Ordinary differential equations for Math

(201.1.0061. Spring 2021. Dmitry Kerner)

Homework 7. Submission date: 07.05.2021

Questions to submit: 1.a.b.c. 2.b.d. 3.a.b.c.e.

Homeworks must be either typed (e.g. in Latex) or written in readable handwriting and scanned in readable resolution.

1. In the following cases write down the general (real) solutions. Are there (un)bounded/periodic solutions? Identify the equilibria points. When are these points (un)stable? For a. and b. draw the phase portraits.

a. $x' = y$, $y' = k \cdot x$. (Distinguish between the cases $k > 0$, $k < 0$.)

b. $x' = \lambda x + y$, $y' = \lambda y$.

c. $\underline{x}' = A \cdot \underline{x}$ for $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$.

2. Consider the system of differential equations $\underline{x}' = A \cdot \underline{x}$, $A \in Mat_{n\times n}(\mathbb{R})$. Prove:

a. If $\underline{x}(t)$ is a solution then all its derivatives are solutions.

b. If $A = A^t$ then there are no (non-constant) periodic solutions.

c. If $A = -A^t$ and the eigenvalues have multiplicity 1 then the space of solutions is spanned by periodic solutions.

d. If A is of odd size then there exists an unbounded solution.

e. What is the necessary and sufficient condition on A to ensure $\lim\limits_{t\to\infty} \underline{x}(t) = \underline{0}$ for each solution?

f. If A is $\mathbb{R}$-diagonalizable and the eigenvalues have the same sign then $\underline{x} = 0$ is a nodal point. (Attracting or repelling)

g. The solutions are globally-analytic, $x(t, t_0, x_0, A) \in C^\omega(\mathbb{R}_t \times \mathbb{R}_{t_0} \times \mathbb{R}^n_{\underline{x}_0} \times Mat_{n\times n}(\mathbb{R}))$.

3. a. Write down the general (real) solutions of the equations

i. $x^{(n+2)} \pm b^2 x^{(n)} = 0$. ii. $x^{(4)} + 2x^{(3)} + 2x^{(2)} = 0$.

b. Find the condition on a, b to ensure: every solution of $x^{(2)} + a\cdot x' + b\cdot x = 0$ satisfies $\lim\limits_{t\to\infty} x(t) = 0$.

c. Find an equation $x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_0 x = 0$ of minimal possible order whose general solution contains the functions $sin(2t)\cdot e^t$, $cos(2t)\cdot e^{2t}$, t^2. Explain why is this order minimal. Prove: the coefficients $\{a_j\}$ are uniquely determined.

d. (Dis)prove: if all the roots of the equation $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 = 0$ are imaginary and simple then every solution of this equation is periodic.

e. Fix a finite set of pairwise distinct complex numbers $\{\lambda_k\}$ and some natural numbers $\{p_k\}$. Prove: the functions $\{t^j \cdot e^{\lambda_k t}\}_{\substack{1\le k\le n \\ 0\le j\le p_k}}$ are $\mathbb{C}$-linearly independent.

(Can you do this in several different ways?)

f. Formulate the $\mathbb{R}$-version of part e., about the linear independence of the functions $\{t^j e^{a_k t} \cdot cos(b_k \cdot t)\}$, $\{t^j e^{a_k t} \cdot sin(c_k \cdot t)\}$.

g. Consider the equation $x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_0 x = 0$. Prove: for any tuple $(a_{n-1}, \ldots, a_1)$ there exists a finite subset $S \subset \mathbb{R}$ such that for $a_0 \in \mathbb{R} \setminus S$ the space of solutions is spanned by exponents. (Hint: when does a polynomial have a multiple root?)

This phenomenon causes statements like "The solutions $t^j e^{\lambda t}$ never appear in laboratory".