# Ordinary differential equations

(201.1.0061. Spring 2021. Dmitry Kerner)

**Homework 0. Not for submission**

Notations/conventions:

- The unit vector in $j$'th direction $\hat{x}_j \in \mathbb{R}^n$. A point in the standard coordinates is $\underline{x}=(x_1,\dots,x_n)\in\mathbb{R}^n$.
- An open subset $\mathcal{U} \subseteq \mathbb{R}^n$. The standard sphere $\mathbb{S}^{n-1} := \{\underline{x} \,|\, ||\underline{x}|| = 1\} \subset \mathbb{R}^n$.
- The partial derivative $\partial_j f$. The (total) $k$'th order derivative at a point $f^{(k)}|_{x_0}$.
- Given $(a,b) \subseteq \mathbb{R}^1$ and $0 \le k \le \infty$ denote by $C^k(a,b)$ the ring of functions with continuous $k$'th derivative. Denote by $C^\omega(a,b)$ the ring of functions analytic at each point of $(a,b)$. For $[a,b] \subset \mathbb{R}^1$ denote by $C^k[a,b] \subset C^k(a,b)$ the ring of functions with finite limits $\lim\limits_{x\to a^+} f^{(k)}|_x$, $\lim\limits_{x\to b^-} f^{(k)}|_x$.
- A function $\mathbb{R}^n \supseteq X \xrightarrow{f} \mathbb{R}$ is called Lipschitz (or uniformly Lipschitz) if $|f(x) - f(x_0)| \le C \cdot ||x - x_0||$ for a (fixed) constant $C \in \mathbb{R}_{>0}$ and any points $x, x_0 \in X$. The function is called locally Lipschitz on $X$ if for each point there exists a neighborhood, $x \in \mathcal{U} \subseteq X$, such that $f$ is Lipschitz on $\mathcal{U}$.
- The max-norm on $C^0[a,b]$ is defined by $||f|| = max_{[a,b]}|f(x)|$.

**1. a.** Suppose $f \in C^\omega(a,b)$ vanishes on a segment $(x_0 - \epsilon, x_0 + \epsilon) \subset (a,b)$. Prove: $f = 0$ on $(a,b)$. Does this hold also for $C^\infty$-functions?

**b.** Fix a function $g \in C^0[a,b]$. For which sub-spaces of $C^0(a,b)$ does the map $f \to \int_a^b g(x)\cdot f(x)\cdot dx$ define an $\mathbb{R}$-linear functional?

**c.** Take a continuous vector valued function on a compact set $\mathbb{R}^n \supset X \xrightarrow{f} \mathbb{R}^m$. Prove: $|| \int_X f(x) d^n x|| \le \int_X ||f(x)|| d^n x$.

**d.** Prove: all the norms on $\mathbb{R}^n$ are equivalent. (You have seen this proof in the previous courses, recall that it is enough to consider the restriction onto $\mathbb{S}^{n-1}$.)

**2. a.** Expand $arctan\frac{x+y}{1+x^2}$ to the Taylor power series at the point $(0,0)$ up to the order 5.

**b.** Take some real numbers $0 < a_1 < \dots < a_k$, $0 < b_1 < \dots < b_l$ and $c_1 < \dots < c_r$. Prove: the functions $\{sin(a_i \cdot x)\}_i$, $\{cos(b_i \cdot x)\}_i$, $\{exp(c_i \cdot x)\}_i$ are $\mathbb{R}$-linearly independent. (Can you do this in several different ways?)

**3.** A function $\mathbb{R}^n \supseteq \mathcal{U} \xrightarrow{f} \mathbb{R}^1$ is called *homogeneous of order* $d \in \mathbb{R}$ if it satisfies $f(t\cdot\underline{x}) = t^d \cdot f(\underline{x})$, $\forall\, t \in \mathbb{R}_{\ge 0}$.

**a.** Given a (not necessarily continuous) function on the standard sphere, $\mathbb{S}^{n-1} \xrightarrow{g} \mathbb{R}^1$ define $f(\underline{x}) := ||\underline{x}||^d \cdot g(\frac{\underline{x}}{||\underline{x}||})$ for $\underline{x} \ne o$, and $f(o) = 0$. Prove: $f$ is homogeneous of order $d$.
Give a condition (on $g$ and $d$) to ensure: i. $f$ is a polynomial. ii. $f \in C^k(\mathbb{R}^n)$.

**b.** Suppose $f \in C^1(\mathbb{R}^n)$ is homogeneous of order $d$. Prove: $\sum x_i \partial_i f = d \cdot f$.

**c.** Let $f$ be homogeneous of order 0. Prove: $f$ is a function of $(n-1)$ variables locally at each point of $\mathbb{R}^n \setminus \{o\}$.

**4. a.** Prove: if $\mathbb{R}^n \supseteq X \xrightarrow{f} \mathbb{R}$ is locally Lipschitz on $X$ then $f$ is Lipschitz on every compact subset of $X$.

**b.** (Dis)Prove: if $f$ is differentiable near $o$ and $f(x) = o(||x||^{1001})$ then $f$ is locally Lipschitz near $o$.

**c.** Suppose one wants to use the following definition. "A function $\mathbb{R}^n \supseteq X \xrightarrow{f} \mathbb{R}$ is called Lipschitz at a point $x_0 \in X$ if $|f(x) - f(x_0)| \le C \cdot ||x - x_0||$, for a constant $C \in \mathbb{R}_{>0}$ and any $x \in X$ close to $x_0$. A function is called locally Lipschitz on $X$ if $f$ is Lipschitz at each point of $X$."
Is this definition equivalent to the initial one?

**d.** Suppose $f \in C^1(\mathcal{U})$ for a convex set $\mathcal{U} \subseteq \mathbb{R}^n$. Prove: $f$ is Lipschitz on $\mathcal{U}$ iff $f'$ is bounded.

**e.** Define $\mathcal{U} \subset \mathbb{R}^2$ by $1 < r < 2$, $\phi \in (0, 2\pi)$, in polar coordinates. Define the function $\mathcal{U} \xrightarrow{f} (0, 2\pi)$ by $f(r,\phi) = \phi$. Prove: $f \in C^1(\mathcal{U})$, with bounded derivative, but $f$ is not Lipschitz.

**5. a.** For which constants $0 < \alpha, \beta$ does the series $\sum_n \frac{1}{n^\alpha \cdot ln^\beta(n)}$ converge?

**b.** Let $f_n \in C^0[a,b]$ be a Cauchy sequence (for the max-norm on $C^0[a,b]$).
Prove: if $x_n$ is a Cauchy sequence on $[a,b]$ then the sequence $f_n(x_n)$ converges.

**c.** Take a sequence of continuous functions $\{\mathbb{R}^n \supseteq X \xrightarrow{f_k} \mathbb{R}\}$. Prove: if the series $\sum f_k$ converges uniformly on $X$ then the limit is a continuous function. (Do not just cite the well-known theorem, write the actual proof.)

**d.** Let $\{[a,b] \xrightarrow{f_n} \mathbb{R}\}$ be a Cauchy sequence of monotonic functions. Suppose $\{f_n\}$ converges to a continuous function. Prove: the convergence is uniform. Is the monotonicity condition necessary here?

**e.** Given a Rieman-integrable function $[a,b] \xrightarrow{f_0} \mathbb{R}$, define the sequence of functions by $f_{k+1}(x) := \int_a^x f_k(t)dt$. Prove: $\{f_k\}$ converges uniformly (and find the limit).
Do the same question for the sequence $f_{k+1}(x) := C + \int_a^x f_k(t)dt$, for a constant $C \in \mathbb{R}$.

**f.** Define the function $\mathbb{R}_{>0} \xrightarrow{f} \mathbb{R}$ by $f(x) = x \cdot sin\frac{1}{x} + \frac{sin(x^2)}{x^2} + \frac{x \cdot ln(x)}{1+x}$. Is it uniformly continuous?

**g.** For a sequence of continuous functions $\mathbb{R} \xrightarrow{\{f_n\}_n} \mathbb{R}$ consider the conditions:
i. Each $f_n$ is uniformly continuous. ii. The sequence is uniformly bounded.
iii. The sequence is equi-continuous. iv. The family converges uniformly.

(Dis)Prove: (i.+ii.+iii.)$\Rightarrow$ iv. (i.+ii.+iv.)$\Rightarrow$ iii. And so on.

**6. a.** Let $A \in Mat_{n\times n}(\mathbb{R})$ and suppose $v_1, \dots, v_k$ are eigenvectors with pairwise distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Prove: $v_1, \dots, v_k$ are linearly independent.

**b.** Prove: $\det[\mathbb{1} + t \cdot A] = 1 + t \cdot trace(A) + O(t^2)$.

**c.** Let $A = \{a_{ij}(t)\} \in Mat_{n\times n}$, here $a_{ij}(t)$ are differentiable functions of one variable. Prove:

$$det[A]' = det \begin{bmatrix} \{a_{1j}(t)'\} \\ \{a_{2j}(t)\} \\ \dots \end{bmatrix} + det \begin{bmatrix} \{a_{1j}(t)\} \\ \{a_{2j}(t)'\} \\ \{a_{3j}(t)\} \\ \dots \end{bmatrix} + \dots + det \begin{bmatrix} \{a_{1j}(t)\} \\ \dots \\ \{a_{nj}(t)'\} \end{bmatrix}$$

**7.** Define the map $Mat_{n\times n}(\mathbb{R}) \xrightarrow{\phi} \mathbb{R}^{n^2}$ by $\phi(A) = \{a_{ij}\}$ (the long vector of all the matrix entries). Define the norm on $Mat_{n\times n}(\mathbb{R})$ by $||A|| = \sqrt{trace(A \cdot A^t)}$. (This is not the operator norm.) Prove: this norm is induced from the standard norm on $\mathbb{R}^{n^2}$, i.e. $||A|| = ||\phi(A)||$. Conclude: $\phi$ is an isomorphism of normed vector spaces. This defines the topology on $Mat_{n\times n}(\mathbb{R})$, and we can speak of $C^k$ functions.

**a.** Consider the functions: $Mat_{n\times n}(\mathbb{R}) \xrightarrow{trace,det} \mathbb{R}$. Prove: these are $C^\infty$.

**b.** Prove: the matrix product, $Mat_{n\times n}(\mathbb{R}) \times Mat_{n\times n}(\mathbb{R}) \to Mat_{n\times n}(\mathbb{R})$, $(A,B) \to A \cdot B$, is a $C^\infty$-function.
Prove: the inverse of a matrix, $GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$, $A \to A^{-1}$, is a $C^\infty$-function.

**c.** Take the characteristic polynomial of $A$ and denote by $\{c_j(A)\}$ its coefficients.
Prove: $Mat_{n\times n}(\mathbb{R}) \xrightarrow{\{c_j(A)\}} \mathbb{R}^{n+1}$ is a $C^\infty$-function.

**d.** Let $\Sigma_{diag} \subset Mat_{n\times n}(\mathbb{R})$ be the subset of all the matrices that are diagonalizable over $\mathbb{C}$. (i.e. $U \cdot A \cdot U^{-1}$ is diagonal for some $U \in GL(n,\mathbb{C})$) Prove: any matrix whose eigenvalues are pairwise distinct complex numbers belongs to the interior $int(\Sigma_{diag})$. (You can use the fact: if all the comlpex roots of a polynomial are distinct then they are $C^\infty$-functions of the coefficients of the polynomial.)

**e.** Conclude: $\overline{\Sigma_{diag}} = Mat_{n\times n}(\mathbb{R})$ and $int(Mat_{n\times n}(\mathbb{R}) \setminus \Sigma_{diag}) = \varnothing$. (Because of this many engineers claim "Any matrix in real life is $\mathbb{C}$-diagonalizable".)

**f.** Is $Mat_{n\times n}(\mathbb{R}) \setminus \Sigma_{diag}$ a closed subset of $Mat_{n\times n}(\mathbb{R})$? (Hint: look at $Mat_{2\times 2}(\mathbb{R})$)

**8. a.** Define the map $Mat_{n\times n}(\mathbb{R}) \xrightarrow{exp} Mat_{n\times n}(\mathbb{R})$ by $exp(A) = \sum_{j=0}^{\infty} \frac{A^j}{j!}$. (Convention: $A^0 = \mathbb{1}$)

- Compute $exp(A)$ for a diagonal matrix. (In particular verify that the series converges)
- Compute $exp(A)$ for $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and for $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

**b.** Prove: the power series of $exp(A)$ converges absolutely, and the convergence is uniform on compact subsets of $Mat_{n\times n}(\mathbb{R})$. You can use $||A \cdot B|| \leq ||A|| \cdot ||B||$.

**c.** Consider $A$ as a complex matrix and take its Jordan form, $A = U^{-1}(D_A + C_A)U$, where $U \in GL(n,\mathbb{C})$, $D_A$ is diagonal and $C_A$ is strictly upper-triangular (corresponding to the Jordan cell structure). Verify: $C_A^n = \mathbb{O}$ and $D_A \cdot C_A = C_A \cdot D_A$.
Prove: $exp(A) = U^{-1} \cdot exp(D_A) \cdot (\sum_{k=0}^{n} \frac{C_A^k}{k!}) \cdot U$. (You will have to open the brackets/to change the order of summation in the series. Justify these steps.)

**d.** Prove: if $A, B$ commute then $exp(A+B) = exp(A)exp(B)$.

**e.** Fix some $A \in Mat_{n\times n}(\mathbb{R})$ and define $\mathbb{R}^1 \xrightarrow{\gamma} Mat_{n\times n}(\mathbb{R})$, by $\gamma(t) = exp(t \cdot A)$. Compute $\frac{d\gamma}{dt}$.

**f.** Can you define the function $ln(A)$ and establish its (corresponding) properties?