Ordinary differential equations for Math

(201.1.0061. Spring 2021. Dmitry Kerner)

Homework 11. Submission date: 10.06.2021

Questions to submit: 1.a. 2.b. 2.c. 2.e. 3.a. 3.b.

Homeworks must be either typed (e.g. in Latex) or written in readable handwriting and scanned in readable resolution.

1. a. Suppose $x(t)$ is a solution of $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = 0$. We have seen how one can use $x(t)$ to pass to an equation $y^{(n-1)} + \tilde{a}_{n-2}(t)y^{(n-2)} + \cdots + \tilde{a}_0(t)y = 0$. Prove: $x(t)$ together with the solutions of this last equations provide the complete system of solutions of the initial equations. In particular, given the independent solutions $y_1(t), \dots, y_{n-1}(t)$ verify: the functions $x(t), x(t)\cdot y_1(t), \dots, x(t)\cdot y_{n-1}(t)$ are $\mathbb{R}$-linearly independent.

b. Suppose the coefficients of $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots = 0$ are periodic. Prove: for any basis of solutions $x_1(t), \dots, x_n(t)$ one can present $\underline{x}(t) = \underline{y}(t)\cdot e^{Rt}$, where $\underline{y}$ is a row of periodic functions, while $R \in Mat_{n\times n}(\mathbb{C})$.

2. Consider the equation $L(x) = g(t)$, where $L(x) = x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_0 x$, with $a_i \in \mathbb{R}$.

a. Write the general solution of $x^{(4)} + 4x = \sum b_j e^{\omega_j t}$, here $\omega_j \in \mathbb{C}$, possibly $\omega_j = 0$ or $\omega_j^3 = -4$.

b. Suppose $\mu \in \mathbb{C}$ is not an eigenvalue of the characteristic polynomial of L. Prove: the equation $L(x) = t^k \cdot e^{\mu t}$, with $k \in \mathbb{N}$, has a solution of the form $g_k(t)\cdot e^{\mu t}$ for a polynomial $g_k(t) \in \mathbb{C}[t]_{\le k}$ of degree k. (Hint. It is enough to show: the operator $L \circlearrowright \mathbb{C}[t]_{\le k}\cdot e^{\mu t}$ acts surjectively. And for this it is enough to verify: L acts injectively.)

c. Suppose $\mu \in \mathbb{C}$ is an eigenvalue of the characteristic polynomial of L, of multiplicity p. Prove: the equation $L(x) = t^k \cdot e^{\mu t}$ has a solution of the form $t^p \cdot g(t)\cdot e^{\mu t}$ for a polynomial $g_k(t) \in \mathbb{C}[t]_{\le k}$ of degree k. Wiki: "Resonance".

d. Write the general solution of $x^{(4)} + 4x = b\cdot t\cdot e^{\mu t}$. (Here $b, \mu \neq 0$ are parameters.)

e. Consider the equation $Lx = p(t)\cdot e^{\mu t}$, here $p(t) \in \mathbb{C}[t]$. What is the necessary and sufficient condition to ensure that the equation has a periodic solution? A bounded solution?

3. a. Find the general solution of $x'' - \frac{4}{t}x' + \frac{6x}{t^2} = t^3 + t$.

b. Find the general solution of $(t^2-1)x'' + 4tx' + 2x = 6t$, given the particular solutions $x_1(t) = t$, $x_2(t) = \frac{t^2+t+1}{t+1}$.

c. Find the general solution of $tx'' - (t+n)x' + nx = 0$, for $n \in \mathbb{N}$, given a solution e^t.

4. a. Write the general solution of the equation $\underline{x}' = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 2 & 3 \end{bmatrix} \cdot \underline{x} + \begin{bmatrix} 3e^t \\ 0 \\ 3e^{-t} \end{bmatrix}$.

b. Consider the equation $\underline{x}' = A\cdot\underline{x} + e^{\mu t}t^k \cdot \underline{b}$, here $\underline{b}$ is a constant vector. Prove:

i. If μ is not an eigenvalue of A then there exists a solution of the form $e^{\mu t}\underline{g}(t)$, where the entries of $\underline{g}$ are polynomials of degree$\le k$.

ii. If μ is an eigenvalue of A then there exists a solution of the form $e^{\mu t}\underline{g}(t)$, where the entries of $\underline{g}$ are polynomials of degree$\le k + Jord.Size.(A)$, here $Jord.Size.(A)$ is the size of the maximal Jordan cell of A.

iii. We have proved: if $x(t)$ is a solution of $\underline{x}' = A(t)\underline{x}$ then $|\underline{x}(t)| \le |x(t_0)|\cdot e^{\int_{t_0}^t ||A(s)||_{op} ds}$. Obtain a similar bound for a solution of $\underline{x}' = A(t)\underline{x} + \underline{b}(t)$.