Ordinary differential equations for Math

(201.1.0061. Spring 2021. Dmitry Kerner)

Homework 5. Submission date: 23.04.2021

Questions to submit: 1.a. 2.c. 3.c. 3.f. 3.h. 4.

Homeworks must be either typed (e.g. in Latex) or written in readable handwriting and scanned in readable resolution.

1. **a.** Prove: the local solution of the IVP $x' = sin\frac{101-t^2}{101-x^2}$, $x(2) = 3$ extends to $x(t) \in C^\omega(-5, 9)$.

b. Prove: $x(t) \in C^1$ is a solution of the IVP $x' = f(t, x)$, $x(t_0) = x_0$ iff $x(t)$ is a solution of the integral equation $x = x_0 + \int_{t_0}^{t} f(s, x)ds$.

c. **i.** Prove: for any sequence of constants $\{c_j\}_{j\in\mathbb{Z}}$ there exists an analytic solution of $x' = 1+x^2$ that satisfies $x(\frac{\sqrt{2}}{ln(3)} + \pi j) = c_j$. Any contradiction to the uniqueness theorem?

ii. Write down Picard's approximation $x_3(t)$ for the IVP $x' = 1 + x^2$, $x(0) = x_0$. Prove: $x_k(t) \in C^\omega(\mathbb{R})$ for each $k \in \mathbb{N}$. Does this contradict the explosion of the solution at finite time?

Prove: $x_k(t)$ converges to the solution for $|t| \leq \frac{1}{2x_0+2\sqrt{1+x_0^2}}$.

2. Let $\{x_k(t)\}$ be Picard's approximations for the IVP $x' = f(t, x)$, $x(t_0) = x_0$, with $|f(t, x) - f(t, y)| \leq L \cdot |x - y|$ and $|f(t, x)| \leq L \cdot C$ on $\mathcal{U}$. Prove:

a. $|x_{k+1}(t) - x_k(t)| \leq \frac{C\cdot L^{k+1}\cdot|t-t_0|^{k+1}}{(k+1)!}$ and $|x(t) - x_k(t)| \leq \frac{C\cdot L^k\cdot|t-t_0|^k}{k!}$, for t near t_0.

b. The series $x_m(t) + \sum_{k=m}^{\infty}[x_{k+1}(t) - x_k(t)]$ converges to the solution $x(t)$.

c. If $f \in C^r(\mathcal{U})$, for $r \leq \infty, \omega$, then $x_k(t) \in C^{r+1}$ for $k \geq 1$. Moreover, if $x_k(t) \rightrightarrows x(t)$ (uniformly) on $[t_0 - \epsilon, t_0 + \epsilon]$ then $x_k(t)^{(i)} \rightrightarrows x(t)^{(i)}$ on $[t_0 - \epsilon, t_0 + \epsilon]$, for all $i \leq r$.

3. Let (X, d) be a metric space.

a. Prove: any convergent sequence is a Cauchy sequence, and the limit is unique.

b. Suppose (X, d) is complete. Prove: a subspace $Y \subset X$ is complete iff Y is closed.

c. Give a metric on $X = (-1, 1)$ such that (X, d) is complete. (Does this contradict part b.?)

d. For a subset $X \subseteq \mathbb{R}^n$ consider $C^0(X)$, with *sup*-norm. Prove: $C^0(X)$ is a complete normed space.

e. Suppose d_1, d_2 are equivalent metrics. Prove: (X, d_1) is complete iff (X, d_2) is complete.

f. Give an example of a non-complete metric space and a contractive map without fixed points.

g. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be convex, open, and suppose the map $\mathcal{U} \xrightarrow{f} \mathcal{U}$ is C^1. Suppose $||f'||\frac{1}{2}$ (the operator norm). Prove: the equation $f(x) = x$ has a solution on $\mathcal{U}$.

h. Can the assumption in the fixed point theorem be weakened to "$d(\Psi(x),\Psi(y))<d(x,y)$ for $x\neq y$"?

4. Define the function $f(x)$ as $x^2 \cdot sin\frac{1}{x^2}$ for $x < 0$, as $\sqrt{x}$ for $0 < x < 1$ and as $e^{-x^2} \cdot sin(e^{x^3})$ for $x > 1$. (Dis)Prove:

a. f is locally Lipschitz at each point where it is defined.

b. f is Lipschitz on $(-\epsilon, \epsilon) \setminus \{0\}$.

c. f is Lipschitz on $(1 - \epsilon, 1 + \epsilon) \setminus \{1\}$.

d. f is Lipschitz on $(-\infty, -1)$ and on $(1, \infty)$.