Ordinary differential equations for Math

(201.1.0061. Spring 2021. Dmitry Kerner)

Homework 6. Submission date: 29.04.2021

Questions to submit: 1. 2.b. 2.c. 2.f. 2.g. 3.b. 3.e.

Homeworks must be either typed (e.g. in Latex) or written in readable handwriting and scanned in readable resolution.

1. Prove: if $A \in Mat_{n\times n}(\mathbb{R})$ is $\mathbb{C}$-diagonalizable then A is $\mathbb{R}$-conjugate to a (real) block-diagonal matrix, with blocks of size ≤ 2. Moreover, each 2×2 block can be assumed in the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

(Hint: the non-real eigenvectors of A come in conjugate pairs. But we want a real basis, ...)

2. a. Prove: the functions $||A|| = \sqrt{trace(A^t \cdot \bar{A})}$, $||A||_{op} := \sup_{||v||\neq 0} \frac{||Av||}{||v||}$ define norms on $Mat_{n\times n}(\mathbb{R})$ and $Mat_{n\times n}(\mathbb{C})$. Moreover, $||A \cdot B||_{op} \leq ||A||_{op} \cdot ||B||_{op}$.

b. Prove: the norms $|| * ||$, $|| * ||_{op}$ are equivalent. Prove: these normed spaces are complete.

c. We have defined $e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}$. Prove: this series converges absolutely, and the convergence is uniform on bounded subsets of $Mat_{n\times n}(\mathbb{C})$.

d. Solve questions 7,8 in homework 0.

e. Prove: if $e^{At}e^{Bt} = e^{(A+B)t}$ holds for all $t \in (-\epsilon, \epsilon)$ then $AB = BA$.

f. Prove: if $e^A = 1\!\!1$ then A is $\mathbb{C}$-diagonalizable. What are the possible eigenvalues?

g. Prove: there exists continuum of 2×2 matrices satisfying $e^A = 1\!\!1$ and $det(t1\!\!1 - A) = t^2 + 4\pi^2$.

3. a. Take the unit ball $Ball_1(\mathbb{O})_{op} := \{A|\ ||A||_{op} < 1\} \subset Mat_{n\times n}(\mathbb{C})$. Prove: if $A \in Ball_1(\mathbb{O})_{op}$ then $1\!\!1 + A \in GL(n, \mathbb{C})$.

b. For a marix $A \in Ball_1(\mathbb{O})_{op}$ define $ln(1\!\!1 + A) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}A^k}{k}$. Prove: the series converges absolutely, and the convergence is uniform on compact subsets of $Ball_1(\mathbb{O})_{op}$.

c. Prove: $exp(ln(1\!\!1 + A)) = 1\!\!1 + A = ln(e^{(1\!\!1+A)})$ for every $A \in Ball_1(\mathbb{O})$. (No long computations are needed here.) Any contradiction to question 2.g?

d. Prove: if $AB = BA$ and $A, B, A + B + AB \in Ball_1(\mathbb{O})_{op}$ then $ln\big[(1\!\!1 + A)(1\!\!1 + B)\big] = ln(1\!\!1 + A) + ln(1\!\!1 + B)$. In particular, $ln[(1\!\!1 + A)^k] = k \cdot ln(1\!\!1 + A)$ for every $k \in \mathbb{Z}$.

e. Compute $\frac{d}{dt} ln(1\!\!1 + At)$. (Do this in two ways, as we did for $\frac{d}{dt}e^{At}$ in the lecture.)

4. Define the functions $Mat_{n\times n}(\mathbb{C}) \overset{sin,cos}{\to} Mat_{n\times n}(\mathbb{C})$ via the Taylor expansion of sin, cos. Prove:

a. These series converge absolutely, and the convergence is uniform on bounded subsets of $Mat_{n\times n}(\mathbb{C})$.

b. Prove: $e^{iA} = cos(A) + i \cdot sin(A)$. $cos(A) = \frac{e^{iA}+e^{-iA}}{2}$. $sin(A) = \frac{e^{iA}-e^{-iA}}{2i}$.

c. Prove: $sin^2(A) + cos^2(A) = 1\!\!1$. If $AB = BA$ then $sin(A+B) = \cdots$, $cos(A+B) = \cdots$.

d. Compute $\frac{d}{dt}cos(At)$ and $\frac{d}{dt}sin(At)$. (Do this in two ways, as we did for $\frac{d}{dt}e^{At}$ in the lecture.)