Introduction to Algebraic Curves

201.2.4451. Spring 2018 (D.Kerner)

Homework 4

(1) Fix a non-constant holomorphic map of compact Riemann surfaces, $X \stackrel{f}{\to} Y$, and a point $x \in X$.
- (a) To define $mult_x(f)$ we have chosen particular local coordinates in X, Y. Prove that the multiplicity does not depend on the choices.
- (b) Suppose $mult_x(f) = 1$. Prove that f is locally a biholomorphism at x. Conclude that f is a local bihilomorphism everywhere except for a finite set of points on X.
- (c) Prove that a map of $deg(f) = 1$ has no ramifications and is a global isomorphism of Riemann surfaces.
- (d) Can the maps $X \stackrel{f,g}{\to} Y$ be added/multiplied? For $X \stackrel{f}{\to} Y \stackrel{g}{\to} Z$ compute $mult_x(g \circ f)$.
- (e) Consider a meromorphic function $f \in \mathcal{M}_X(\mathcal{U})$ as a map of Riemann surfaces, denote it by F. What is the relation between $mult_x(F)$ and $ord_x(f)$? (Distinguish between the zeros/poles of f and other points.)
- (f) Prove/disprove:
  - (i) There exist neighborhoods $x \in \mathcal{U}_x \subset X$, $f(x) \in \mathcal{U}_{f(x)} \subset Y$ such that for any $y \in \mathcal{U}_{f(x)}$ holds: $\sum\limits_{x_i \in \mathcal{U}_x \cap f^{-1}(f(x))} mult_{x_i} f = mult_x(f)$.
  - (ii) For any neighborhood $x \in \mathcal{U}_x \subset X$ there exists a neighborhood $f(x) \in \mathcal{U}_{f(x)} \subset Y$ such that for any $y \in \mathcal{U}_{f(x)}$ holds: $\sum\limits_{x_i \in \mathcal{U}_x \cap f^{-1}(f(x))} mult_{x_i} f = mult_x(f)$.

(2)
- (a) What is the minimal triangulation of the cylinder?
- (b) Does there exist a triangulation of S^2 with just three triangles? Does there exist a triangulation of S^2 with 7 faces, 12 edges and 8 vertices?
- (c) Construct some simple triangulations of S^2 with g handles. (Here 'simple' is not 'the minimal possible'.)

(3)
- (a) Let $f(z) = \frac{4z^2(z-1)^2}{(2z-1)^2}$ a meromorphic function on $\mathbb{C}$, consider the corresponding holomorphic map $\mathbb{P}^1_{\mathbb{C}} \stackrel{F}{\to} \mathbb{P}^1_{\mathbb{C}}$. Describe its ramification data. What is $deg(F)$?
- (b) Project the curves $\{xy = 1\} \subset \mathbb{C}^2$, $\{y = x^2\} \subset \mathbb{C}^2$ onto $\hat{y}$-axis. Are the degrees of this projection constant?
- (c) Fix a smooth real algebraic curve (not necessarily connected), $C = \{f(x, y) = 0\} \subset \mathbb{R}^2$. Suppose the degree of the projection $C \stackrel{\pi_x}{\to} \mathbb{R}$ is constant, i.e. the total number of preimages $\pi_x^{-1}(x)$ (counted with their multiplicities) does not depend on x. Prove that all the multiplicities of $\{\pi_x\}_{x \in X}$ are odd. What are the possible topological types of C?

(4)
- (a) Prove: the action $\mathbb{P}GL(3, \mathbb{C}) \circlearrowright \mathbb{P}^2_{\mathbb{C}}$ preserves the degrees and genera of smooth algebraic curves.
- (b) Prove the Riemann-Hurwitz formula.
- (c) Compute the genus of a smooth plane projective cubic. (Recall: any such curve can be brought, by a $\mathbb{P}GL(3, \mathbb{C})$ transformation, to the Weierstraß form, which in the affine coordinates is $\{y^2 = x^3 + ax + b\}$.)
- (d) Let $X \stackrel{f}{\to} Y$ be a non-constant holomorphic map or compact Riemann surfaces. Prove:
  - (i) $g(X) \geq g(Y)$.
  - (ii) If $g(X) = g(Y) = 1$ then f is unramified.
  - (iii) If $g(X) = g(Y) = 2$ then f is an isomorphism.
  - (iv) The sum of ramification indices of f is even. (The ramification index at a point x is $(mult_x(f) - 1)$.)

(5) In the lecture we saw how to "plug the holes" in a punctured Riemann surface.
- (a) Prove that plugging the holes preserves Hausdorffness and path-connectedness.
- (b) A curve $\{f(x, y) = 0\} \subset \mathbb{C}^2$ is said to "have a node" at $(0, 0)$ if $f(0, 0) = 0 = \partial_x f|_{(0,0)} = \partial_y f|_{(0,0)}$ and the Hessian matrix of f at $(0, 0)$ is non-degenerate. Prove that by a local holomorphic change of coordinates at $(0, 0)$ we can bring f to the form $x^2 - y^2$.
- (c) Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a singular algebraic curve and let X be the Riemann surface obtained by puncturing the singular points of C and plugging the holes. Take the natural projection $X \stackrel{\pi}{\to} C$. Fix a point of C and take some local coordinates on $\mathbb{P}^2$ at this point: (x, y). Prove: any (holomorphic/meromorphic) function $f(x, y)$ on $\mathbb{C}^2$ induces a local holomorphic map: $X \supset \mathcal{U} \stackrel{f|_C \circ \pi}{\to} \mathbb{P}^1$.
- (d) Compute the genera of the Riemann surfaces obtained from the curves $\{y^2 = x^3\} \subset \mathbb{C}^2$, $\{y^2 = x^3 + x^2\} \subset \mathbb{C}^2$ by compactifying (in $\mathbb{P}^2$), puncturing the singularities, and plugging the holes.
- (e) Let X be a Riemann surface with punctures, so that the surface $\overline{X}$, obtained by plugging all the holes in X, is a compactification of X. Prove that this compactification is unique, i.e. for any two compact Riemann surfaces $\overline{X}_1 \supset X \subset \overline{X}_2$ holds: $\overline{X}_1 \stackrel{\sim}{\to} \overline{X}_2$.