Ordinary differential equations for Math

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Homework 4. Submission date: 09.04.2021

Questions to submit: 1.b. 1.d. 2.a. 2.d. 2.f. 4.a. 4.b.
Homeworks must be either typed (e.g. in Latex) or written in readable handwriting and scanned in readable resolution.

1. a. Prove: the solutions of $x' = e^{x^2} - t$ have no local minima. ($\exists$ at least two different solutions.)

b. Prove: every local solution of $x' = sin^2(t) \cdot e^{t \cdot cos(x)}$ extends (uniquely) to $x(t) \in C^\omega(\mathbb{R})$, this global solution has infinite number of critical points, and all the critical points are flexes (i.e. neither maxima nor minima).

c. Prove: the local solution of $x' = \frac{(x-1)sin(t \cdot x)}{t^2+x^2+1}$, $x(0) = \frac{1}{2}$ extends (uniquely) to the global solution, $x(t) \in C^\omega(\mathbb{R})$. Moreover it satisfies: $0 < x(t) < 1$.

d. Prove: the IVP $x' = \sum_{m=1}^{\infty} \frac{sin(m \cdot x) \cdot cos(m \cdot t)}{m^{\sqrt{5}}}$, $x(t_0) = x_0$ admits the unique local solution for any $(t_0, x_0) \in \mathbb{R}^2$. Moreover, this solution extends (uniquely) to $x(t) \in C^\omega(\mathbb{R})$.

2. a. Prove: the sums/products in $\mathbb{R}[[\underline{x}]]$,$C^\omega(\mathcal{U})$ are well defined. (Therefore $\mathbb{R}[[\underline{x}]]$,$C^\omega(\mathcal{U})$ are commutative rings.) For $C^\omega(\mathcal{U})$ don't forget to check: the product of locally convergent series is locally convergent.

b. Strengthen the statement of Abel's theorem for the power series $\sum a_{\underline{m}} \underline{x}^{\underline{m}}$ to: "If for some $\underline{x}_0 \in \mathbb{R}^n$ the set $\left\{|a_{\underline{m}} \underline{x}_0^{\underline{m}}|\right\}_{\underline{m}}$ is 'sub-exponentially' bounded, i.e. $\lim\limits_{|\underline{m}| \to \infty} \frac{ln(1+|a_{\underline{m}} \underline{x}_0^{\underline{m}}|)}{|\underline{m}|} = 0$, then ...".

c. Suppose the series $\sum a_{\underline{m}} \underline{x}^{\underline{m}}$ converges uniformly on $\mathcal{U} \subset \mathbb{R}^n$. Prove: $\partial_{x_j} \sum a_{\underline{m}} \underline{x}^{\underline{m}} = \sum a_{\underline{m}} \partial_{x_j} (\underline{x}^{\underline{m}})$ and $\int (\sum a_{\underline{m}} \underline{x}^{\underline{m}}) dx_j = \sum a_{\underline{m}} (\int \underline{x}^{\underline{m}} dx_j)$.

d. Tthe *set of convergence* of a series is defined by $\mathfrak{S} := \{\underline{x} | \ \sum a_{\underline{m}} \underline{x}^{\underline{m}} \text{ converges}\} \subseteq \mathbb{R}^n$.

i. Find $\mathfrak{S}$ for $\sum c_m (x^a y^b)^m$, here $a, b \in \mathbb{N}$, the sequence $\{c_m\}$ is bounded and $c_m \not\to 0$. Among all the open boxes $(-x_0, x_0) \times (-y_0, y_0) \subset \mathfrak{S}$ does there exist "the largest box"? Namely, does there exist an open box that contains all the other boxes lying inside $\mathfrak{S}$?

ii. Recall that for $n = 1$ one has $(-R, R) \subseteq \mathfrak{S} \subseteq [-R, R]$, where R is the radius of convergence. For $n > 1$ we try to establish weaker properties. (Dis)Prove:

- $\mathfrak{S} \subseteq \overline{Int(\mathfrak{S})}$ (the closure of the interior); (Hint: $f(x_1, x_2) = \frac{x_1}{1-x_2}$)
- $\mathfrak{S}$ is of "star-type", i.e. for any $x \in \mathfrak{S}$ the segment $[o, x] \subset \mathbb{R}^n$ lies in $\mathfrak{S}$.

e. Why is the expansion $Taylor_{t_0}[f]$ presented as $e^{(t_1-t_0)\frac{d}{dt}} f|_{t=t_1}$ and not just $e^{(t-t_0)\frac{d}{dt}} f$?

f. Let $x(t)$ be the (local) solution of $x' = e^{tx^2}$, $x(0) = 0$. Find $Taylor_0[x(t)]$ up to order 6.

3. Define the distance between two sets $S_1, S_2 \subset \mathbb{R}^n$ by $d(S_1, S_2) := inf\{d(s_1, s_2) | \ s_i \in S_i\}$. Prove:

a. $d(x, S) = 0$ iff $x \in \overline{S}$. (Give an example with $x \notin S$.)

b. If S is closed then $d(x, S) = d(x, s)$ for some $s \in S$. (What can happen if S is not closed?)

c. If $S_1, S_2 \subset \mathbb{R}^n$ are bounded then $d(S_1, S_2) = 0$ iff $\overline{S_1} \cap \overline{S_2} \neq \varnothing$.
Give an example of bounded sets with $S_1 \cap S_2 = \varnothing$ but $d(S_1, S_2) = 0$.
Give an example of unbounded sets with $\overline{S_1} \cap \overline{S_2} = \varnothing$ but $d(S_1, S_2) = 0$.

d. If S_1, S_2 are compact then there exist $s_1 \in S_1$, $s_2 \in S_2$ such that $d(s_1, s_2) = d(S_1, S_2)$.

4. a. (Comparison test) Consider two equations $x' = f_i(t, x)$, $i = 1, 2$ and their solutions $x_i(t)$, both defined on $[t_0, t_1)$. Suppose $f_1(t, x) \geq f_2(t, x)$ for every $(t, x) \in \mathcal{U}$ and $x_1(t_0) \geq x_2(t_0)$. Prove: $x_1(t) \geq x_2(t)$ on $[t_0, t_1)$.

b. Prove: the local solution of 2.f extends (uniquely) to $x(t) \in C^\omega(-\infty, \epsilon)$ but explodes (in finite time) on the interval $(\epsilon, 2)$.