

Ordinary differential equations for Math

(201.1.0061. Spring 2021. Dmitry Kerner)

Homework 1. Submission date: 18.03.2021

Questions to submit: 1 b. 1 d. 2 c. 3 a. 3 b. 4 b.

Homeworks must be either typed (in Latex) or written in readable handwriting and scanned in readable resolution.



1.
 - a. The earth is defined by $\{\underline{x} \mid x_n \leq 0\} \subset \mathbb{R}^n$. Suppose every freely falling particle (with $x_n > 0$) experiences the constant acceleration, $\vec{a} = -g \cdot \hat{x}_n$. (The law of Gallileo, $g = 9.8m/sec^2$.) A frog jumps at the moment t_0 from the point $0 \in \mathbb{R}^n$, with the initial velocity \vec{v} (here $v_n > 0$). Find the trajectory $\underline{x}(t)$, the total time of the jump (i.e. $t_1 - t_0$) and the total displacement, i.e. $\underline{x}(t_1) - \underline{x}(t_0)$.
 - b. A particle moves on the standard sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Prove: $\vec{v}(t) \perp \underline{x}(t)$ for each t . Suppose $n = 2$ and $\|\vec{v}\| = const$. Prove: $\vec{a}(t) \parallel \underline{x}(t)$ for each t . (wiki: centrifugal force)
 - c. How does a linear coordinate change, $\mathbb{R}^n \xrightarrow{A} \mathbb{R}^n$, $\underline{x} \rightarrow A \cdot \underline{x}$, with $A \in GL(n, \mathbb{R})$. affect \vec{v} , \vec{a} ?
 - d. Take the second law of Newton in dimension 1, i.e. $m \cdot \vec{a} = \vec{F}(x)$, for a continuous vector field $\mathbb{R}^1 \xrightarrow{\vec{F}} \mathbb{R}^1$. Multiply this 2'nd order ODE by \vec{v} and integrate. You get a 1'st order ODE, called the energy conservation law. Here $\frac{m \cdot \vec{v}^2}{2}$ is the kinetic energy, while $-\int F(x) dx$ is the potential energy. (To extend this to $n > 1$ one needs some ingredients from Geometric Calculus 2.)

2. Consider the equation $x' + f(t) \cdot x = g(t)$, for $f, g \in C^0(a, b)$.
 - a. In the lecture we have obtained the complete solution. Rederive the general formula. What is the dimensionality of the family of solutions? (It seems to depend on two constants?)
 - b. Impose the initial condition $x(t_0) = x_0$. Prove: the solution is unique and is globally defined.
 - c. For which pairs $(t_0, x_0) \in \mathbb{R}^2$ does the IVP $t \cdot x' + x = \sin(t)$, $x(t_0) = x_0$ posses a solution? When is it unique? Draw the integral curves.

3. The hare runs over \mathbb{R}^2 with constant velocity $(0, v_h)$, starting at $t = 0$ from the origin. The dog starts to run from the point $(x_0, 0)$ at $t = 0$ with velocity \vec{v} satisfying: $\|\vec{v}\| = v_d$, and at each moment \vec{v} is directed towards the hare. Present the trajectory of the dog as $y = y(x)$.
 - a. Prove: the function $y(x)$ satisfies the equation $x \cdot y'' = \frac{v_h}{v_d} \cdot \sqrt{1 + (y')^2}$.
(Hint: write all the equations on the functions $x(t), y(t)$. Observe that these functions are invertible. Exclude t from this system of equations.)
 - b. Substitute $z = y'$ to convert this equation to the first order ODE. Solve it.
 - c. Give the necessary and sufficient condition (on v_h, v_d) for the dog to catch the hare in finite time. (Verify that the natural guess means the converges of some improper integral.)
 - d. Wiki: The curve of pursuit. (The french version is better than the english one, 26.02.2021.)

4. Suppose $x(t)$ is a solution of the equation $x' = f(t, x)$, for some $f \in C^\infty(\mathcal{U})$.
 - a. Prove: if t_0 is an extremum of the function $x(t)$ then $f(t_0, x(t_0)) = 0$.
Is this also a sufficient condition for an extremum of the solution?
 - b. Suppose $f(t_0, x(t_0)) = 0$ and $\partial_t f|_{(t_0, x(t_0))} > 0$. Prove: $x(t)$ has a local minimum at t_0 .
 - c. Suppose $\partial_t^j f|_{(t_0, x(t_0))} = 0$ for $j = 0, \dots, r-1$ and $\partial_t^r f|_{(t_0, x(t_0))} < 0$. Is t_0 a local min/max of $x(t)$?