Ordinary differential equations for Math

(201.1.0061. Spring 2021. Dmitry Kerner)

Homework 13. Not for submission

1. a. Denote by $\mathbb{R}[[t]][t^{-1}]$ the set of (formal) Laurent power series with finite number of negative powers. Prove: $\mathbb{R}[[t]][t^{-1}]$ is closed under products, and every non-zero element is invertible. (Thus $\mathbb{R}[[t]][t^{-1}]$ is a field.)

b. For $f, g \in \mathbb{R}[[t]][t^{-1}]$ express $ord(f \cdot g)$, $ord(\frac{1}{f})$, $ord(f^{(n)})$, $ord(e^f)$ via $ord(f)$, $ord(g)$. What about $ord(f+g)$?

c. Given a Laurent power series $\sum_{-\infty}^{\infty} c_j t^j$ prove: there exist $0 \le r \le R \le \infty$ such that the series converges absolutely for $r < |t| < R$, and diverges at each point where $|t| > R$ or $|t| < r$. How to compute r, R via the coefficients $\{c_j\}$?

2. Consider the system $\underline{x}' = \frac{A}{t^{q+1}}\underline{x}$, for $q \in \mathbb{Z}_{\ge 0}$ and $A \in Mat(\mathbb{R})$.

a. Write down the real(!) solutions for $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, with $a, b \in \mathbb{R}$.

b. Give the necessary and sufficient conditions on q, A to ensure:

i. This is a regular singular point.

ii. There exists a solution bounded near $t = 0$.

iii. There exists a solution with a pole.

iv. A solution is bounded for $t > 0$ if and only if it is bounded for $t < 0$.

v. All the solutions are power series with integer exponents.

3. a. Consider the system $\underline{x}' = (\frac{A_0}{t} + A_1)\underline{x}$. When does there exist a solution bounded near $t = 0$? When are all the solutions power series with integer exponents?
Suppose $A_0 A_1 = A_1 A_0$. Write the solutions.

b. Prove: the system $x_1' = x_2$, $x_2' = 2\frac{x_1}{t^2}$ has a regular singular point. (Hint: look for monomial solutions.)

c. Suppose $A(t)$ has a singularity at $t = 0$. Is every (non-zero) solution of $\underline{x}' = A(t) \cdot \underline{x}$ necessarily singular at t=0?

d. Suppose the matrix $A(t)$ has a singularity at $t = 0$, but there exists an analytic fundamental matrix $\mathbb{X}(t)$ for $\underline{x}' = A(t)\underline{x}$. Prove: $det[\mathbb{X}(0)] = 0$.

4. a. Consider the equation $t^2 x'' + t a_1 x' + a_0 x = 0$. Give the necesary and sufficient conditions on $a_1, a_0 \in \mathbb{R}$ to ensure:

i. There exists a bounded solution.

ii. There exists a global solution $x(t) \in C^0(\mathbb{R}^1)$.

iii. Every solution has at most a pole at $t = 0$. Express the order of the pole in terms of a_0, a_1.

b. Write the power series solutions for Bessel's equation $t^2 x'' + t x' + (t^2 - a^2)x = 0$. What is the domain of convergence?

c. Write the power series solutions at $t = \pm 1$ for Legendre's equation $(1-t^2)x'' - 2tx' + 4(a+1)x = 0$. What is the domain of convergence for each point? (Wiki: Legendre polynomials)

d. Prove: the series $\sum_{j=0}^{\infty} j! t^j$ is a formal solution of the equation $t^2 x'' + (3t-1)x' + x = 0$. Can this equation be transformed into Bessel's equation by time change, $t = g(\tau)$, and rescaling, $x \to h(t) \cdot x$, for some functions $g, h \in C^\omega$?

e. Write the general solution of $(2x - 3x^2)y'' + 4y' + 6xy = 0$. (Hint: there exists a polynomial solution) What are the types of the singular points?