\underline{f}(\underline{x})$ have the same phase potraits. (What happens for $g<0$?)

b. Prove: the phase curves of the system $\underline{x}'=\underline{f}(\underline{x})$, $\underline{f}\in C^1(\mathcal{U})$ for $\mathcal{U}\subseteq\mathbb{R}^n$, cover the whole $\mathcal{U}$ and either coincide or do not intersect.

c. Consider the system $x'=sin(x)\cdot(e^{y^2}+x^4)$, $y'=sin(cos(y))\cdot(e^{x^2}+y^3)$.

i. Find the equilibria points.

ii. Prove: there exist infinity of phase curves that are parallel to $\hat{y}$-axis. Moreover, each of these curves is an ope interval of length$<\pi$. (And the same for $\hat{x}$-axis.)

iii. Prove: any local solution extends (uniquely) to the global solution $x(t),y(t)\in C^{\omega}(\mathbb{R})$.

3. **a.** Verify: $e^{\underline{a}\cdot\nabla}f(\underline{x})=f(\underline{x}+\underline{a})$, here $\nabla=(\partial_{x_1},\dots,\partial_{x_n})$.

b. Write down the Taylor expansion of a solution $\underline{x}(t)$ of $\underline{x}'=A\cdot\underline{x}$ using the general formula for Taylor power series, as was given in the class. Verify that you get $\underline{x}(t)=e^{A(t-t_0)}\cdot\underline{x}_0$.

c. Let $\underline{x}(t)$ be the solution of $\underline{x}'=A(t)\cdot\underline{x}$, $\underline{x}(t_0)=\underline{x}_0$. Compute the Taylor expanion of $\underline{x}(t)$ up to order 3. (Attention, the matrices $A(t)$, $A'(t)$ do not necessarily commute.)

4. Consider the system $\underline{x}'=\underline{f}(t,\underline{x})$, with $\underline{f}\in C^r((a,b)\times\mathbb{R}^n)$. We have proved: If $|\underline{x}\cdot\underline{f}(t,\underline{x})|\le g(t)\cdot(1+||\underline{x}||^2)$ then any solution extends to $C^{r+1}(a,b)$.

a. Instead of $|\underline{x}\cdot\underline{f}(t,\underline{x})|\le g(t)\cdot(1+||\underline{x}||^2)$ one could take the condition $|\underline{x}\cdot\underline{f}(t,\underline{x})|\le g_0(t)+g_1(t)\cdot||\underline{x}||+g_2(t)\cdot||\underline{x}||^2$, for some g_0,g_1,g_2. Prove: this condition is not essentially weaker. Namely, this condition holds for some g_0,g_1,g_2 iff the previous condition holds for some g.

b. Suppose the bound $|\underline{x}\cdot\underline{f}(t,\underline{x})|\le g(t)\cdot(1+\phi(||\underline{x}||^2))$ holds for some function $g(t)$ and a function $\phi(y)\ge 0$ satisfying: $\int_0^\infty\frac{dy}{1+\phi(y)}=\infty$. Prove: any solution extends to $C^{r+1}(a,b)$. For which function ϕ do we get the criterion proved in the class? For which functions ϕ we get a stronger criterion?

5. **a.** Let $A(t)\in Mat_{n\times n}(C^r(\mathbb{R}))$, for $1\le r\le\infty,\omega$. Prove: any local solution of $\underline{x}'=A(t)\cdot\underline{x}$ extends (uniquely) to a global solution $\underline{x}(t)\in C^{r+1}(\mathbb{R})$.

b. Let $\underline{x}(t)$, $\underline{y}(t)$ be solutions of $\underline{x}'=A(t)\cdot\underline{x}$. Prove: $||\underline{x}(t)-\underline{y}(t)||\le||\underline{x}(t_0)-\underline{y}(t_0)||\cdot e^{\int_{t_0}^t||A(s)||_{op}ds}$.

c. Consider the system $\underline{x}'=f(t,\underline{x})$ for $f\in C^0(\mathcal{U})$. Suppose $|(\underline{x}-\underline{y})\cdot(\underline{f}(t,\underline{x})-\underline{f}(t,\underline{y}))|\le g(t)\cdot e^{||\underline{x}-\underline{y}||^2}$ in $\mathcal{U}$. Prove: any solutions $\underline{x}(t)$, $\underline{y}(t)\in C^1(a,b)$ satisfy $|\underline{x}(t)-\underline{y}(t)|^2\le|\underline{x}(0)-\underline{y}(0)|^2-ln\Big(1-e^{|\underline{x}(0)-\underline{y}(0)|^2}\cdot\int_{t_0}^t g(s)ds\Big)$. (We assume here $e^{|\underline{x}(0)-\underline{y}(0)|^2}\cdot\int_{t_0}^t g(s)ds<1$.)