Introduction to Algebraic Curves

201.2.4451. Spring 2018 (D.Kerner)

Homework 8

(1) Let $0 \to V_1 \to \cdots \to V_n \to 0$ be an exact sequence of finite-dimensional $\Bbbk$-vector spaces. Show: $\sum_i (-1)^j dim_{\Bbbk}(V_j) = 0$.

(2) In the proof of "$mult_{pt}(f) = \mathfrak{m}^j_{pt}/\mathfrak{m}^{j+1}_{pt}$ for $j \gg 1$", we have used the exact sequence $0 \to \mathfrak{m}^j/\mathfrak{m}^{j+1} \to \mathcal{O}_{(C,pt)}/\mathfrak{m}^{j+1} \to \mathcal{O}_{(C,pt)}/\mathfrak{m}^j \to 0$. For an arbitrary algebraic subset $X \subset \Bbbk^n$ the Hilbert-Samuel function of the local ring at $pt \in X$ is defined as $\chi(j) := dim_{\Bbbk} \mathcal{O}_{(X,pt)}/\mathfrak{m}^j_{pt}$.

  (a) Prove that the function $\chi(j) := dim_{\Bbbk} \mathcal{O}_{(\Bbbk^n,pt)}/\mathfrak{m}^j_{pt}$ is a polynomial in j, of degree n, whose leading coefficient is $\frac{1}{n!}$.
  (b) Let $X = \{f(\underline{x}) = 0\} \subset \Bbbk^n$ be a hypersurface. Prove: for $j \gg 1$ the function $\chi(j) := dim_{\Bbbk} \mathcal{O}_{(X,pt)}/\mathfrak{m}^j_{pt}$ is a polynomial in j, of degree $(n-1)$, whose leading coefficient is $\frac{ord_{pt}(f)}{(n-1)!}$.

(3) Here we assume $\Bbbk = \bar{\Bbbk}$.

  (a) Go over all the details in the definition/proof "$i_{pt}(f_1, f_2) = dim\, \Bbbk[x,y]_{\mathfrak{m}_{pt}}/(f_1, f_2)$".
  Using this proof compute $i_0(f_1, f_2)$ for $f_1(x,y) = (x^2+y^2)^2 + 3x^2y - y^3$, $f_2(x,y) = (x^2+y^2)^3 - 4x^2y^2$.
  (b) Suppose $pt \in C \subset \Bbbk^2$ is a smooth point and fix a uniformizer $t \in \mathcal{O}_{(C,pt)}$. Let $D = \{f(x,y) = 0\}$. Prove: $i_{pt}(C, D) = ord_t(f(x(t), y(t)))$.
  (c) Go over all the details of the proof of "$i_{pt}(C,D) \geq mult_{pt}(C) \cdot mult_{pt}(D)$ with equality iff ...".
  In particular, prove: If $\mathfrak{m}^{m_1+m_2-1} \subseteq (f,g) + \mathfrak{m}^{m_1+m_2} \subseteq \Bbbk[x,y]_{\mathfrak{m}}$ then $\mathfrak{m}^{m_1+m_2-1} \subseteq (f,g) \subseteq \Bbbk[x,y]_{\mathfrak{m}}$.
  This is immediate if one uses the Nakayama lemma. For a proof without Nakayama see Fulton, page 39.
  (d) Let $pt \in \{f(x,y) = 0\}$ be a smooth point. Prove: $i_{pt}(f, g+h) \geq min(i_{pt}(f,g), i_{pt}(f,h))$.
  Does this hold also for non-smooth points?
  (e) Fix a curve $C = \{f(x,y) = 0\}$ and a parameterized line $L = \{(x(t), y(t)) = (a_x t + b_x, a_y t + b_y)\}$. Check that the roots of the equation $f(x(t), y(t)) = 0$ correspond to the intersection points, $C \cap L$, with their multiplicities.
  Prove: $\sum i_{pt}(L, C) \leq deg(f)$. When is the equality realized?
  (f) Give an example of two (smooth) conics both passing through $(0,0) \in \Bbbk^2$, whose local intersection multiplicity at $(0,0)$ equals 4.
  (g) Suppose C is smooth at pt and $L = T_{(C,pt)}$. Check: $i_{pt}(C, L) \geq 2$. This point is called an inflection point (or "a flex") if $i_{pt}(C, L) > 2$. What are the flexes of the curve $\{y = x^n\} \subset \Bbbk^2$?
    (i) Does this definition depend on the choice of local coordinates?
    (ii) Prove: a smooth conic in $\Bbbk^2$ has no flexes. (Hint: no heavy computations are needed here)
    (iii) Prove: $pt \in \{f(x,y) = 0\}$ is an inflection point iff the Hessian matrix satisfies:

$$(-\partial_y f, \partial_x f)|_{pt} \cdot \partial^2 f|_{pt} \cdot \begin{pmatrix} -\partial_y f \\ \partial_x f \end{pmatrix} |_{pt} = 0.$$

(4)
  (a) Let $\{\mathcal{U}_i\}$ be the standard affine charts on $\mathbb{P}^n$. Describe the sets $X_i := \mathbb{P}^n \setminus \{\cup_{j \neq i} \mathcal{U}_j\}$, $X_{12} := \mathbb{P}^n \setminus \{\cup_{j \neq 1,2} \mathcal{U}_j\}$.
  (b) Prove: through any two points of $\mathbb{P}^2$ passes unique (projective) line.
  (c) Prove: given any $\binom{d+2}{2} - 1$ points in $\mathbb{P}^k$ there is a curve of degree d passing through them. Moreover, if the points are "mutually generic" then such a curve is unique. (Here the genericity condition is formulated as the non-degeneracy of some matrix, what is this matrix?)
  (d) Let $L_1, L_2 \subset \mathbb{P}^n$ be planes of dimensions d_1, d_2. Prove: $dim(L_1 \cap L_2) \geq d_1 + d_2 - n$. Does this hold also in $\Bbbk^n$? (Which condition should be added?)

(5)
  (a) Prove: $f(\underline{x}) \in \Bbbk[x_1, \ldots, x_n]$ is irreducible iff $x_0^{deg(f)} f(\frac{x_1}{x_0}, .., \frac{x_n}{x_0}) \in \Bbbk[x_0, \ldots, x_n]$ is irreducible.
  (b) Prove: $I \subset \Bbbk[x_0, \ldots, x_n]$ has a system of homogeneous generators iff for any $f \in I$ all the homogeneous components of f belong to I.
  (c) Prove: an algebraic set $X \subset \mathbb{P}^n$ is irreducible iff the ideal $I(X) \subset \Bbbk[x_0, \ldots, x_n]$ is prime.
  (d) Let $I \subset \Bbbk[\underline{x}]$ be a homogeneous prime ideal. Prove: if $f \cdot g \in I$ then either all homogeneous components of f belong to I or the same for g. (Thus I is "homogeneously prime".)
  (e) Prove: if I is a homogeneous ideal then $\sqrt{I}$ is homogeneous.
  (f) Suppose $\Bbbk$ is infinite. Prove: if $X \subset \mathbb{P}^n$ is an algebraic subset and $f \in \Bbbk[x_0, \ldots, x_n]$ vanishes identically on X, then all the homogeneous components of f vanish on X.
  (g) Prove: any intersection/finite union of algebraic subsets of $\mathbb{P}^n$ is algebraic.
  (h) The group $\mathbb{P}GL(n+1, \Bbbk) = GL(n+1, \Bbbk)/\Bbbk^\times$ acts on $\mathbb{P}^n$, these are called: projective transformations. Prove: $\mathbb{P}GL(n+1, \Bbbk)$ sends algebraic subsets to algebraic, hypersurfaces to hypersurfaces, smooth points of plane curves to smooth points.