Explicit Motivic Chabauty-Kim Method

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Background: The Unit Equation

Let Z be an integer ring with a finite set of primes inverted ($= \mathcal{O}_k[1/S]$) and $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Theorem

There are finitely many $z, w \in Z^\times$ such that $z + w = 1$
Equivalently, $|X(Z)| < \infty$.

Originally proven by Siegel using Diophantine approximation around 1929.

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Problem

Find $X(Z)$ for various Z, or even find an algorithm.

In 2004, Minhyong Kim gave a proof in the case $k = \mathbb{Q}$ using a non-abelian version of Chabauty's method.

Refined Problem (Chabauty-Kim Theory)

Find p-adic analytic (Coleman) functions on $X(\mathbb{Z}_p)$ that vanish on $X(Z)$.

(p-adic) Polylogarithms

- The p-adic analytic functions that appear are p-adic polylogarithms.
- We now recall the definition of the k-logarithm for $k \in \mathbb{Z}_{\geq 1}$

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$$\mathrm{Li}_k(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^k}$$

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- These functions satisfy the recursive differential equation

$$\frac{d}{dz}\mathrm{Li}_k(z) = \frac{1}{z}\mathrm{Li}_{k-1}(z),$$

  with $\mathrm{Li}_1(z) = -\log(1-z)$ and $\mathrm{Li}_k(0) = 0$ for all k.
- p-adic polylogarithms $\mathrm{Li}_k^p(z)$ are defined as p-adic analytic functions satisfying the same differential equations
- As the p-adics are totally disconnected, one must use Coleman's theory to fix the constants of integration

Recent Explicit Results

Kim's method defines a set $\mathcal{I}^Z$ of $\mathbb{Q}_p$-valued analytic functions on $X(\mathbb{Z}_p)$ that vanish on $X(Z)$ and have finitely many zeroes (for any $p \notin S$).

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Theorem (Dan-Cohen, Wewers, 2013)

- For $Z = \mathbb{Z}[1/\ell]$ and all p, the following Coleman function is in $\mathcal{I}^Z$:

$$2\mathrm{Li}_2^p(z) - \log^p(z)\mathrm{Li}_1^p(z)$$

- For $Z = \mathbb{Z}[1/2]$ and $p \neq 2$, the following Coleman function is in $\mathcal{I}^Z$:

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- For $Z = \mathbb{Z}[1/2]$ and $p \neq 2$, the following Coleman function is in $\mathcal{I}^Z$:

$$24\log^p(2)\zeta^p(3)\mathrm{Li}_4^p(z) + \frac{8}{7}\left(\log^p(2)^4 + 24\mathrm{Li}_4^p(\frac{1}{2})\right)\log^p(z)\mathrm{Li}_3^p(z)$$
$$+\left(\frac{4}{21}\log^p(2)^4 + \frac{32}{7}\mathrm{Li}_4^p(\frac{1}{2}) + \log^p(2)\zeta^p(3)\right)\log^p(z)^3\log^p(1-z)$$

Recent Results, cont.

- In 2015, Dan-Cohen posted a preprint showing that this could be made into an algorithm, whose halting is conditional on refinements of conjectures due to Kim and Goncharov.

Theorem (C, Dan-Cohen, 2017)

For $Z = \mathbb{Z}[1/3]$ and $p \neq 2, 3$, the following Coleman function is in $\mathcal{I}_4^Z$:

$$\zeta^p(3)\log^p(3)\mathrm{Li}_4^p(z) - \left(\frac{18}{13}\mathrm{Li}_4^p(3) - \frac{3}{52}\mathrm{Li}_4^p(9)\right)\log^p(z)\mathrm{Li}_3^p(z)$$
$$-\frac{(\log^p(z))^3\mathrm{Li}_1^p(z)}{24}\left(\zeta^p(3)\log^p(3) - 4\left(\frac{18}{13}\mathrm{Li}_4^p(3) - \frac{3}{52}\mathrm{Li}_4^p(9)\right)\right),$$

Recent results, cont.

- Dan-Cohen–Wewers and C–Dan-Cohen have used these to verify a conjecture of Kim in special cases:

Conjecture (Kim et al., 2014)

The set of common zeroes of elements of $\mathcal{I}^Z$ is precisely $X(Z)$.

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- Another arXiv preprint of C–Dan-Cohen presents an improved algorithm
- If the algorithm halts, then it provably gives the correct answer
- Kim's conjecture and some standard conjectures about mixed motives imply the algorithm halts

Next Goal: Chabauty-Kim for a Punctured Elliptic Curve

Let Z be an integer ring with a finite set of primes inverted ($= \mathcal{O}_k[1/S]$) and $E' = E \setminus \{O\}$ for some elliptic curve $E/\mathbb{Q}$ with good reduction outside S.

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$|E'(Z)| < \infty$

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Problem

Extend the previous method from $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ to E' using mixed elliptic motives in place of mixed Tate motives (more on this later).

- Some of what we need in this case is conjectural, but we can still do computations.
- Eventual goal: show that standard conjectures about mixed motives plus Kim's conjecture imply an effective Faltings

How does one cut out $X(Z)$ inside $X(\mathbb{Z}_p)$?

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- But, when $z \in X(Z)$, we can do better:
- There is a graded Hopf algebra $A(Z)$ over $\mathbb{Q}$, finite-dimensional in each degree, with a (conjecturally injective) ring homomorphism

$$\mathrm{per}_p \colon A(Z) \to \mathbb{Q}_p$$

- Each $z \in X(Z)$ gives rise to an element $\mathrm{Li}_k^{\mathfrak{u}}(z) \in A(Z)$ such that

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- Furthermore, using deep results in arithmetic (about algebraic K-theory), we can describe $A(Z)$ abstractly as a graded Hopf algebra
- This extra structure on $A(Z)$ along with precise information about its size greatly limits which $z \in X(\mathbb{Z}_p)$ can actually be in $X(Z)$

Using the extra structure to define $\mathcal{I}^Z$

Definition

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- Each $z \in X(Z)$ defines a homomorphism $\kappa(z) \colon \mathcal{O}(\pi_1^{\mathrm{PL}}(X)) \to A(Z)$ sending $\mathrm{Li}_k^{\mathfrak{u}}$ to $\mathrm{Li}_k^{\mathfrak{u}}(z)$.
- There is furthermore a graded Hopf algebra structure on $\mathcal{O}(\pi_1^{\mathrm{PL}}(X))$, in which $\mathrm{Li}_k^{\mathfrak{u}}$ has degree k, $\log^{\mathfrak{u}}$ has degree 1, and the reduced coproduct Δ' is given by:

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Fact

For $z \in X(Z)$, the homomorphism $\kappa(z)$ is a homomorphism of graded Hopf algebras.

Using the extra structure to define $\mathcal{I}^Z$, cont.

For a prime p, this gives us a diagram:

$$\begin{array}{ccc} X(Z) & \longrightarrow & X(\mathbb{Z}_p) \\ \kappa \big\downarrow & & \\ \mathrm{Hom}_{\mathrm{GrHopf}}(\mathcal{O}(\pi_1^{\mathrm{PL}}(X)), A(Z)) & & \end{array}$$

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- In addition, an arbitrary $z \in X(\mathbb{Z}_p)$ induces a homomorphism $\mathcal{O}(\pi_1^{\mathrm{PL}}(X)) \to \mathbb{Q}_p$ sending $\mathrm{Li}_k^{\mathfrak{u}}$ to $\mathrm{Li}_k^p(z)$.

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- We may upgrade the bottom horizontal morphism to a map of schemes, as follows:
- We define a scheme $Z_{\mathrm{PL}}^{1,\mathbb{G}_m}$ over $\mathbb{Q}$ by $Z_{\mathrm{PL}}^{1,\mathbb{G}_m}(R) = \mathrm{Hom}_{\mathrm{GrHopf}}(\mathcal{O}(\pi_1^{\mathrm{PL}}(X)), A(Z)) \otimes R)$ for a $\mathbb{Q}$-algebra R.
- The bottom arrow may then be viewed as a map of $\mathbb{Q}_p$-schemes

$$Z_{\mathrm{PL}}^{1,\mathbb{G}_m} \otimes \mathbb{Q}_p \to \pi_1^{\mathrm{PL}}(X) \otimes \mathbb{Q}_p$$

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- Dimension counts show that the bottom horizontal arrow is non-dominant, which is what proves Siegel's theorem.
- Therefore, there is a nonzero ideal $\mathcal{I}_{\mathrm{PL}}^{Z} \subseteq \mathcal{O}(\pi_1^{\mathrm{PL}}(X)) \otimes \mathbb{Q}_p$ vanishing on the image of the bottom arrow, known as the *(polylogarithmic) Chabauty-Kim ideal*.
- The right-hand vertical map is Coleman analytic, so elements of $\mathcal{I}_{\mathrm{PL}}^{Z}$ pull back to Coleman functions on $X(\mathbb{Z}_p)$ that vanish on $X(Z)$; these are elements of $\mathcal{I}^{Z}$

Where does $A(Z)$ come from?

- There is a $\mathbb{Q}$-linear Tannakian category $\mathcal{MT}(Z)$ of *mixed Tate motives* over Z
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- $\mathcal{MT}(Z)$ is the category of representations of the pro-algebraic group $\pi_1^{\mathrm{MT}}(Z)$ over $\mathbb{Q}$.
- This group has a semidirect product decomposition

$$\pi_1^{\mathrm{MT}}(Z) = \pi_1^{\mathrm{un}}(Z) \rtimes \mathbb{G}_m,$$

  where $\pi_1^{\mathrm{un}}(Z)$ is pro-unipotent.
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- Then $A(Z) := \mathcal{O}(\pi_1^{\mathrm{un}}(Z))$, with grading coming from the $\mathbb{G}_m$-action.
- $A(Z)$ is known as the ring of *mixed Tate motivic periods over Z*
- Its elements are formal versions of numbers that arise from (p-adic) integrals of algebraic differential forms

Polylogarithms as Motivic Periods

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- These integrals show up in relative cohomology of powers $(\mathbb{P}^1 \setminus \{0, 1, \infty\})^n$
- The relevant relative cohomology groups give objects in $\mathcal{MT}(Z)$

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- The relevant relative cohomology groups give objects in $\mathcal{MT}(Z)$
- One can abstractly describe the graded Hopf algebra $A(Z)$ using our knowledge of

$$\mathrm{Ext}^1_{\mathcal{MT}(Z)}(\mathbb{Q}(-n), \mathbb{Q}(0)) \cong K_{2n-1}(Z)_{\mathbb{Q}}$$

- This is given in more detail on the next slide:

Abstract Structure of $A(Z)$

- As a graded vector space, $A(\mathbb{Z}[1/S])$ is the free vector space over $\mathbb{Q}$ on symbols f_w, where w is a word in the set $\{\tau_\ell\}_{\ell \in S} \cup \{\sigma_{2n+1}\}_{n \geq 1}$, with τ_ℓ in degree 1 and σ_{2n+1} in degree $2n+1$.

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where ℓ denotes the length of a word, $Ш(\ell(w_1), \ell(w_2)) \subseteq S_{\ell(w_1)+\ell(w_2)}$ denotes the group of shuffle permutations of type $(\ell(w_1), \ell(w_2))$, and $w_1 w_2$ denotes concatenation.

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- E.g., $f_{\tau_2 \sigma_3} f_{\tau_2} = 2 f_{\tau_2 \tau_2 \sigma_3} + f_{\tau_2 \sigma_3 \tau_2}$.

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- For words w_1, w_2, the product is given as follows:

$$f_{w_1} f_{w_2} := \sum_{\sigma \in Ш(\ell(w_1), \ell(w_2))} f_{\sigma(w_1 w_2)},$$

where ℓ denotes the length of a word, $Ш(\ell(w_1), \ell(w_2)) \subseteq S_{\ell(w_1)+\ell(w_2)}$ denotes the group of shuffle permutations of type $(\ell(w_1), \ell(w_2))$, and $w_1 w_2$ denotes concatenation.
- E.g., $f_{\tau_2 \sigma_3} f_{\tau_2} = 2 f_{\tau_2 \tau_2 \sigma_3} + f_{\tau_2 \sigma_3 \tau_2}$.
- The coproduct Δ is given by

$$\Delta f_w := \sum_{w_1 w_2 = w} f_{w_1} \otimes f_{w_2} = f_w \otimes 1 + \Delta' f_w + 1 \otimes f_w.$$

- E.g., $\Delta f_{\tau_2 \sigma_3} = 1 \otimes f_{\tau_2 \sigma_3} + f_{\tau_2} \otimes f_{\sigma_3} + f_{\tau_2 \sigma_3} \otimes 1$, so $\Delta' f_{\tau_2 \sigma_3} = f_{\tau_2} \otimes f_{\sigma_3}$.

Computing Explicit Generators for $A(Z)$

- To compute the functions, we need to write the elements f_w explicitly in the form $\mathrm{Li}^{\mathfrak{u}}_n(z)$.
- One may choose:

$$f_{\tau_\ell} = \log^{\mathfrak{u}}(\ell)$$

$$f_{\sigma_{2n+1}} = \zeta^{\mathfrak{u}}(2n+1) = \mathrm{Li}^{\mathfrak{u}}_{2n+1}(1),$$

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  where $f_{\sigma_{2n+1}}$ corresponds to a generator of $K_{4n+1}(Z)$.
- Up to degree 4, $A(\mathbb{Z}[1/\ell])$ has the basis

$$\{1, f_{\tau_\ell}, f_{\tau_\ell^2}, f_{\tau_\ell^3}, f_{\sigma_3}, f_{\tau_\ell^4}, f_{\tau_\ell\sigma_3}, f_{\sigma_3\tau_\ell}\}.$$

- As $f_{\tau_\ell}^n = n! f_{\tau_\ell^n}$, we explicitly understand everything except the last two basis elements.
- As $f_{\tau_\ell} f_{\sigma_3} = f_{\tau_\ell\sigma_3} + f_{\sigma_3\tau_\ell}$, we need to understand only the last element.

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- As $1/2 \in X(Z)$, a natural element to try is $\mathrm{Li}_4^{\mathfrak{u}}(1/2)$.
- The coproduct formula expresses $\Delta' \mathrm{Li}_4^{\mathfrak{u}}(1/2)$ in terms of $\mathrm{Li}_3^{\mathfrak{u}}(1/2)$, which one must then write in terms of $f_{\sigma_3} = \zeta^{\mathfrak{u}}(3)$ and $f_{\tau_\ell^3} = \log^{\mathfrak{u}}(2)^3/6$.
- Using the coproduct formula and the fact that $\log^{\mathfrak{u}}(1/2) = -\log^{\mathfrak{u}}(2)$, one may check that

$$\Delta' \left(\mathrm{Li}_3^{\mathfrak{u}}(1/2) - \frac{(\log^{\mathfrak{u}}(2))^3}{6} \right) = 0.$$

- This implies that $\mathrm{Li}_3^{\mathfrak{u}}(1/2) - \frac{(\log^{\mathfrak{u}}(2))^3}{6}$ is a rational multiple of $f_{\sigma_3} = \zeta^{\mathfrak{u}}(3)$.

Finding $f_{\sigma_3\tau_2}$, cont.

- By p-adically approximating

$$\frac{\mathrm{Li}_3^p(1/2) - \frac{(\log^p(2))^3}{6}}{\zeta^p(3)}$$

in SAGE for various small p, one seems to get the value $7/8$.

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- The identity $\mathrm{Li}_3^{\mathfrak{u}}(1/2) = \frac{(\log^{\mathfrak{u}}(2))^3}{6} + \frac{7}{8}\zeta^{\mathfrak{u}}(3)$ is verified in the appendices of Dan-Cohen–Wewers using identities for polylogarithms.
- Using this identity, one may check that

$$\Delta'\left(-\frac{8}{7}\left(\frac{\log^{\mathfrak{u}}(2)^4}{24} + \mathrm{Li}_4^{\mathfrak{u}}(1/2)\right)\right) = \zeta^{\mathfrak{u}}(3) \otimes \log^{\mathfrak{u}}(2),$$

as desired.

Finding $f_{\sigma_3\tau_3}$

- In C–Dan-Cohen, we show that for $\ell = 3$, we have

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- To deal with this, we must consider the larger Hopf algebra $A(\mathbb{Z}[1/6])$.
- The abstract coordinates tell us how $A(\mathbb{Z}[1/3])$ sits inside $A(\mathbb{Z}[1/6])$. Indeed, $\mathrm{Li}_4^{\mathfrak{u}}(3), \mathrm{Li}_4^{\mathfrak{u}}(9) \in A(\mathbb{Z}[1/6]) \setminus A(\mathbb{Z}[1/3])$.

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- $A(\mathbb{Z}[1/6])$ is already more complicated in weight 2, as $\log^{\mathfrak{u}}(2)^2, \log^{\mathfrak{u}}(3)^2, \log^{\mathfrak{u}}(2)\log^{\mathfrak{u}}(3)$ do not span the weight 2 part. By computing its coproduct, one may show that $\mathrm{Li}_2^{\mathfrak{u}}(-2) = -f_{\tau_3 \tau_2}$.
- Similar to the $\ell = 2$ case, one must figure out the rational number $\frac{\mathrm{Li}_3^{\mathfrak{u}}(9) - 12\mathrm{Li}_3^{\mathfrak{u}}(3)}{\zeta^{\mathfrak{u}}(3)}$ by p-adic approximation; we have found that it is about $-\frac{26}{3}$.

Summary: What ingredients did we need?

In order to see how one might generalize to the elliptic case, let's review some key ingredients:

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2. We understand the abstract structure of its fundamental group $\pi_1^{\mathcal{MT}}(Z)$ as a pro-algebraic group (via the graded Hopf algebra $A(Z)$); this comes from our knowledge of algebraic K-theory of integer rings
3. We know how to write down explicit elements of $\mathcal{O}(\pi_1^{\mathrm{un}}(X_{\mathbb{Q}}))$ and have a formula for their coproducts; these are our $\mathrm{Li}^{\mathfrak{u}}_k$ (or more generally, $\mathrm{Li}^{\mathfrak{u}}_{k_1,\cdots,k_r}$)

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4. We know how to write down explicit elements of $\mathcal{O}(\pi_1^{\mathcal{MT}}(Z))$ and have a formula for their coproducts; these are our $\mathrm{Li}^{\mathfrak{u}}_k(z)$ for $z \in X(Z)$ (or more generally, $\mathrm{Li}^{\mathfrak{u}}_{k_1,\cdots,k_r}(z)$)
5. We know how to compute (up to p-adic approximation) p-adic periods of these elements and p-adic power series for the corresponding functions

1. Motives for Elliptic Curves

- The de Rham fundamental group of E' lies not in $\mathcal{MT}(Z)$, but in a larger category $\mathcal{ME}_{E'}(Z)$ of mixed motives generated by Tate motives and the Tate module $H_1(E')$ of E'
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- We know this category to exist only by assuming the Beilinson-Soulé Vanishing Conjecture; but we may still use candidate categories constructed by Owen Patashnick and others
- For a non-CM elliptic curve, the Tannakian fundamental group $\pi_1^{\mathrm{ME},E'}(Z)$ of $\mathcal{ME}_{E'}(Z)$ is an extension of GL_2 by a pro-unipotent group
- In particular, the coordinate ring $A^{E'}(Z)$ of this pro-unipotent group is "graded" by a GL_2-action
- To understand the pro-unipotent part, we must compute dimensions of motivic Ext groups, or equivalently, of algebraic K-theory, as described on the next slide

2. Algebraic K-Theory for Elliptic Curves

- For each irreducible representation V of GL_2, we need to know the dimension of $\mathrm{Ext}^1_{\mathcal{ME}_{E'}(Z)}(V, \mathbb{Q}(0))$
- For this, we must use the Bloch-Kato conjecture for powers of the elliptic curve E along with Euler characteristic formulas in Galois cohomology.

Conjecture (Bloch-Kato)

For a smooth projective variety $Z/\mathbb{Q}$, $n > 0$, and $2r \neq n+1$, the natural map

$$K_{2r-1-n}(Z)^{(r)}_{\mathbb{Q}_p} \to H^1_f(G_{\mathbb{Q}}, H^n_{\text{ét}}(Z_{\overline{\mathbb{Q}}}, \mathbb{Q}_p(r)))$$

is bijective.

- Note that $\mathrm{Ext}_{\mathcal{ME}_{E'}(Z)}(H^1(E), \mathbb{Q}(0))$ has dimension the rank of the Mordell-Weil group $E(\mathbb{Q})$.
- *In particular, it is an arithmetic invariant that depends heavily on which elliptic curve E we choose*

3. Functions on the de Rham Fundamental Group

- We need a way to explicitly write down regular functions on $\pi_1^{\mathrm{un}}(E'_{\mathbb{Q}}, O)$
- The story is much more complicated than for $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ because there is no canonical de Rham path (because unipotent vector bundles can be non-trivial)
- Therefore, we must restrict a quotient of $\pi_1^{\mathrm{un}}(E'_{\mathbb{Q}}, O)$ defined using the Hodge filtration
- Such a function should correspond to a function on E' defined by a (homotopy-invariant) iterated integral based at O
- The analogous object for $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is $\mathrm{Li}^{\mathfrak{u}}_{k_1, \cdots, k_r}(z)$, where z is viewed as a variable endpoint for the iterated integral (aka $\mathrm{Li}^{\mathfrak{u}}_{k_1, \cdots, k_r}$)
- We need to write them down in such a way that we can compute their coproducts and know which GL_2-isotypic component they lie in.

4. Functions on the Motivic Galois Group

- Once one has functions on the appropriate quotient of $\pi_1^{\mathrm{un}}(E'_{\mathbb{Q}}, O)$, we can input specific values of $z \in E'(Z)$ to get elements of $A^{E'}(Z)$
- We need a coproduct formula for these elements, analogous to the Ihara-Goncharov coproduct formula
- Patashnick's category of mixed elliptic motives is defined using a bar construction on Bloch's cycle complexes (similar to the approach of Bloch-Kriz for mixed Tate motives)
- Bloch-Kriz express multiple zeta values in terms of cycles, so one might also be able to write such elements down in terms of cycles on powers of E

- Given an iterated integral on E', there are methods due to Balakrishnan for computing its p-adic version

Useful References/Credits

The following are on arXiv:

- Mixed Tate Motives and the Unit Equation, Ishai Dan-Cohen and Stefan Wewers
- Mixed Tate Motives and the Unit Equation II, Ishai Dan-Cohen
- Single-Valued Motivic Periods, Francis Brown
- Motivic Periods and $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, Francis Brown
- Notes on Motivic Periods, Francis Brown
- Integral Points on Curves, the Unit Equation, and Motivic Periods, Francis Brown.
- The polylog quotient and the Goncharov quotient in computational Chabauty-Kim theory I, David Corwin and Ishai Dan-Cohen
- The polylog quotient and the Goncharov quotient in computational Chabauty-Kim theory II, David Corwin and Ishai Dan-Cohen

The elliptic case involves ongoing discussions with Ishai Dan-Cohen, Stefan Wewers, Owen Patashnick, and others.

Thank You!