Ordinary differential equations for Math

(201.1.0061. Spring 2021. Dmitry Kerner)

Homework 3. Submission date: 02.04.2021

Questions to submit: 1.b. 2.i. 2.iii. 2.v. 4.c. 5.a. 6.b.
Homeworks must be either typed (e.g. in Latex) or written
in readable handwriting and scanned in readable resolution.

1. The Bernoulli equation, $x' = g(t)x + h(t)x^{n+1}$, $g, h \in C^0(a, b)$, $n \neq 0, -1$, is important for the fluid transition between two reservoirs. The standard approach is to convert it to a linear equation, by substition $x = z^k$ near a point $x_0 \neq 0$.
   a. Find the needed k. (Address both cases $x_0 > 0$, $x_0 < 0$.)
   b. Solve the IVP: $x' = x \cdot tan(t) + x^4 \cdot cos(t)$, $x(\pi) = -1$.

2. In the following cases write the general solution (at least in the form $F(x, t) = const$). Draw the integral curves (you can use any software). For which initial conditions the local solution exists/is unique? What are the singular points of these curves (the points with $grad(F) = \vec{0}$)?
   i. $x' = \frac{2t + 3t^2 x}{3x^2 - t^3}$
   ii. $x' = \frac{2x}{3t} + \frac{2t}{x}$
   iii. $3t^2 x + x^2 + 2t^3 x' + 3tx \cdot x' = 0$
   iv. $x' = x(1 + xe^t)$
   v. $tx' - 2t^2 \sqrt{x} = 4x$

3. Take two linear functions $l_i(t, x) = a_i t + b_i x - c_i$, $i = 1, 2$. Consider the equation $x' = \frac{l_2(t,x)}{l_1(t,x)}$.
   a. Suppose the lines $\{l_i(t, x) = 0\} \subset \mathbb{R}^2$ are not parallel. Shift the coordinates $t \to t - c_t$, $x \to x - c_x$ to move the line intersection to the origin. In the new coordinates the equation becomes homogeneous. Write the details. Solve the IVP: $x' = \frac{4x - 2t - 6}{x + t - 3}$, $x(2) = 2$.
   b. If the two lines are parallel (but do not coincide) then change the coordinates $x \to x - c \cdot t$ to get an autonomous equation. Write the details.
   Write the general solution of the equation $x' = \frac{t - x - 1}{t - x - 2}$.

4. For a differentiable function $\mathbb{R}^n \supseteq \mathcal{U} \xrightarrow{F} \mathbb{R}$ define the total differential $dF := \sum_j \partial_j F \cdot dx_j$.
   a. Is $xdy - ydx$ the total differential of a C^2-function?
   b. Prove: if $\sum a_j \cdot dx_j$ is the total differential of a C^2 function then $\partial_j a_i = \partial_i a_j$ for all i, j.
   c. Prove that the differential forms $\frac{xdx + ydy}{(x^2 + y^2)^p}$, $\frac{ydx - xdy}{(x^2 + y^2)}$ are locally exact at each point of $\mathbb{R}^2 \setminus \{0, 0\}$. Let $\mathcal{U}$ be one of: $\mathbb{R}^2 \setminus \{0, 0\}$, $\mathbb{R}^1_{x>0} \times \mathbb{R}^1_y$, $\mathbb{R}^2 \setminus \{y = 0, x \leq 0\}$. In which cases are these forms exact on $\mathcal{U}$?

5. Pass from the (x, t)-coordinates to the polar coordinates (r, ϕ). Accordingly transform the equation $x' = f(\frac{x}{t})$ into $\frac{dr}{d\phi} = r \cdot \Phi(\phi)$. Here the function Φ is 2π-periodic and possibly not defined on some (discrete) set of points. Define $p := \int_0^{2\pi} \Phi(\phi)d\phi$, suppose this integral converges.
   a. Describe the integral curves in the cases $p > 0$, $p = 0$, $p < 0$. Prove: any local solution extends to a global one, $\mathbb{R}^1 \xrightarrow{r(\phi)} \mathbb{R}^1_{\geq 0}$. Conclude: in (x, t) coordinates no local solution extends to a global $\mathbb{R}^1 \xrightarrow{x(t)} \mathbb{R}^1$, and for some initial conditions (which?) there is no existence/uniqueness.
   b. As an explicit example analyze the solutions of $x' = \frac{C \cdot x - t}{x + C \cdot t}$, for all constants $C \in \mathbb{R}$.

6. Consider the equation $f \cdot dx + g \cdot dy = 0$. Suppose $g(x_0, y_0) \neq 0$.
   a. Prove: There exists a local integrating factor of the form $\lambda(x)$ iff $\frac{f - g}{g}$ is locally independent of y. In this case $\lambda = C \cdot e^{\int^x (\frac{f-g}{g})ds}$. (We have proved this in the class.)
   b. Obtain similar results for the existence of integrating factors in the forms $\lambda(x - y)$, $\lambda(x^2 + y^2)$.
   c. Solve the equations: i. $(t^2 - x^2 + 1) + (t^2 - x^2 - 1)x' = 0$, ii. $ydx - (x^2 + y^2 + x)dy = 0$.