# Ordinary differential equations for Math

(201.1.0061. Spring 2021. Dmitry Kerner)

**Homework 10. Not for submission**

**1. a.** Write the general solution of the system $x' = \frac{x}{1+t^2} + y \cdot sin(2t)$, $y' = y \cdot cos(t)$.

**b.** Write the general solution of the equation $(\frac{d}{dt} - c_1(t)) \circ (\frac{d}{dt} - c_2(t))x = 0, \ c_1(t), c_2(t) \in C^0(a,b)$.

**c.** Find the full Taylor expansion of the solution of $x'' + t^p \cdot x = 0$, $p \in \mathbb{N}$, $x(0) = 0$, $x'(0) = 1$.

**2. a.** Suppose the matrices $A(t)$ and $\int_{t_0}^t A(s)ds$ commute for each $t \in (a,b)$.
Prove: the (unique) solution of $\underline{x}' = A(t) \cdot \underline{x}, \ \ \underline{x}(t_0) = \underline{x}_0$ is given by $\underline{x}(t) = e^{\int_{t_0}^t A(s)ds} \underline{x}_0$.

**b.** Let $\{A_j\}$ be some constant pairwise commuting matrices. Let $\{g_j(t)\}$ be $C^0(a,b)$.
Solve the system $\underline{x}' = (\sum g_j(t) A_j)\underline{x}, \ \ \underline{x}(t_0) = \underline{x}_0$.

**c.** Prove that the assumption in **a.** implies: the matrices $A(t)'$ and $\int_{t_0}^t A(s)ds$ commute for each $t \in (a,b)$.

**d.** Give an example of equation $\underline{x}' = A(t) \cdot \underline{x}$ for which $e^{\int_{t_0}^t A(s)ds} \underline{x}_0$ is not a solution.

**3. a.** Let $\mathbb{X}(t) := [\underline{x}_1(t), \dots, \underline{x}_n(t)] \in Mat_{n \times n}(C^1(a,b))$, here the columns are some solutions of $\underline{x}' = A(t) \cdot \underline{x}$. Prove: $det[\mathbb{X}(t)] \not\equiv 0$ iff $\mathbb{X}(t)$ is non-degenerate for all $t \in (a,b)$.

**b.** Find a system $\underline{x}' = A(t) \cdot \underline{x}$ of minimal size whose solutions are $\underline{x}_1(t) = [e^t cos(t), e^t sin(t)]$ and $\underline{x}_2(t) = [-sin(t), cos(t)]$.

**c.** Prove: if $\lim_{t \to \infty} \int^t trace[A(s)]ds = \infty$ then at least one solution of $\underline{x}' = A(t)\underline{x}$ is unbounded. Show by an example that the conclusion "$|x(t)| \to \infty$ for at least one solutions" does not necessarily hold.

**d.** Prove: the rescaling $\underline{x} \to e^{-\int^t \frac{trace[A(s)]}{n} ds} \underline{x}$ transforms $\underline{x}' = A(t) \cdot \underline{x}$ into the system $\underline{x}' = \tilde{A}(t) \cdot \underline{x}$ with $trace[\tilde{A}(t)] = 0$.

**4. a.** Prove: there exists a fundamental matrix $\mathbb{X}(t)$ of the equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots = 0$ satisfying $\mathbb{X}(t_0) = \mathbb{1}$.

**b.** Verify: the functions $sin(t^2)$, $cos(t^2)$ are (linearly independent) solutions of $tx'' - X' + 4t^3 x = 0$, but the Wronskian of these functions vanishes at a point. Any contradiction to **3.a**?

**c.** Write the general solution of $tx'' + 2x' - tx = 0$. (Hint: one solution is $x(t) = \frac{e^t}{t}$.)

**d.** Find a linear ODE whose space of solutions is spanned by $sin\frac{1}{t}$, $cos\frac{1}{t}$.

**e.** Prove: the rescaling $x \to e^{-\int^t \frac{a_{n-1}(s)}{n} ds} x$ transforms the equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = 0$ into an equation with $\tilde{a}_{n-1}(t) = 0$.