Ordinary differential equations for Math

(201.1.0061. Spring 2021. Dmitry Kerner)

Homework 12. Submission date: 17.06.2021

Questions to submit: 1.a. 1.b. 1.c. 1.d. 2.a. 2.b.

Homeworks must be either typed (e.g. in Latex) or written in readable handwriting and scanned in readable resolution.

1. a. Suppose $e^t, sin(t), t^{17}$ are solutions of a linear non-homogeneous equation of 2'nd order. Write the general solution. Find the solution satisfying: $x(0) = a$, $x'(0) = b$.

b. Given two solutions te^t, $(t-2)e^t$ of the equation $tx'' - (t+1)x' + x = (t-1)e^x$, find the general solution.

c. Find the general solution of Bessel's equation $t^2x'' + tx' + (t^2 - \frac{1}{4})x = 3t^{\frac{3}{2}} sin(t)$, $t > 0$, given a particular solution of the corresponding homogeneneous equation, $x(t) = \frac{sin(t)}{\sqrt{t}}$.

d. Suppose $A(t) = \begin{bmatrix} sin(t) & arctan(cos(2t)) \\ cos(sin(2t)) & \frac{1}{10} - cos(t) \end{bmatrix}$. Prove: the system $\underline{x}' = A(t) \cdot \underline{x}$ has at least one unbounded solution.

2. a. Let $x_1(t)$, $x_2(t)$ linearly independent solutions of the equation $x'' + a_1(t)x' + a_0x = 0$. Prove: between any two zeros of $x_1(t)$ lies exactly one zero of $x_2(t)$.

b. Consider the equation $x'' + a(t)x = 0$ on the interval (t_0, ∞). Prove:

i. If $0 < m \le a(t) \le M$ then the distance between any two consecutive zeros is $\frac{\pi}{\sqrt{M}} \le dist \le \frac{\pi}{\sqrt{m}}$.

ii. (Kneser's theorem) If $a(t) > \frac{1+\epsilon}{4t^2}$, for some $\epsilon > 0$, then any solution has infinite number of zeros. (Hint: compare to $x'' + \frac{1}{4t^2}x = 0$.)

iii. If $a(t) < \frac{1}{4t^2}$ then any non-zero solution has a finite number of zeros.

3. The Bessel function $J_n(t)$ is defined as the solution of the equation $x'' + (1 + \frac{1/4-n^2}{t^2})x = 0$.

a. Prove: $J_n(t)$ has infinite number of zeros, denote $J_n(t_k) = 0$, and $|t_{k+1} - t_k| \to \pi$ as $t \to \infty$.

b. When does one have $|t_{k+1} - t_k| > \pi$, $|t_{k+1} - t_k| < \pi$?

c. Prove: if $n > m$ then between every two zeros or J_m lies a zero of J_n.