TANNAKIAN SELMER VARIETIES

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Abstract. This article serves in part as an exposition and an overview of my joint work with Ishai Dan-Cohen.

This paper develops the theory of Tannakian Selmer varieties, which aims to render the explicit motivic Chabauty-Kim method developed in papers of Dan-Cohen–Wewers and Dan-Cohen and the author applicable to non-rational curves. We replace the Tannakian category of motives by a Tannakian category of p-adic Galois representations, whose definition (unlike that of motives in general) is non-conjectural, and then describe Selmer varieties in terms of these categories. We sketch how this might lead to a general method for computing rational points on hyperbolic curves.

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1. Introduction

1.1. **Extended Abstract.** This article serves in part as an exposition and an overview of my joint work with Ishai Dan-Cohen.

The main point of the paper is to adapt the explicit motivic Chabauty-Kim method developed in [DCW16] and [CDC20] to non-rational curves. The key insight is that while we don't have abelian categories of mixed motives in general, we can use (for the purposes of Chabauty-Kim) Tannakian categories of p-adic Galois representations in place of motives. Thus it might be appropriate to call this "explicit Tannakian Chabauty-Kim". Some important themes include:

- Use of the Bloch-Kato conjectures and Poitou-Tate duality for explicitly bounding dimensions of Selmer groups (§3.2, §7.3)
- Use of $\mathbb{Q}_p$-linear categories of Galois representations in place of $\mathbb{Q}$-linear motives (§4)
- A setup for Tannakian Selmer varieties (§5) and a description in terms of cocycles (Theorem A.4) generalizing [DCW16, Proposition 5.2]

In [Cor21], we study the *mixed elliptic* (c.f. Remark 4.9) case and calculate the abstract form of an element of the Chabauty-Kim ideal for $\mathbb{Z}[1/\ell]$-points on a punctured elliptic curve.

Much of the content of this paper originally appeared with the restriction $\dim A = 1$ in an earlier version of [Cor21], which now serves as a concrete realization of these ideas.

1.2. **The Problem of Effective Faltings.** Let X be a smooth proper curve of genus $g \geq 2$ over a number field k. The theorem of Faltings states that $X(k)$ is finite. A major open question is to find an algorithm for determining the finite set $X(k)$ given X/k.

More generally, the combination of the theorems of Faltings and Siegel imply that whenever X is a smooth curve with negative (geometric) Euler characteristic, and S is a finite set of places of k, we have $\mathcal{X}(\mathcal{O}_{k,S})$ finite, for an $\mathcal{O}_{k,S}$-model[i] $\mathcal{X}$ of X. This formalism includes the case of rational points, as $\mathcal{X}(\mathcal{O}_{k,S}) = X(k)$ whenever X is proper.[ii] We are thus interested in the general question of determining the set $\mathcal{X}(\mathcal{O}_{k,S})$ for X/k and a finite set S of places of k.

In practice, one may often conjecturally find the set $\mathcal{X}(\mathcal{O}_{k,S})$ by searching over points of bounded height. This produces a finite set of elements of $\mathcal{X}(\mathcal{O}_{k,S})$, and one hopes, after a dilligent enough search, that this is all of $\mathcal{X}(\mathcal{O}_{k,S})$. The challenge is in proving that one has found all of $\mathcal{X}(\mathcal{O}_{k,S})$.

1.3. **The Chabauty-Skolem Method.** Before Faltings' proof in 1983, the primary method for proving finiteness of $\mathcal{X}(\mathcal{O}_{k,S})$ was via the method of Chabauty-Skolem ([Cha41]). In the 1980's, Chabauty's method was upgraded to an effective method by Coleman ([Col85]), using his theory known as "Coleman integration." More specifically, using the generalized Jacobian[iii] $X \hookrightarrow J$, a basepoint $b \in \mathcal{X}(\mathcal{O}_{k,S})$,[iv] and $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_{k,S}$ with $k_{\mathfrak{p}} \cong \mathbb{Q}_p$, one constructs a diagram:

$$\begin{tikzcd}
\mathcal{X}(\mathcal{O}_{k,S}) \arrow[r] \arrow[d] & \mathcal{X}(\mathcal{O}_{\mathfrak{p}}) \arrow[d] \arrow[dr, "\int_{\mathfrak{p}}"] & \\
\mathcal{J}(\mathcal{O}_{k,S}) \arrow[r, "\mathrm{loc}"'] & \mathcal{J}(\mathcal{O}_{\mathfrak{p}}) \arrow[r, "\log"'] & \operatorname{Lie} J_{\mathbb{Q}_p}
\end{tikzcd}$$

for an appropriate integral model $\mathcal{J}$ of J when X is not proper.

By definition, $\operatorname{Lie} J_{k_{\mathfrak{p}}}$ is the tangent space to $J_{k_{\mathfrak{p}}}$ at the identity. When J is proper, it is the linear dual of $H^0(J_{k_{\mathfrak{p}}}, \Omega^1)$, and more generally, it is dual to the subspace $H^0(J_{k_{\mathfrak{p}}}, \Omega^1)^J$ of translation-invariant differential 1-forms. The map $\int_{\mathfrak{p}}$ sends $z \in \mathcal{X}(\mathcal{O}_{\mathfrak{p}})$ to the functional sending $\omega \in H^0(J_{k_{\mathfrak{p}}}, \Omega^1)^J$ to the Coleman integral

$$\int_b^z \omega.$$

If $r \coloneqq \operatorname{rank}_{\mathbb{Q}} J(k) < g$, then the image of $J(k)$ in $\operatorname{Lie} J_{k_{\mathfrak{p}}}$ lies in a vector subspace of positive codimension. Therefore, the annihilator $\mathcal{I}_J$ of $\operatorname{loc}(\mathcal{J}(\mathcal{O}_{k,S}))$ in $(\operatorname{Lie} J_{k_{\mathfrak{p}}})^{\vee} = H^0(J_{k_{\mathfrak{p}}}, \Omega^1)^J$ is nonzero, so its pullback $\int_{\mathfrak{p}}^{\#}(\mathcal{I}_J)$ is a nonzero set of functions on $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})$ that vanish on $\mathcal{X}(\mathcal{O}_{k,S})$. By the theory of Coleman, each nonzero $f \in \int_{\mathfrak{p}}^{\#}(\mathcal{I}_J)$ has finitely many zeroes, and the theory

[i]We define an $\mathcal{O}_{k,S}$-model as a finite type, separated, faithfully flat scheme $\mathcal{X}$ over $\mathcal{O}_{k,S}$ with an isomorphism $\mathcal{X}_k \to X$.

[ii]One might then prefer to replace $\mathcal{X}$ by X and $\mathcal{O}_{k,S}$ by k, but as discussed in §2.4.2 and §4.10, it is best to set $S = \emptyset$ and thus consider $\mathcal{O}_k$-points in the proper case.

[iii]This is characterized by the fact that J is a semi-Abelian variety and that the closed embedding $X \hookrightarrow J$ is an isomorphism on first homology.

[iv]More generally, an integral tangential basepoint is allowed when X is not proper.

of Newton polygons allows one to p-adically estimate the set $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_J$ of common zeroes of all $f \in \int_{\mathfrak{p}}^{\#}(\mathcal{I}_J)$. More details may be found in [MP12].

If $r \geq g$, one is usually out of luck with Chabauty's method. Moreover, even if $r < g$, $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_J \setminus \mathcal{X}(\mathcal{O}_{k,S})$ might be nonempty, and the fact that computations of zeroes are p-adic approximations means that one cannot then use $\int_{\mathfrak{p}}^{\#}(\mathcal{I}_J)$ to determine $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})$ (though it can be successful in conjunction with other methods; see [Poo02, §5.3]).

1.4. **Non-Abelian Chabauty's Method.** The non-abelian Chabauty's method of Minhyong Kim ([Kim05], [Kim09]), also known as Chabauty-Kim, allows one to remove this restriction. For $\mathcal{X}$ and $\mathfrak{p}$ as in §1.3, an integer n,[v] and a basepoint $b \in \mathcal{X}(\mathcal{O}_{k,S})$, Kim constructs a diagram ((4) for $\Pi = U_n$):

$$\begin{tikzcd} \mathcal{X}(\mathcal{O}_{k,S}) \arrow[r] \arrow[d, "\kappa"] & \mathcal{X}(\mathcal{O}_{\mathfrak{p}}) \arrow[d, "\kappa_{\mathfrak{p}}"'] \arrow[dr, "\int_{\mathfrak{p}}"] & \\ \mathrm{Sel}_{S,n}(\mathcal{X}) \arrow[r, "\mathrm{loc}_n"'] & \mathrm{Sel}_n(\mathcal{X}/\mathcal{O}_{\mathfrak{p}}) \arrow[r, "\sim", "\log_{\mathrm{BK}}"'] & U_n^{\mathrm{dR}}/F^0U_n^{\mathrm{dR}} \end{tikzcd} \tag{1}$$

The set $\mathrm{Sel}_{S,n}(\mathcal{X})$ is a global non-abelian Galois cohomology set of the form $H^1_{f,S}(G_k; U_n)$ or a finite union thereof,[vi] where f refers to the Selmer conditions of Bloch-Kato, and U_n is the nth descending central series quotient of the $\mathbb{Q}_p$-unipotent geometric fundamental group of X (based at b). More details may be found in [Cor20, §4].

For $n = 1$, this is essentially the same as the diagram in classical Chabauty's method. More precisely, $\mathrm{Sel}_1(\mathcal{X})$ is the p-adic Selmer group of J, and we have an embedding

$$\kappa_J \colon \mathcal{J}(\mathcal{O}_{k,S}) \otimes \mathbb{Q}_p \hookrightarrow \mathrm{Sel}_1(\mathcal{X})$$

that is conjecturally (by finiteness of $Ш(J)$) an isomorphism, and verifiably so in practice.

We define the *Chabauty-Kim locus*:

$$\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_n := \kappa_{\mathfrak{p}}^{-1}(\mathrm{Im}(\mathrm{loc}_n)) = \int_{\mathfrak{p}}^{-1}(\mathrm{Im}(\log_{\mathrm{BK}} \circ \mathrm{loc}_n)).$$

The *Chabauty-Kim ideal*

$$\mathcal{I}_{CK,n} = \mathcal{I}_{CK,n}(\mathcal{X})$$

of regular functions vanishing on the image of $\log_{\mathrm{BK}} \circ \mathrm{loc}_n$ pulls back to a set $\int_{\mathfrak{p}}^{\#}(\mathcal{I}_{CK,n})$ of functions on $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})$ vanishing on $\mathcal{X}(\mathcal{O}_{k,S})$ with $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_n$ defined as its set of common zeroes. As long as κ_J is an isomorphism, $\mathcal{I}_{CK,1}$ is the ideal generated by $\mathcal{I}_J$, and $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_1 = \mathcal{X}(\mathcal{O}_{\mathfrak{p}})_J$. For $n \geq 1$,

$$\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_{n+1} \subseteq \mathcal{X}(\mathcal{O}_{\mathfrak{p}})_n.$$

When

$$\dim_{\mathbb{Q}_p} \mathrm{Sel}_{S,n}(\mathcal{X}) < \dim_{\mathbb{Q}_p} \mathrm{Sel}_n(\mathcal{X}/\mathcal{O}_{\mathfrak{p}}), \tag{2}$$

[v] Often called the *depth*, although this conflicts with the notion of *depth* in the theory of multiple zeta values, so we prefer the term *level*.

[vi] Technically, one must either expand S to include all places of bad reduction of X as in [Kim09] or take a finite union of twists as in Definition 2.6.

the set $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_n$ is finite, a consequence of the fact that $\kappa_{\mathfrak{p}}$ has Zariski dense image ([Kim09, Theorem 1]).

Kim shows ([Kim09, Theorem 2]) that this inequality holds for sufficiently large n if a part of the Bloch-Kato Conjecture (see Conjecture 3.5 below) holds.

The following appears as [BDCKW18, Conjecture 3.1] for $S = \emptyset$ and in [BDCKW18, §8] as a remark about what one "might conjecture":

Conjecture 1.1 (Kim's Conjecture). *For $k = \mathbb{Q}$, a regular minimal model*[vii] *$\mathcal{X}$ and n sufficiently large, we have*

$$\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_n = \mathcal{X}(\mathcal{O}_{k,S}).$$

Remark 1.2. It is mentioned in [BDCKW18, Remark 3.2] that there should be a suitable generalization to all number fields k. One might expect such a conjecture to follow from the recent ideas of [Dog20], although no such conjecture is contained therein.

The intuition behind this conjecture is that a random p-adic analytic function should not vanish at a given point unless it has a very good reason to.

This conjecture implies that if we can compute $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_n$ up to arbitrary p-adic precision for all n, then there is an effective version of Faltings' Theorem. More precisely, if we have a subset F of $\mathcal{X}(\mathcal{O}_{k,S})$, then to check that $F = \mathcal{X}(\mathcal{O}_{k,S})$, we need only find some n for which $|\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_n| = |F|$. Given a collection of p-adic analytic functions, the theory of Newton polygons then allows us to determine the total number of common zeroes.[viii] Thus effective Faltings over $\mathbb{Q}$ is reduced modulo Conjecture 1.1 to the problem of computing, up to sufficient p-adic precision, the set of functions on U_n/F^0U_n that vanish on the image of $\log_{\mathrm{BK}} \circ \mathrm{loc}_n$.

1.5. **Quadratic Chabauty.** The most successful method to-date for computing with non-abelian Chabauty is the Quadratic Chabauty method of Balakrishnan et al ([BBM16], [BD18a]). This method essentially computes part of loc_2 using the observation of Kim that a certain coordinate of $\mathrm{Sel}_2(\mathcal{X})$ corresponds to the p-adic height pairing on J, while a similar coordinate of $\mathrm{Sel}_(\mathcal{X}/\mathcal{O}_{\mathfrak{p}})$ corresponds to the local component at $\mathfrak{p}$ of the p-adic height pairing. (More precisely, [BBM16] covers the case of integral points on affine curves using the Coleman-Gross p-adic height pairing, while [BD18a] covers the case of rational points on proper curves using a whole slew of pairings defined relative to a certain kind of divisor on $X \times X$.)

In particular, let $\rho := \mathrm{rank}_{\mathbb{Q}} \mathrm{NS}(J)$, with X proper. Then [BD18a, Lemma 3.2] shows that $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_2$ is finite whenever

$$r < g + \rho - 1,$$

and moreover, [BD18a, §8] gives a method to p-adically approximate a set containing $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_2$ in this case. In [BD18b], the authors relax the condition (partly dependent on Conjecture 3.5), but still restrict to the case $n = 2$.

[vii] Let g denote the genus of the smooth projective closure $\overline{X}$ of X. Then $\mathcal{X}$ is a *regular minimal model* if it is the complement of a reduced horizontal divisor in the regular minimal model $\overline{\mathcal{X}}$ of $\overline{X}$ over $\mathcal{O}_{k,S}$ (resp. in $\mathbb{P}^1/\mathcal{O}_{k,S}$) when $g \geq 1$ (resp. when $g = 0$).

[viii] Technically it allows only to determine the number of zeroes *with multiplicity*, but a stronger version of Conjecture 1.1 resolves this by stating that $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_n$ as an analytic space is reduced for sufficiently large n.

1.6. **Explicit Motivic Non-Abelian Chabauty.** The only computed cases of Chabauty-Kim for $n > 2$ are for S-integral points on $\mathcal{X} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. More specifically, the case of $S = \emptyset, \{2\}$ with $n = 4$ appeared in [DCW16], while $S = \{3\}$, $n = 4$ appeared in [CDC20]. Both cases were also done in [Bro17a].

To compute $\mathcal{X}(\mathcal{O}_\mathfrak{p})_n$, one needs to understand $\mathrm{Sel}(\mathcal{X})_n$ and the map loc_n concretely. As mentioned above, (the set of $\mathbb{Q}_p$-points of) $\mathrm{Sel}(\mathcal{X})_n$ is (modulo technicalities described in §2.4)

$$H^1_{f,S}(G_k; U_n),$$

the set of cohomology classes of G_k with coefficients in U_n that are crystalline at $v \in \{p\}_k$ and unramified outside $S \cup \{p\}_k$. Both the group G_k and the local conditions are hard to understand explicitly.

We now explain the method of [DCW16, CDC20] from the viewpoint of this paper. The key insight is that one need understand only the *category* of continuous p-adic representations of G_k that appear in U_n and its torsors. This category is Tannakian, and $H^1_{f,S}(G_k; U_n)$ may be described as group cohomology of the Tannakian fundamental group with coefficients in U_n.

In the case of $\mathcal{X} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, the relevant category is the category

$$\mathbf{Rep}^{\mathbf{MT}}_{\mathbb{Q}_p}(\mathcal{O}_{k,S})$$

of mixed Tate geometric Galois representations with good reduction over $\mathcal{O}_{k,S}$. Its Tannakian fundamental group

$$\pi_1^{\mathbf{MT}}(\mathcal{O}_{k,S})$$

is isomorphic to an extension of $\mathbb{G}_m$ by a pro-unipotent group. The pro-unipotent group may be determined by computing the Bloch-Kato Selmer groups $H^1_f(G_k; \mathbb{Q}_p(n))$ for each n, and these are known ([Sou79]). In this way, the set

$$H^1_{f,S}(G_k; U_n),$$

becomes simply the group cohomology set

$$H^1(\pi_1^{\mathbf{MT}}(\mathcal{O}_{k,S}); U_n),$$

with no further local conditions other than those encoded in the group $\pi_1^{\mathbf{MT}}(\mathcal{O}_{k,S})$ itself.

Remark 1.3. To explain the relationship to [DCW16, CDC20], we note that $\mathbf{Rep}^{\mathbf{MT}}_{\mathbb{Q}_p}(\mathcal{O}_{k,S})$ has a $\mathbb{Q}$-form

$$\mathbf{MT}(\mathcal{O}_{k,S}, \mathbb{Q}),$$

the category of mixed Tate motives over $\mathcal{O}_{k,S}$ with coefficients in $\mathbb{Q}$. This latter category was defined in [DG05] and first applied to Selmer varieties in [Had11] and [DCW16]. The group $\pi_1^{\mathbf{MT}}(\mathcal{O}_{k,S})$ is $\pi_1(\mathbf{MT}(\mathcal{O}_{k,S}, \mathbb{Q}))$, while the analogue of what we do in this paper is its tensorization with $\mathbb{Q}_p$.

As such a $\mathbb{Q}$-linear category is only conjectural in the mixed elliptic case (see [Pat13]), we work with categories of Galois representations. While it is in some ways nicer to work over $\mathbb{Q}$, it is not necessary, as the end result of the Chabauty-Kim method is still p-adic.

1.7. **This Work: Tannakian Selmer Varieties.** The main goal of this paper is to extend the methods of [DCW16] and [CDC20] (see also [DC20] and [DCC20]) to arbitrary curves. We develop general foundations that we expect to apply to all curves, leaving explicit computations to future work.

More precisely, we replace $\mathbf{Rep}^{\mathbf{MT}}_{\mathbb{Q}_p}(\mathcal{O}_{k,S})$ with a category (Definition 4.7):

$$\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, A)$$

of S-integral mixed Abelian Galois representations for a fixed Abelian variety A; i.e., iterated extensions of Galois representations appearing in tensor powers of the Tate module

$$h_1(A) := H_1^{\text{ét}}(A_{\overline{k}}; \mathbb{Q}_p)$$

of A. The fundamental group (Definition 5.9)

$$\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A) := \pi_1(\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, A))$$

of this Tannakian category is now an extension of a a reductive group $\mathbb{G} = \mathbb{G}(A)$ by a pro-unipotent group $U(\mathcal{O}_{k,S}, A)$, the latter of which is determined (§5.5) by the Bloch-Kato Selmer groups

$$H^1_{f,S}(G_k; M),$$

for $M \in \mathrm{Irr}(\mathbf{Rep}_{\mathbb{Q}_p}(\mathbb{G}))$.

Assuming the Bloch-Kato conjectures, one may explicitly compute these ranks using Fact 3.8 as described in §7.3, which can be used to explicitly bound the dimension of the level n Selmer variety of an arbitrary curve X. In some cases, one may use Iwasawa theory to prove specific cases of these conjectures.

We explain the basic setup of Selmer varieties in the Tannakian formalism in §5-6. Most notably, we prove an explicit description of cohomology sets, generalizing [DCW16, Proposition 5.2], in Theorem A.4. This is necessary in order to carry out explicit motivic Chabauty-Kim in general.

In [Cor21], we use the general setup to compute an element of $\mathcal{I}_{CK,n}(\mathcal{X})$ for $n = 3$ and $\mathcal{X}$, $\mathcal{O}_{k,S} = \mathbb{Z}[1/\ell]$, and $\mathcal{X}$ the punctured minimal Weierstrass model of an elliptic curve, as follows:

Theorem 1.4 ([Cor21, Theorem 10.1])**.** *Let $\mathcal{E}$ be the minimal Weierstrass model of an elliptic curve E over $\mathbb{Q}$ with p-Selmer rank* 1*, let α denote a choice of component of the Néron model of E at each place of $\mathbb{Q}$, and let $S = \{\ell\}$ for some prime $\ell \neq p$. Then assuming Conjecture 3.5 for $h^1(A)$, there is a function of the form*

$$c_1 J_4 + c_2 J_3 + c_3 J_1 J_2 + c_4 J_1^3 + c_5 J_1$$

vanishing on the subset $\mathcal{E}'(\mathbb{Z}[1/S])_\alpha$ of $\mathcal{E}'(\mathbb{Z}[1/S])$ reducing to α at each bad prime of $\mathcal{E}$, in which:

- *The $c_i \in \mathbb{Q}_p$ arising as periods of elements of $\mathcal{O}(W)$ (c.f. §6), not all of which are zero, and*
- *The J_i are explicit iterated integrals on E' defined in §*[Cor21, ??].

1.7.1. *Paper Outline.* In §2, we define the diagram (1) for a general Galois-equivariant quotient Π of $U(X)$, with an emphasis on the relationship between $\mathrm{Sel}_{S,\Pi}(\mathcal{X})$ and $H^1_{f,S}(G_k;\Pi)$ as described in [BD20].

§3-§5 together describe $H^1_{f,S}(G_k;\Pi)$ in Tannakian terms. In §3, we review some background on Galois representations and Bloch-Kato Selmer groups. This includes the statement of the part of the Bloch-Kato Conjecture we need (Conjecture 3.5). In §4, we define $\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k,A)$ and related objects. In §5, we define a Tannakian fundamental group $\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S},A)$ and write Selmer varieties in terms of it. We also explain the relevance of an analogue of [DCW16, Proposition 5.2], proven in the appendix (Theorem A.4).

In §6, we explain how to understand the composition $\log_{\mathrm{BK}}\circ\mathrm{loc}_\Pi$ more explicitly. For this, we describe a universal cocycle evaluation map (§6.1) and a p-adic period map (§6.2). Our main result is that functions vanishing on the image of a more accessible map $\mathfrak{ev}^I_{\Pi,F_{k,S,A}}/F^0$ pullback to functions vanishing on the image of $\log_{\mathrm{BK}}\circ\mathrm{loc}_\Pi$.

In §7, we explain a general procedure for determining the semisimplification of U_n. We then explain how this may be used to bound the dimension of the Selmer variety.

In §8, we explain how one might put specific coordinates on the objects and maps of (4) and compute the image of $\log_{\mathrm{BK}}\circ\mathrm{loc}_\Pi$ and outline how a future algorithm might look.

1.8. **Notation.** When we use the term "motive", we are thinking of a system of realizations (as in [BK90, Definition 5.5]), but working primarily with the p-adic Galois representation realization. Thus when we say "pure motive," we mean "semisimple p-adic Galois representation." We use "$\{p\}_k$" to denote the set of places k above a place p of $\mathbb{Q}$; to justify this notation when p is not inert in k, one may think of it as the subscheme of $\mathcal{O}_k$ defined by $\{p=0\}$. We use H^1_f and H^1_g as in [BK90] and [FPR94], recalled also in §3.2. We write $H^1_{f,S}$ for the subset of H^1_g unramified/crystalline at all $v\notin S$.

For a p-adic representation V of G_k, we write

$$h^i(G_k;V) := \dim H^i(G_k;V)$$

and

$$h^i_\bullet(G_k;V) := \dim H^i_\bullet(G_k;V)$$

for k a local or global field and $\bullet\in\{f,g\}$ or k a global field and $\bullet=f,S$.

We always work over a number field k and with a chosen rational prime p. We write G_k for the absolute Galois group of k. If v is a place of k, then k_v denotes the completion of k at v, $\mathcal{O}_{k_v}$ its integer ring, G_v its absolute Galois group, and I_v the inertia subgroup. We write $\mathcal{O}_{k,S}$ for the subset of k that is integral at all $v\notin S$. For a set T of places, we write $G_{k,T}$ for the Galois group of the maximal extension of k unramified outside $T\cup\{\infty\}$.

If Y is a variety over k and i a non-negative integer, we let $h^i(Y)$ (resp. $h_i(Y)$) denote the continuous p-adic representation of G_k given by $H^i_{\text{ét}}(Y_{\overline{k}};\mathbb{Q}_p)$ (resp. $H_i^{\text{ét}}(Y_{\overline{k}};\mathbb{Q}_p)$). We let $h^\bullet(Y)$ (resp. $h_\bullet(Y)$) denote $\bigoplus_{i\geq 0}h^i(Y)$ (resp. $\bigoplus_{i\geq 0}h_i(Y)$).

For a smooth geometrically irreducible variety Y over k, we let

$$U(Y) := \pi_1^{\text{ét,un}}(Y_{\overline{k}})_{\mathbb{Q}_p},$$

the $\mathbb{Q}_p$ pro-unipotent completion of the geometric étale fundamental group of Y. It has a continuous algebraic action of G_k, and $U(Y)^{ab}\cong h_1(Y)$. We always take it relative to basepoint $b\in Y(k)$, but we suppress this basepoint in the notation.

For a smooth curve X, we let $\overline{X}$ denote its smooth compactification, and $D_X := \overline{X}\setminus X$.

We use "$*$" to denote the point, in the context of objects of a pointed category.

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2. Selmer Varieties

We use this section to precisely specify our notation and definitions and to review the differing notions of Selmer variety or scheme in the literature. Almost none of the material here is new.

Let k be a number field, X/k, Π a finite-dimensional Galois-equivariant quotient of $U = U(X)$ based at $b \in X(k)$, and $\mathfrak{p} \in \{p\}_k$ for which $k_v \cong \mathbb{Q}_p$ at which X has good reduction.

There are local and global unipotent Kummer maps[ix] fitting into a diagram

$$\begin{tikzcd} X(k) \arrow[r] \arrow[d, "\kappa"] & X(k_{\mathfrak{p}}) \arrow[d, "\kappa_v"] \\ H^1(G_k;\Pi) \arrow[r, "\mathrm{loc}_{\mathfrak{p}}"'] & H^1(G_{\mathfrak{p}};\Pi). \end{tikzcd} \tag{3}$$

2.1. **Selmer Schemes as Schemes.** Let $G = G_v$ or $G = G_{k,T}$ for a finite place v of k or a finite set T of places of k, and let Π be a unipotent group over $\mathbb{Q}_p$ with continuous action of G.

It is proven in [Kim05, Proposition 2], [Kim09, §2], and [BD20, Remark 2.2.7] that

$$H^1(G;\Pi)$$

is naturally the set of $\mathbb{Q}_p$-points of a scheme over $\mathbb{Q}_p$ sending a $\mathbb{Q}_p$-algebra R to $H^1(G;\Pi_R)$. This scheme has the following important properties:

- $H^1(G;\Pi)$ is the affine space underlying the $\mathbb{Q}_p$-vector space $H^1(G;\Pi)$ when Π is abelian,

[ix]Also called *unipotent Albanese maps* in some parts of the literature, such as [Kim09] and [KT08].

- When $1 \to \Pi' \to \Pi \to \Pi'' \to 0$ is a short exact sequence of unipotent groups with G-action,
$$* \to H^1(G;\Pi') \to H^1(G;\Pi) \to H^1(G;\Pi'') \to *$$
is a short exact sequence of pointed schemes,
- The map $\mathrm{loc}_v\colon H^1(G_{k,T};\Pi) \to H^1(G_v;\Pi)$ is a map of $\mathbb{Q}_p$-schemes, and
- The subspaces $H^1_g(G_v;\Pi)$ and $H^1_f(G_v;\Pi)$ (Definition 2.2) are subschemes of $H^1(G_v;\Pi)$.

We will not be particularly careful in our notation about the difference between a Selmer scheme and its set of $\mathbb{Q}_p$-points; see Remark 2.4 for technical comments on this.

2.2. **Good Reduction and Local Conditions.**

Definition 2.1. We define good reduction as follows:

- For a place v of k, we say that X *has good reduction at* v if there is a model $\mathcal{X}$ of X over $\mathcal{O}_v$ sitting inside a smooth proper curve $\overline{\mathcal{X}}$ over $\mathcal{O}_v$ with étale boundary divisor.[x] We say more precisely that the *model* $\mathcal{X}$ has good reduction at v.
- We say that X has *potentially good reduction* at v if there is a finite extension l_v/k_v for which X_{l_v} has good reduction at v. We say more precisely that the model $\mathcal{X}$ has potentially good reduction at v if $\mathcal{X}_{\mathcal{O}_{l_v}}$ is dominated[xi] by a good model of X_{l_v}.

Definition 2.2. We set
$$H^1_f(G_v;\Pi) := \mathrm{Ker}(H^1(G_v;\Pi) \to H^1(I_v;\Pi))$$
$$H^1_g(G_v;\Pi) := H^1(G_v;\Pi)$$
for $v \notin \{p\}_k$ and
$$H^1_f(G_v;\Pi) := \mathrm{Ker}(H^1(G_v;\Pi) \to H^1(G_v;\Pi \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}))$$
$$H^1_g(G_v;\Pi) := \mathrm{Ker}(H^1(G_v;\Pi) \to H^1(G_v;\Pi \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}))$$
for $v \in \{p\}_k$.

Fact 2.3. *If $v \notin \{p\}_k$, then*

(1) ([BD20, Corollary 2.1.9]*)*
$$H^1_f(G_v;V) = *$$

(2) ([BD20, Proposition 1.2]*) if $\mathcal{X}$ has potentially good reduction at v, then*
$$\kappa_v(\mathcal{X}(\mathcal{O}_v)) = H^1_f(G_v;V) = *$$

If $v \in \{p\}_k$, then

(3) ([Kim09, Theorem 1]*) if $\mathcal{X}$ has good reduction at v, then*
$$\kappa_v(\mathcal{X}(\mathcal{O}_v))^{\mathrm{Zar}} = H^1_f(G_v;V).$$

[x] As an example of why the condition on the divisor is necessary, $\mathbb{P}^1 \setminus \{0,1,2,\infty\}$ has bad reduction at 2 even though it has a smooth model over $\mathbb{Z}_2$.

[xi] In this context, "dominate" means that the good model is the complement in a blowup of an integral compactification of the strict transform of the boundary. In particular, the good model does not necessarily map to $\mathcal{X}_{\mathcal{O}_{l_v}}$ as a scheme over $\mathcal{O}_{l_v}$.

If $\mathcal{X}$ has good reduction at v, the action of G_v on Π is unramified if $v \notin \{p\}_k$ and crystalline if $v \in \{p\}_k$.

We set

$$\mathrm{Sel}_\Pi(\mathcal{X}/\mathcal{O}_\mathfrak{p}) := H^1_f(G_\mathfrak{p}; \Pi).$$

As Π is crystalline at $\mathfrak{p}$, there is a unipotent group $\Pi^{\mathrm{dR}} := \mathrm{D}_{\mathrm{cris}}(\Pi)$ in the Tannakian category of (admissible) filtered ϕ-modules, while every torsor T representing an element of $H^1_f(G_\mathfrak{p}; \Pi)$ corresponds to a torsor $T^{\mathrm{dR}} := \mathrm{D}_{\mathrm{cris}}(T)$ under $\mathrm{D}_{\mathrm{cris}}(\Pi)$ in this category. Then T^{dR} has a unique ϕ-invariant element p_T^{cr}, and after choosing $p_T^H \in F^0 T^{\mathrm{dR}}$, we get a well-defined class

$$[(p_T^{\mathrm{cr}})^{-1} p_T^H] \in \Pi^{\mathrm{dR}}/F^0\Pi^{\mathrm{dR}}.$$

This association defines a map

$$\log_{\mathrm{BK}} \colon \mathrm{Sel}_\Pi(\mathcal{X}/\mathcal{O}_\mathfrak{p}) \to \Pi^{\mathrm{dR}}/F^0\Pi^{\mathrm{dR}},$$

which is an isomorphism by [Kim09, Proposition 1]. The importance of $\log_{\mathrm{BK}}$ is that map $\int_\mathfrak{p} := \log_{\mathrm{BK}} \circ \kappa_\mathfrak{p}$ of (4) may be computed explicitly via iterated Coleman integrals.

2.3. **The Chabauty-Kim Diagram.** Let S denote a finite set of places of k not containing $\mathfrak{p}$. For a model $\mathcal{X}$ of X over $\mathcal{O}_{k,S}$ with good reduction at $\mathfrak{p}$ and $b \in \mathcal{X}(\mathcal{O}_{k,S})$, we will describe in §2.4 a Selmer variety $\mathrm{Sel}_{S,\Pi}(\mathcal{X})$ fitting into a diagram

$$\begin{tikzcd}
\mathcal{X}(\mathcal{O}_{k,S}) \arrow[r] \arrow[d, "\kappa"] & \mathcal{X}(\mathcal{O}_\mathfrak{p}) \arrow[d, "\kappa_\mathfrak{p}"'] \arrow[dr, "\int_\mathfrak{p}"] & \\
\mathrm{Sel}_{S,\Pi}(\mathcal{X}) \arrow[r, "\mathrm{loc}_\Pi"'] & \mathrm{Sel}_\Pi(\mathcal{X}/\mathcal{O}_\mathfrak{p}) \arrow[r, "\sim", "\log_{\mathrm{BK}}"'] & \Pi^{\mathrm{dR}}/F^0\Pi^{\mathrm{dR}}
\end{tikzcd} \tag{4}$$

where loc_Π always refers to $\mathrm{loc}_\mathfrak{p}$ for the chosen place $\mathfrak{p} \in \{p\}_k$ with respect to the fundamental group quotient Π, and

$$\mathrm{Sel}_\Pi(\mathcal{X}/\mathcal{O}_\mathfrak{p}) := H^1_f(G_\mathfrak{p}; \Pi).$$

Remark 2.4. Assuming $\mathcal{X}(\mathcal{O}_{k,S})$ is finite, we may view it as a (rigid) analytic space (a finite union of copies of $\mathrm{Spm}\,\mathbb{Q}_p$), and the entire diagram is a commutative diagram of analytic spaces, while loc_Π and $\log_{\mathrm{BK}}$ are induced by maps of $\mathbb{Q}_p$-schemes. Without this assumption (e.g., if one wants to prove finiteness), we may view κ as a map from $\mathcal{X}(\mathcal{O}_{k,S})$ to the set $\mathrm{Sel}_{S,\Pi}(\mathcal{X})(\mathbb{Q}_p)$, while the other maps are of rigid spaces.

We set the *Chabauty-Kim locus*:

$$\mathcal{X}(\mathcal{O}_\mathfrak{p})_\Pi := \kappa_\mathfrak{p}^{-1}(\mathrm{Im}(\mathrm{loc}_\Pi)) = \int_\mathfrak{p}^{-1}(\mathrm{Im}(\log_{\mathrm{BK}} \circ \mathrm{loc}_\Pi))$$

as an analytic subspace of $\mathcal{X}(\mathcal{O}_\mathfrak{p})$.

The *Chabauty-Kim ideal*

$$\mathcal{I}_{CK,\Pi} = \mathcal{I}_{CK,\Pi}(\mathcal{X})$$

of regular functions vanishing on the image of loc_Π pulls back to a set $\kappa_\mathfrak{p}^\#(\mathcal{I}_{CK,\Pi})$ of functions on $\mathcal{X}(\mathcal{O}_\mathfrak{p})$ vanishing on $\mathcal{X}(\mathcal{O}_{k,S})$ with $\mathcal{X}(\mathcal{O}_\mathfrak{p})_\Pi$ as its locus of common zeroes.

For $\Pi = U_n$, we write $\mathrm{Sel}_{S,n}(\mathcal{X})$, $\mathrm{Sel}_n(\mathcal{X}/\mathcal{O}_\mathfrak{p})$, $\mathcal{I}_{CK,n}(\mathcal{X})$, and $\mathcal{X}(\mathcal{O}_\mathfrak{p})_n$.

As described in [Kim09, p.96], when the *dimension inequality*

$$\dim \mathrm{Sel}_{S,\Pi}(\mathcal{X}) < \dim \mathrm{Sel}_{\Pi}(\mathcal{X}/\mathcal{O}_{\mathfrak{p}}) \tag{5}$$

holds, we may conclude that loc_{Π} is non-dominant and therefore that

$$\mathcal{I}_{CK,\Pi}(\mathcal{X})$$

is nonzero, hence by [Kim09, Theorem 1] that

$$\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_{\Pi}$$

is finite. The statement of [Kim09, Theorem 2] is that this happens for $k = \mathbb{Q}$ and n sufficiently large if Conjecture 3.5 is true.

In §7.3, we discuss how to check (5).

2.4. **Global Selmer Varieties.** Let T_0 denote the set of places at which X does not have potentially good reduction, and let $T' = S \cup \{p\}_k \cup T_0$. Let T_1 denote the set of places at which X has bad but potentially good reduction, and let $T = T' \cup T_1$. Then X has good reduction outside T, so the action of G_k on Π factors through $G_{k,T}$, and our Selmer varieties will be subvarieties of

$$H^1_g(G_{k,T}; \Pi) := \{\alpha \in H^1(G_{k,T}; \Pi) \mid \mathrm{loc}_v(\alpha) \in H^1_g(G_v; \Pi)\, \forall\, v\}.$$

We first discuss the case in which $T_0 \subseteq S$, which is much simpler and already applies for example to the case of $\mathbb{Z}[1/2]$-points on the elliptic curve "128a2" as described in [Cor21, §12.1]. The general case is in §2.4.2.

2.4.1. *Good Reduction Outside S.*

Definition 2.5. We suppose $\mathcal{X}$ has potentially good reduction at all $v \in \operatorname{Spec} \mathcal{O}_{k,S}$, i.e., that $T_0 \subseteq S$. As in [Kim09], we define

$$\mathrm{Sel}_{S,\Pi}(\mathcal{X}) := H^1_{f,S}(G_k; \Pi) := \{\alpha \in H^1_g(G_{k,T}; \Pi) \mid \mathrm{loc}_v(\alpha) \in H^1_f(G_v; \Pi)\, \forall\, v \notin S\}.$$

If $T_0 \nsubseteq S$, we may expand S to $S' = S \cup T_0$.[xii] We may also modify p and $\mathfrak{p}$ to ensure that $S' \cap \{p\}_k = \emptyset$. One might then hope to apply the Chabauty-Kim method to compute $\mathcal{X}(\mathcal{O}_{k,S'})$, then find the subset $\mathcal{X}(\mathcal{O}_{k,S}) \subseteq \mathcal{X}(\mathcal{O}_{k,S'})$ by hand. This is the approach of [Kim09],[xiii] which is enough to show that Conjecture 3.5 implies finiteness of $\mathcal{X}(\mathcal{O}_{k,S})$ when $k = \mathbb{Q}$.

2.4.2. *Bad Reduction Outside S.* Nonetheless, it is often more practical to work with a Selmer scheme $\mathrm{Sel}_{S,\Pi}(\mathcal{X})$ defined even when $\mathcal{X}/\mathcal{O}_{k,S}$ has permanent bad reduction at some $v \in \operatorname{Spec} \mathcal{O}_{k,S}$. This is because we often have

$$\dim \mathrm{Sel}_{S,\Pi}(\mathcal{X}) < \dim \mathrm{Sel}_{S',\Pi}(\mathcal{X}),$$

which means in practice that one may need to pass to a larger Π to get finiteness for the Chabauty-Kim locus from $\mathrm{Sel}_{S',\Pi}(\mathcal{X})$ than from $\mathrm{Sel}_{S,\Pi}(\mathcal{X})$. This is especially true when X is proper, for which one hopes to set $S = \emptyset$.

We now recall the definition of Selmer variety from [BDCKW18, 2.7]:

[xii] If one is worried about having a single model $\mathcal{X}$ over $\mathcal{O}_{k,S'}$ with potentially good reduction, or even good reduction, one may, by spreading out, expand S' to ensure this is the case.

[xiii] In [Kim09], it is further assumed that $\mathcal{X}$ has good reduction at all $v \in \operatorname{Spec} \mathcal{O}_{k,S}$, but this does not appear necessary.

Definition 2.6. Given a model $\mathcal{X}$ of X over $\mathcal{O}_{k,S}$, we define

$$\mathrm{Sel}_{S,\Pi}(\mathcal{X}) := \{\alpha \in H^1_g(G_{k,T};\Pi) \mid \alpha \in \kappa_v(\mathcal{X}(\mathcal{O}_v))^{\mathrm{Zar}} \,\forall\, v \notin S\},$$

where superscript Zar denotes Zariski closure.

Remark 2.7. The *refined Selmer scheme* of [BD20, Definition 1.2.2] is defined by the additional condition that $\alpha \in \kappa_v(\mathcal{X}(k_v))^{\mathrm{Zar}} \,\forall\, v \in S$.

Then κ of (3) restricts to a map

$$\kappa\colon \mathcal{X}(\mathcal{O}_{k,S}) \to \mathrm{Sel}_{S,\Pi}(\mathcal{X}).$$

It follows from Fact 2.3 that Definition 2.6 agrees with Definition 2.5 when $\mathcal{X}$ has potentially good reduction at all $v \in \operatorname{Spec} \mathcal{O}_{k,S}$ and good reduction at all $v \in \{p\}_k$.

In general, $\mathrm{Sel}_{S,\Pi}(\mathcal{X})$ may differ from $H^1_{f,S}(G_k;\Pi)$ when $T_0 \setminus S \neq \emptyset$. This is because of the fact that

$$\kappa_v(\mathcal{X}(\mathcal{O}_v))$$

may be nontrivial for $v \in T_0 \setminus S$.

Nonetheless, when $v \notin \{p\}_k$, the image is finite by [KT08, Corollary 0.2]. We therefore assume that $\mathcal{X}$ has good reduction at all $v \in \{p\}_k$ (previously we assumed this only for $v = \mathfrak{p}$), i.e. that $(T_0 \cup T_1) \cap \{p\}_k = \emptyset$. This may be arranged by an appropriate choice of p.

We now explain, under the assumptions above (that $\{p\}_k$ is disjoint from $S \cup T_0 \cup T_1$), why the Selmer variety

$$\mathrm{Sel}_{S,\Pi}(\mathcal{X})$$

is a disjoint union of copies of

$$H^1_{f,S}(G_k;\Pi)$$

indexed by the (finite) image of the map

$$\mathrm{Sel}_{S,\Pi}(\mathcal{X}) \to \prod_{v \in T_0 \setminus S} \kappa_v(\mathcal{X}(\mathcal{O}_v)) = \prod_{v \in T_0 \setminus S} \kappa_v(\mathcal{X}(\mathcal{O}_v))^{\mathrm{Zar}}.$$

As in [BD18a] and [BD18b], let $\alpha_1, \cdots, \alpha_N$ denote a set of representatives in $\mathrm{Sel}_{S,\Pi}(\mathcal{X})$ for this image. For $i = 1, \cdots, N$, let

$$\mathrm{Sel}_{S,\Pi}(\mathcal{X})_{\alpha_i} := \{\alpha \in \mathrm{Sel}_{S,\Pi}(\mathcal{X}) \mid \mathrm{loc}_v(\alpha) = \mathrm{loc}_v(\alpha_i) \,\forall\, v \in T_0 \setminus S\},$$
$$\mathcal{X}(\mathcal{O}_{k,S})_{\alpha_i}\{z \in \mathcal{X}(\mathcal{O}_{k,S}) \mid \prod_{v \in T_0 \setminus S} \kappa_v(\alpha) = \alpha_i\}$$

so that

$$\mathrm{Sel}_{S,\Pi}(\mathcal{X}) = \bigsqcup_{i=1}^{N} \mathrm{Sel}_{S,\Pi}(\mathcal{X})_{\alpha_i} \tag{6}$$
$$\mathcal{X}(\mathcal{O}_{k,S}) = \bigsqcup_{i=1}^{N} \mathcal{X}(\mathcal{O}_{k,S})_{\alpha_i}.$$

Then by [BD18a, Lemma 2.6] (c.f. also [BD18b, Lemma 2.1] and [Dog20, Lemma 3.1]), we have a natural isomorphism

$$\mathrm{Sel}_{S,\Pi}(\mathcal{X})_{\alpha_i} \cong H^1_{f,S}(G_k;\Pi^{\alpha_i}), \tag{7}$$

where Π^{α_i} denotes the twist of Π by an element of $\mathrm{Sel}_{S,\Pi}(\mathcal{X})$ mapping to α_i. Note that if $\kappa(b')$ equals this element, then Π^{α_i} is just the fundamental group based at b' in place of b.

Remark 2.8. In particular, $\mathrm{Sel}_{S,\Pi}(\mathcal{X})$ has the same dimension as $H^1_{f,S}(G_k; \Pi^{\alpha_i})$, so its dimension may be computed by the methods of §7.3.

2.5. **Local Geometry at Bad Places.** Given $z \in \mathcal{X}(\mathcal{O}_{k,S})$, we may use [BD20, Theorem 1.3.1] to determine the i for which $\kappa(z) \in \mathrm{Sel}_{S,\Pi}(\mathcal{X})_{\alpha_i}$. For $v \in T_0 \setminus S$, let l_v/k_v be a finite extension over which $\overline{X}$ has semistable reduction, and let $\overline{\mathcal{X}}^{r-ss}$ denote a regular semistable integral model of $\overline{X}_{l_v}$. Let $\Gamma_v = \Gamma_v(\overline{\mathcal{X}}^{r-ss})$ denote the reduction graph of $\overline{X}$, so that

$$E(\Gamma_v)$$

is the set of irreducible components of the special fiber of $\overline{\mathcal{X}}^{r-ss}$.

Then there is a natural map $\overline{\mathcal{X}}^{r-ss}(\mathcal{O}_{l_v}) \to E(\Gamma_v)$, hence a map

$$\mathrm{red}_v \colon X(k) \to X(l_v) \to \overline{X}(l_v) = \overline{\mathcal{X}}^{r-ss}(\mathcal{O}_{l_v}) \to E(\Gamma_v)$$

Fact 2.9 ([BD20, Theorem 1.3.1])**.** *Let $\mathcal{X}^{r-ss}$ be an integral model of X_{l_v} equal to the complement in $\overline{\mathcal{X}}^{r-ss}$ of a horizontal divisor $\mathcal{D}$. Let $x, y \in X(k_v) \cap \mathcal{X}^{r-ss}(\mathcal{O}_{l_v}) \subseteq X(l_v)$. Then*

(1) If $\mathrm{red}_v(x) = \mathrm{red}_v(y)$, then $\kappa_v(x) = \kappa_v(y)$
(2) If Π dominates $U_3(X)$, and $\kappa_v(x) = \kappa_v(y)$, then $\mathrm{red}_v(x) = \mathrm{red}_v(y)$.

Remark 2.10. [BD20, Theorem 1.3.1] also requires the boundary divisor $\mathcal{D}$ to be étale over $\mathcal{O}_{l_v}$. One may arrange this by blowing up points on the boundary, but this does not change $\mathrm{red}_v\big|_{\mathcal{X}^{r-ss}(\mathcal{O}_{l_v})}$ and is therefore not strictly necessary.

Remark 2.11. If x or y is not in $\mathcal{X}^{r-ss}(\mathcal{O}_{l_v})$, then the truth of "$\mathrm{red}_v(x) = \mathrm{red}_v(y)$" might depend on the chosen integral model $\overline{\mathcal{X}}^{r-ss}$. An example, suggested to us by A. Betts, is provided by $\mathcal{X} = \mathcal{X}^{r-ss} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $v = v_3$, $x = 2$, and $y = 3$. Then $\mathrm{red}_v(x) = \mathrm{red}_v(y)$ because $|E(\Gamma_v(\overline{\mathcal{X}}^{r-ss}))| = 1$, but $\kappa_v(x) \neq \kappa_v(y)$, at least for Π dominating U_3.

Remark 2.12. Suppose $x, y \in X(l_v) \setminus \mathcal{X}^{r-ss}(\mathcal{O}_{l_v})$, and $\mathrm{red}_v(x) = \mathrm{red}_v(y)$. Then we still have $\kappa_v(x) = \kappa_v(y)$ for all Π factoring through the quotient map

$$U(X) \to U(\overline{X})$$

but not for general quotients Π of $U(X)$.

Relatedly, if $\kappa_v(x) = \kappa_v(y)$ for any Π dominating $U_3(\overline{X})$, then we have $\mathrm{red}_v(x) = \mathrm{red}_v(y)$, regardless of whether x, y are integral.

In practice, we will choose a model $\mathcal{X}$ for X over $\mathcal{O}_v$ (or even $\mathcal{O}_{k,S}$) for which it is clear that all elements of $\mathcal{X}(\mathcal{O}_v)$ extend to elements of $\mathcal{X}^{r-ss}(\mathcal{O}_{l_v})$.

In general, one may do this as follows. We suppose that $\mathcal{X}$ is given with a compactification $\mathcal{X} \subseteq \overline{\mathcal{X}}$. First, choose l_v/k_v for which X has semistable reduction. Then we may apply Lipman's resolution of singularities to $\overline{\mathcal{X}}_{\mathcal{O}_{l_v}}$ to obtain a regular semistable model $\overline{\mathcal{X}}^{r-ss}$. At each stage, we choose a model of X_{l_v} in our model of $\overline{X}_{l_v}$ by taking the strict transform (not the preimage) of the boundary divisor.

Example 2.13. In one example described in [Cor21], $\mathcal{X}$ is the minimal Weierstrass model of a punctured elliptic curve with semistable reduction at all $v \in T_0 \setminus S = \{3, 17\}$, so that $l_v = k_v$. The model is already regular at 17, but at 3 we obtain a regular model $\overline{\mathcal{X}}^{r-ss}$ by blowing up at the unique singular point. Then the smooth locus of $\overline{\mathcal{X}}^{r-ss}$ is the Néron model of $E = \overline{X}$, and the reduction of a point is its image in the Néron component group.

3. Bloch-Kato Groups and Conjectures

In this section, we go over background on Galois representations and the Bloch-Kato Selmer groups and conjectures.

3.1. **Categories of Galois Representations.** Let k be a number field and p a prime number, and let

$$\mathbf{Rep}^{\mathrm{g}}_{\mathbb{Q}_p}(G_k)$$

denote the category of p-adic representations of G_k that are unramified almost everywhere and de Rham at every place of k above p. We let

$$\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)$$

denote the subcategory of representations V that are *strongly geometric*, meaning V has a finite increasing filtration $W^{\bullet}V$, known as the *motivic weight filtration*, for which

$$\mathrm{Gr}^W_n V$$

is semisimple and pure of weight n at all unramified places in the sense of [Del80, 1.2.3].

Definition 3.1. For an integer n and full subcategory

$$\mathcal{C} \subseteq \mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k),$$

denote by $\mathcal{C}_{w \le n}$ (resp. $\mathcal{C}_{w \ge n}$, $\mathcal{C}_{w=n}$) the full subcategory of $V \in \mathcal{C}$ for which $W^n V = V$ (resp., $W_{n-1}V = 0$, resp. $\mathrm{Gr}^n V = V$).

We let $\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k)$ denote subcategory of $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)$ of semisimple objects (equivalently, objects for which the weight filtration splits).

The conjecture of Fontaine-Mazur states that every irreducible object of $\mathbf{Rep}^{\mathrm{g}}_{\mathbb{Q}_p}(G_k)$ is a subquotient of an étale cohomology group of some variety over k. A mixed version of Fontaine-Mazur, stated as [Fon92, "Conjecture" 12.4], predicts that every object of $\mathbf{Rep}^{\mathrm{g}}_{\mathbb{Q}_p}(G_k)$ is such a subquotient.[xiv] By resolution of singularities and the Weil conjectures, any such subquotient has a motivic weight filtration for which the graded pieces are pure of the appropriate weight. A corollary of this mixed Fontaine-Mazur along with the Grothendieck-Serre semisimplicity conjecture[xv] is thus:

Conjecture 3.2. *The categories* $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)$ *and* $\mathbf{Rep}^{\mathrm{g}}_{\mathbb{Q}_p}(G_k)$ *are equal.*[xvi]

[xiv] More precisely, it states that every object of $\mathbf{Rep}^{\mathrm{g}}_{\mathbb{Q}_p}(G_k)$ is contained in the Tannakian subcategory generated by realizations of motives ([Fon92, 6.6]).

[xv] that the étale cohomology of a smooth projective variety over a finitely generated field of characteristic 0 is semisimple as a Galois representation, and it follows from the Tate conjecture by [Moo19].

[xvi] We really mean equal, not equivalent or isomorphic, as the former is defined as an explicit subcategory of the latter.

3.2. **Bloch-Kato Selmer Groups.** Let v be a place of k, G_v a decomposition group of v in G_k, and I_v the inertia subgroup. We recall the local and global Selmer groups of [BK90].

For $v \notin \{p\}_k$ and a p-adic representation V of G_v, we set

$$H^1_g(G_v;V) := H^1(G_v;V)$$
$$H^1_f(G_v;V) := H^1_{ur}(G_k;V) := \ker(H^1(G_v;V) \to H^1(I_v;V)).$$

For $v \in \{p\}_k$, let B_{cris} and B_{dR} denote the crystalline and de Rham period rings associated to k_v, respectively. We set

$$H^1_g(G_v;V) := \ker(H^1(G_v;V) \to H^1(G_v;V \otimes_{\mathbb{Q}_p} B_{dR}))$$
$$H^1_f(G_v;V) := \ker(H^1(G_v;V) \to H^1(G_v;V \otimes_{\mathbb{Q}_p} B_{cris})).$$

Let V be a p-adic representation of G_k. For a place v of k, we let Res_v denote the restriction map $H^1(G_k;V) \to H^1(G_v;V)$. We then set

$$H^1_g(G_k;V) := \{\alpha \in H^1(G_k;V) \mid \mathrm{Res}_v(\alpha) \in H^1_g(G_v;V)\, \forall\, v\}.$$
$$H^1_f(G_k;V) := \{\alpha \in H^1(G_k;V) \mid \mathrm{Res}_v(\alpha) \in H^1_f(G_v;V)\, \forall\, v\}.$$

In terms of the category $\mathbf{Rep}^{\mathrm{g}}_{\mathbb{Q}_p}(G_k)$, we have

$$H^1_g(G_k;V) = \mathrm{Ext}^1_{\mathbf{Rep}^{\mathrm{g}}_{\mathbb{Q}_p}(G_k)}(\mathbb{Q}_p(0), V),$$

for any $V \in \mathbf{Rep}^{\mathrm{g}}_{\mathbb{Q}_p}(G_k)$, while in terms of $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)$, we have

$$H^1_g(G_k;V) = \mathrm{Ext}^1_{\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)}(\mathbb{Q}_p(0), V) \tag{8}$$

for $V \in \mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)_{w \leq -1}$.

Remark 3.3. Conjecturally, there is a category $\mathcal{MM}(k,\mathbb{Q})$ of mixed motives over k with $\mathbb{Q}$-coefficients. It should have a realization functor

$$\mathcal{MM}(k,\mathbb{Q}) \xrightarrow{\mathrm{real}_p} \mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)$$

for which the induced functor $\mathcal{MM}(k,\mathbb{Q}_p) := \mathcal{MM}(k,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)$[xvii] is an equivalence. This equivalence amounts to the Fontaine-Mazur, Tate, and Bloch-Kato conjectures. The latter implies that the map induced by the realization functor

$$\mathrm{Ext}^1_{\mathcal{MM}(k,\mathbb{Q})}(\mathbb{Q}(0), V) \otimes_{\mathbb{Q}} \mathbb{Q}_p \to H^1_g(G_k;V)$$

is an isomorphism.

As in [FPR94, II.1.3.1], for a set S of places of k, we set

$$H^1_{f,S}(G_k;V) := \{\alpha \in H^1_g(G_k;V) \mid \mathrm{Res}_v(\alpha) \in H^1_f(G_v;V)\, \forall\, v \notin S\}.$$

Remark 3.4. If we let Π_V denote the (abelian) p-adic unipotent group with Galois action corresponding to V, then comparing Definition 2.5 with this section, we have

$$H^1_{f,S}(G_k;\Pi_V) = H^1_{f,S}(G_k;V).$$

Conjecture 3.2 implies the following conjecture:

[xvii] The tensor product here is base extension from $\mathbb{Q}$-linear categories to $\mathbb{Q}_p$-linear categories; in particular, changes the class of objects as well as the Hom-sets.

Conjecture 3.5 ([Bel09, Prediction 4.1])**.** *For an irreducible geometric p-adic representation V of G_k of non-negative weight, the group*

$$H^1_g(G_k; V)$$

vanishes.

As explained in [Bel09, §4.1.3] as part of Grothendieck's 'yoga of weights', this conjecture follows from the mixed Fontaine-Mazur conjecture.

Remark 3.6. There are two other philosophical reasons behind this conjecture:

(1) The dimension of the group should be (by [FPR94, Conjectures 3.4.5(i)]) the order of vanishing of an L-function in the region of convergence of the Euler product.
(2) The group corresponds (by [BK90, Conjecture 5.3(i)]) to part of an algebraic K-theory group in negative degree.

Remark 3.7. [Bel09, Conjecture 4.1] refers to [FPR94, Conjectures 3.4.5(i)] as the "Bloch-Kato" conjecture, but they are really due to Fontaine–Perrin-Riou, who formulated the ideas of Bloch–Kato for Galois representations rather than varieties. A related conjecture, phrased in terms of conjectural Ext-groups of motives, is [Sch91, Conjecture B].

We also need the following consequence of Poitou-Tate duality in order to compute $h^1_f(G_k; V)$. For a de Rham representation V, let $\mathrm{D}^+_{\mathrm{dR}} V$ denote the 0th Hodge filtered piece of $\mathrm{D}_{\mathrm{dR}} V$.

Fact 3.8 ([FPR94, II.2.2.2][xviii])**.** *For a geometric Galois representation V, we have*

$$h^1_f(G_k; V) = h^0_f(G_k; V) + h^1_f(G_k; V^\vee(1)) - h^0_f(G_k; V^\vee(1)) + \sum_{v \in \{p\}_k} \dim_{\mathbb{Q}_p}(\mathrm{D}_{\mathrm{dR}} V_v / \mathrm{D}^+_{\mathrm{dR}} V_v) - \sum_{v \in \{\infty\}_k} h^0(G_v; V), \tag{9}$$

where $V_v := V\big|_{G_v}$.

The local dimensions are much simpler to compute (and are known non-conjecturally). Let $v \in \{p\}_k$. Then we have:

Fact 3.9 ([BK90, Corollary 3.8.4][xix])**.** *For a de Rham representation V of G_v, we have*

$$h^1_f(G_v; V) = \dim_{\mathbb{Q}_p}(\mathrm{D}_{\mathrm{dR}} V_v / \mathrm{D}^+_{\mathrm{dR}} V_v) + h^0(G_v; V).$$

3.3. **S-integral Versions.** Often, we would like to generalize (8) to $H^1_{f,S}$ for some finite set of primes S in place of H^1_g. For this purpose, we introduce the following category:

Definition 3.10. We define $\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k)$ to be the subcategory of $V \in \mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)$ such that for every place $v \notin S$:

- If $v \notin \{p\}_k$, the weight filtration of V splits as a representation of I_{k_v}.
- If $v \in \{p\}_k$, the weight filtration of $V \otimes B_{cris}$ splits as a representation of G_v.

Then for $V \in \mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k)_{w \le -1}$, we have

$$H^1_{f,S}(G_k; V) = \mathrm{Ext}^1_{\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k)}(\mathbb{Q}_p(0), V).$$

Remark 3.11. The categories $\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k)$ and $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)$ have the same semisimple objects.

[xviii] See also [Bel09, Theorem 2.2]

[xix] See also [Bel09, Proposition 2.8] or [FPR94, I.3.3.11]

3.4. **Change in S and Weight-Monodromy.** Let us analyze the extent to which $H^1_{f,S}(G_k;V)$ depends on S.

Proposition 3.12. *Let $S' = S \cup \{v\}$, with V a p-adic representation of V and $v \notin \{p\}_k$. Then*

$$h^1_{f,S'}(G_k;V) = h^1_{f,S}(G_k;V) + h^0(G_v;V^\vee(1))$$

In particular, $\dim H^1_{f,S}(G_k;V) = \dim H^1_{f,S'}(G_k;V)$ *if* $V\big|_{G_v}$ *has no quotient isomorphic to* $\mathbb{Q}_p(1)$.

Proof. By [FPR94, Proposition III.3.3.1(b)], we have an exact sequence

$$0 \to H^1_{f,S}(G_k;V) \to \dim H^1_{f,S'}(G_k;V) \to H^1_{g/f}(G_v;V) \to 0,$$

where $H^1_{g/f}(G_v;V) := H^1_g(G_v;V)/H^1_f(G_v;V)$.

By Tate-Poitou duality and because $v \notin \{p\}_k$, we have an isomorphism

$$H^1_{g/f}(G_v;V) = H^1(G_v;V)/H^1_f(G_v;V) \cong H^1_f(G_v;V^\vee(1))^\vee.$$

The inequality then follows by $\dim H^1_f(G_v;V^\vee(1)) = \dim H^0(G_v;V^\vee(1))$. $\square$

Supposing still that $v \notin \{p\}_k$, we recall the theory of weights for possibly-ramified p-adic representations of G_v. By Grothendieck's Monodromy Theorem ([ST68, Appendix]), after restricting to an open subgroup G'_v of G_v, the action of inertia is unipotent, so the irreducible pieces in the Jordan-Hölder series of $V\big|_{I'_v}$ are unramified. We may therefore talk about the Frobenius weight of such a piece, and the *(Frobenius) weights*[xx] of $V\big|_{G_v}$ is the set of weights that appear in these subquotients.

Let $V \in \mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)$, so that there is a filtration $W_\bullet V$ by motivic weight. By our definition,

$$\mathrm{Gr}^W_n V$$

is then unramified and pure of weight n at almost all places v. Let us recall what happens if we ask for a description at *all* v. After restricting to an open subgroup of I_v that acts unipotently, the inertia action may be described by a nilpotent operator $N\colon V \to V$. By Jacobson-Morosov ([Del80, Proposition 1.6.1]), there is a unique filtration $\mathrm{Fil}^N_\bullet V$ for which $N(\mathrm{Fil}^N_i V) \subseteq \mathrm{Fil}^N_{i-2} V$ and $N^i\colon \mathrm{Gr}^N_i V \to \mathrm{Gr}^N_{-i} V$ is an isomorphism for all $i \geq 0$ (c.f. [Sch12, Conjecture 1.13]). If we assume that V comes from geometry (as is implied by the Fontaine-Mazur conjecture), then the weight-monodromy conjecture of Deligne ([Del71]) states that:

Conjecture 3.13. $\mathrm{Gr}^N_j \mathrm{Gr}^W_i V$ *is pure of weight $i+j$ as a representation of V.*

This conjecture is the Weight-Monodromy Theorem of Grothendieck ([ST68, Appendix]) for a Galois representation coming from an abelian variety. As a corollary, it holds for any representation coming from first homology, or even more generally, from the p-adic unipotent fundamental group, of any variety. In particular, it holds for all the representations we care about.

If V is potentially unramified at v, then $N = 0$, so that the Frobenius weights of $V\big|_{G_v}$ are the motivic weights of V. For example, the set of weights of $\mathbb{Q}_p(n)$ is $\{-2n\}$.

[xx]We write "Frobenius weight" in full when we must distinguish from motivic weight.

Proposition 3.14. *If 0 is not one of the Frobenius weights of $V\big|_{G_v}$, then $h^0(G_v; V) = 0$. More generally, if $V\big|_{G_v}$ satisfies the weight-monodromy conjecture at V, and*

$$\operatorname{Gr}_j^N \operatorname{Gr}_i^W V$$

is trivial whenever $i + j = 0$ and $i \geq 0$.

Remark 3.15. The condition is satisfied if $\operatorname{Gr}_W^i V = 0$ for $i \geq 0$, i.e., if $V \in \mathbf{Rep}_{\mathbb{Q}_p}^{\mathrm{sg}}(G_k)_{w \leq -1}$. This corresponds to the condition "($\mathrm{WM}_{<0}$)" of [BD20, §2.1].

Proof. It suffices to prove this for V of pure motivic weight. By the definition of $\operatorname{Fil}_i^N V$, the kernel of N is contained in $\operatorname{Fil}_1^N V$. Therefore, if $H^0(G_v; V)$ is nontrivial, it projects nontrivially onto $\operatorname{Gr}_j^N \operatorname{Gr}_W^i V$ for some $j \leq 0$. This piece then has a nontrivial Galois-fixed part, so we have $i + j = 0$. Then $i \geq 0$, hence $\operatorname{Gr}_j^N \operatorname{Gr}_i^W V$ is trivial by hypothesis, a contradiction. □

We now consider what happens when we dualize and twist. The monodromy operator N of $V^\vee$ is the negative dual of the monodromy operator of V; in particular, $\operatorname{Fil}_\bullet^N V^\vee$ is dual to $\operatorname{Fil}_\bullet^N V$. Finally, $W_\bullet V^\vee$ is dual to $W_\bullet V$. In particular, we find that

$$\operatorname{Gr}_j^N \operatorname{Gr}_i^W V$$

is dual to

$$\operatorname{Gr}_{-j}^N \operatorname{Gr}_{-i}^W V^\vee$$

as a G_v/I_v-representation. Since duality negates the Frobenius weights, so that the weight-monodromy conjecture holds for V iff it does for $V^\vee$.

A Tate twist shifts the motivic weight filtration down by 2 and does not affect $\operatorname{Fil}_\bullet^N$. It also shifts the Frobenius weights down by 2 and thus preserves the truth of the weight-monodromy conjecture. We have

$$\dim \operatorname{Gr}_j^N \operatorname{Gr}_i^W V = \dim \operatorname{Gr}_{-j}^N \operatorname{Gr}_{-i-2}^W V^\vee(1).$$

As a corollary of Proposition 3.12 and Proposition 3.14, we get

Corollary 3.16. *In the notation of Proposition 3.12, if -2 is not one of the Frobenius weights of $V\big|_{G_v}$, then*

$$\dim H^1_{f,S}(G_k; V) = \dim H^1_{f,S'}(G_k; V).$$

This is true more generally if $V\big|_{G_v}$ satisfies the weight-monodromy conjecture at V, and

$$\operatorname{Gr}_j^N \operatorname{Gr}_i^W V$$

is trivial whenever $i + j = -2$ and $i \leq -2$.

Remark 3.17. The Weight-Monodromy Theorem for abelian varieties states more precisely that the weights of the Tate module are contained in $\{0, -1, -2\}$. The same is therefore true for the first homology of any smooth variety.

4. Mixed Abelian Galois Representations

Let k, X/k, Π, b, and v be as in §2, and suppose X has good reduction outside a finite set of places S not containing v. Then $\operatorname{Lie}\Pi$ is a (Lie algebra) object in

$$\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k),$$

while its universal enveloping algebra $\mathcal{U}\Pi$ and coordinate ring $\mathcal{O}(\Pi)$ are Pro- and Ind-objects, respectively, of that category.

More importantly, if G denotes the fundamental group of $\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k)$ with respect to some fiber functor ω, then

$$H^1_{f,s}(G_k;\Pi) = H^1(G;\omega(\Pi))$$

since Π has all negative weights.

However, the category $\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k)$ is much too large for us to get a concrete handle on it. We therefore need to define a much smaller subcategory that contains both Π and all torsors under it.

4.1. Semisimple Representations.

If Y is projective and smoooth, the (Grothendieck–Serre) semisimplicity conjecture ([Moo19, (S)]) implies that $h_\bullet(Y)$ (c.f. §1.8) is semisimple as a Galois representation.

Definition 4.1. Suppose that Y is smooth and projective and satisfies the semisimplicity conjecture. Then we let

$$\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, Y)$$

denote the Tannakian subcategory of $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)$ generated by $h_\bullet(Y)$.

Definition 4.2. Let $\operatorname{Irr}(\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, Y))$ denote the set of irreducible objects of $\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, Y)$. Note that by the Weil conjectures, each $V \in \operatorname{Irr}(\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, Y))$ has a well-defined weight, denoted $w(V)$.

4.2. Mixed Representations.

We now define the category of interest:

Definition 4.3. In the notation of Definition 4.1, we let

$$\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k, Y)$$

denote the smallest Serre subcategory of $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)$ containing $\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, Y)$. Equivalently, it is the subcategory of $V \in \mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)$ with $\operatorname{Gr}^W V \in \mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, Y)$.

Definition 4.4. In the notation of Definition 4.3 and for a finite set S of places of k, we let

$$\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, Y)$$

denote the full subcategory on the intersection of $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k, Y)$ with $\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k)$.

Then for any $V \in \mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, Y)_{w\leq -1}$, we have

$$H^1_g(G_k;V) = \operatorname{Ext}^1_{\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k,Y)}(\mathbb{Q}_p(0), V)$$

$$H^1_{f,S}(G_k;V) = \operatorname{Ext}^1_{\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k,Y)}(\mathbb{Q}_p(0), V).$$

Remark 4.5. We could instead use $\mathbf{Rep}^{\mathrm{g}}_{\mathbb{Q}_p}(G_k, Y)$, defined as the corresponding Serre subcategory of $\mathbf{Rep}^{\mathrm{g}}_{\mathbb{Q}_p}(G_k)$. This is conjecturally the same as $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k, Y)$ by Conjecture 3.2. Then we would know $H^1_g(G_k; V) = \mathrm{Ext}^1_{\mathbf{Rep}^{\mathrm{g}}_{\mathbb{Q}_p}(G_k,Y)}(\mathbb{Q}_p(0), V)$ for all $V \in \mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, Y)$ without Conjecture 3.5. Nonetheless, we find it more convenient in §5 to use the category $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k, Y)$, as we will make use of the weight filtration. Doing so presents no trouble essentially because the fundamental group of a curve has strictly negative weights, as proved in Theorem 4.11.

Remark 4.6. In the notation of Remark 3.3, the subcategory of $\mathcal{MM}(k, \mathbb{Q})$ corresponding under real_p to $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k, A)$ for an Abelian variety A is known as the category of *mixed abelian motives.* A candidate for this category when A is an elliptic curve was constructed in [Pat13]. See also [Com19, 4.11] for a discussion of abelian motives.

4.3. **S-Integral Representations.** When X is proper, the question of determining $X(k)$ is the same as that of determining $\mathcal{X}(\mathcal{O}_k)$, so we use H^1_f to define Selmer varieties as in §2.4. If X is not proper, we may consider $\mathcal{O}_{k,S}$-points, and we must then replace H^1_f by $H^1_{f,S}$.

Definition 4.7. We set

$$\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, Y)$$

to be the full subcategory on the intersection of $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k, Y)$ with $\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k)$.

Then

$$H^1_{f,S}(G_k; V) = \mathrm{Ext}^1_{\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k,Y)}(\mathbb{Q}_p(0), V) \tag{10}$$

for $V \in \mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, Y)_{w \leq -1}$.

4.4. **The Case of a Curve.** As in §2, let X/k be a smooth curve, Π a finite-dimensional Galois-equivariant quotient of $U = U(X)$ based at $b \in X(k)$, and $v \in \{p\}_k$ at which X has good reduction. We usually take A to be the generalized Jacobian variety J of X as in §1.3.

Remark 4.8. More generally, we define two abelian varieties B, B' to be *power isogenous* if some power of B is isogenous to some power of B'. In this case, we have $\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, B) = \mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, B')$, hence

$$\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, B) = \mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, B').$$

Therefore, we may equivalently take A to be any abelian variety that is power isogenous to J.

Remark 4.9. When the Jacobian of X is power isogenous to an elliptic curve, we refer to X as *mixed elliptic.* This is the topic of [Cor21].

If X has good reduction over $\mathcal{O}_{k,S}$ or if Π is semisimple as a Galois representation, then Π is a unipotent algebraic group in the Tannakian category $\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, A)$ in the sense of [Del89, 5.14(iv)].

Remark 4.10. It is best to take S as small as possible; in theory, one should take $S = \emptyset$ for any projective curve, although this does not seem possible in the Tannakian formalism if Π is not semisimple.

Then

Theorem 4.11. *For Π a unipotent group in $\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, A)$,*

$$H^1_{f,S}(G_k;\Pi) = H^1(\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, A);\Pi),^{\mathrm{xxi}}$$

where $H^1(\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, A);\Pi)$ denotes the set of torsors under Π in the Tannakian category $\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, E)$ in the sense of [Del89, 5.4].

We start with a lemma about weight filtrations on the coordinate rings of torsors:

Lemma 4.12. *For every $T \in H^1_g(G_k;\Pi)$, the coordinate ring $\mathcal{O}(T)$ has a motivic weight filtration in the sense of §3.1 (i.e., is an algebra in $\mathrm{Ind}-\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)$) supported in non-negative weights with $W_0\mathcal{O}(T) = \mathbb{Q}_p \subseteq \mathcal{O}(T)$.*

In particular, the map

$$H^1(\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k, A);\Pi) \hookrightarrow H^1_g(G_k;\Pi)$$

defined by sending a torsor in the category $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k, A)$ to its underlying torsor with G_k-action is an isomorphism.

Proof. We choose arbitrarily a point $b \in T(\mathbb{Q}_p)$. This defines an isomorphism $\varphi_b \colon \Pi \xrightarrow{\sim} T$ of schemes over $\mathbb{Q}_p$, thus an isomorphism of rings $\phi = \phi_b \colon \mathcal{O}(T) \xrightarrow{\sim} \mathcal{O}(\Pi)$. This isomorphism is not G_k-equivariant, but we will show that it is on the associated graded for a certain filtration. We set $W_n\mathcal{O}(T) \coloneqq \phi(W_n\mathcal{O}(\Pi))$. It thus suffices to show that the induced map $\mathrm{Gr}^W_n(\phi_b) \colon \mathrm{Gr}^W_n \mathcal{O}(\Pi) \to \mathrm{Gr}^W_n \mathcal{O}(T)$ is G_k-equivariant. Equivalently, if $f \in W_n\mathcal{O}(\Pi)$ and $g \in G_k$, then $g(\phi_b(f)) - \phi_b(g(f)) \in W_{n-1}\mathcal{O}(T)$.

We claim first that $W_n\mathcal{O}(T)$ is independent of b. For this, suppose we chose $b' \in T(\mathbb{Q}_p)$. Then there is $\pi \in \Pi$ such that $b' = b\pi$. Then $\varphi_{b'} = \varphi_b \circ \mathrm{ltr}_\pi$, where ltr_π denotes left translation by π on Π, so $\phi_{b'} = \mathrm{ltr}_\pi^\# \circ \phi_b$. It thus suffices to show that $\mathrm{ltr}_\pi^\#$ respects the weight filtration.

The map $\mathrm{ltr}_\pi^\#$ is given by the composition

$$\mathcal{O}(\Pi) \xrightarrow{\Delta_\Pi} \mathcal{O}(\Pi) \otimes \mathcal{O}(\Pi) \xrightarrow{b^\# \otimes \mathrm{id}} \mathcal{O}(\Pi).$$

Because $\mathcal{O}(\Pi)$ is concentrated in non-negative degrees, the image of $W_n\mathcal{O}(\Pi)$ under Δ_Π is contained in

$$\sum_{i=0}^n W_{n-i}\mathcal{O}(\Pi) \otimes W_i\mathcal{O}_\Pi.$$

Applying $b^\# \otimes \mathrm{id}$ to this thus lands in $\sum_{i=0}^n W_i\mathcal{O}(\Pi) = W_n\mathcal{O}(\Pi)$. The same argument shows that $\mathrm{ltr}_\pi{}^{-1} = \mathrm{ltr}_{\pi^{-1}}$ respects the weight filtration, so that in fact $\mathrm{ltr}_\pi^\#$ sends $W_n\mathcal{O}(\Pi)$ isomorphically onto itself. This implies that the weight filtration defined by $\phi_{b'}$ in place of ϕ_b is the same.

This implies in particular that the action of Π on T preserves the weight filtration we have defined on $\mathcal{O}(T)$.

□

Proof of Theorem 4.11. We phrase the argument in terms of $\mathbb{Q}_p$-points, but note that it may be upgraded to an isomorphism of functors on $\mathbb{Q}_p$-algebras.

[xxi] Note that this equality does not require Conjecture 3.2, because Π has negative weights. Furthermore, the semisimplicity condition in the definition of $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)$ follows from [Fal83]. C.f. Remark 4.5.

In view of the isomorphism

$$H^1(\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k, A); \Pi) \cong H^1_g(G_k; \Pi)$$

of Lemma 4.12, we must show that the subspaces $H^1(\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k, A); \Pi)$ and $H^1_{f,S}(G_k; \Pi)$ correspond.

We show for each place v of k that the local conditions on either side are equivalent. For a $\mathbb{Q}_p$-variety V with G_k-action and v a place of k, we set

$$R_v V := \begin{cases} V\big|_{\{\mathrm{id}\}}, & v \in S \setminus \{p\}_k \\ V\big|_{I_v}, & v \notin S \cup \{p\}_k \\ V\big|_{G_v} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}, & v \in \{p\}_k \setminus S \\ V\big|_{G_v} \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}, & v \in \{p\}_k \cap S \end{cases}$$

Suppose $T \in H^1(\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, A); \Pi)$. For all v, the weight filtration on $R_v\mathcal{O}(T)$ splits. By Lemma 4.12, there is thus a Galois-equivariant splitting $R_v\mathcal{O}(T) = R_v\mathbb{Q}_p \oplus R_v\mathcal{O}(T)_{>0}$. Projection onto $R_v\mathbb{Q}_p$ is thus a Galois-invariant point of R_vT, showing that R_vT is trivial.

Conversely, suppose that R_vT is trivial. Then $R_v\Pi_T \cong R_v\Pi \oplus R_v\mathbb{G}_a$. Thus $R_v\mathrm{Lie}\,\Pi_T \cong R_v\mathrm{Lie}\,\Pi \oplus R_v\mathrm{Lie}\,\mathbb{G}_a$. Since the weight filtration splits on $R_v\mathrm{Lie}\,\Pi$ and $R_v\mathrm{Lie}\,\mathbb{G}_a$, it does on $R_v\mathrm{Lie}\,\Pi_T$ and therefore $R_v\mathcal{O}(\Pi_T)$. Since $\mathcal{O}(T)$ is a Galois-equivariant quotient of $\mathcal{O}(\Pi_T)$, and R_v is exact, the weight filtration on $R_v\mathcal{O}(T)$ splits, so that $T \in H^1(\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, A); \Pi)$. $\square$

In order to express this set as the group cohomology of a pro-algebraic group, we introduce fiber functors in the next section.

5. Fiber Functors and Fundamental Groups

The goal of this section is to express

$$H^1_{f,S}(G_k; \Pi)$$

as the group cohomology of a pro-algebraic group and to describe this group. We refer back to §2 for the relationship between $H^1_{f,S}(G_k; \Pi)$ and $\mathrm{Sel}_{S,\Pi}(\mathcal{X})$.

5.1. Fiber Functors.

We define two important fiber functors on the category $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)$, known as the *de Rham* and *graded de Rham* fiber functors. We will use their restrictions to categories of the form $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k, A)$ and $\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, A)$ to define Tannakian Selmer varieties. We note first that there is a tensor functor $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k) \to \mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k)$ sending a representation V to its associated graded $\mathrm{Gr}^W_\bullet V$ for the motivic weight filtration. The restriction of this functor to $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k, A)$ lands in $\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A)$.

Definition 5.1. For $v \in \{p\}_k$, we define the *de Rham* fiber functor $\omega = \omega_v \colon \mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k) \to \mathbf{Vect}_{k_v}$ by

$$V \mapsto V^{\mathrm{dR}} := \mathrm{D}_{\mathrm{dR}}\, V.$$

We define the *graded de Rham* fiber functor $\omega^{\mathrm{Gr}} = \omega^{\mathrm{Gr}}_v \colon \mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k) \to \mathbf{Vect}_{k_v}$ by

$$V \mapsto V^{\mathrm{Gr\,dR}} := \mathrm{D}_{\mathrm{dR}}\, \mathrm{Gr}^W_\bullet V.$$

Remark 5.2. If we think of D_{dR} as a functor from $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)$ to filtered k_v-vector spaces, then $V^{\mathrm{Gr\,dR}} = \mathrm{Gr}^W_\bullet(V^{\mathrm{dR}})$. By exactness of D_{dR} and [Zie15, Lemma 4.1] for $S = \operatorname{Spec} k_v$, the graded de Rham functor is exact (and therefore is, in fact, a fiber functor). Note that a fiber functor detects exactness, which means that the associated graded functor is then also exact by $V^{\mathrm{Gr\,dR}} = \mathrm{D}_{\mathrm{dR}} \mathrm{Gr}^W_\bullet V$.

Remark 5.3. Note that both fiber functors are canonically isomorphic when restricted to the subcategory $\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k) \subseteq \mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)$.

5.1.1. *Reductive Monodromy Groups.* Fix a variety Y as in Definition 4.1.

Definition 5.4. We let

$$\mathbb{G} = \mathbb{G}(Y)$$

denote the Tannakian fundamental group of $\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, Y)$ under ω (equivalently, by Remark 5.3, under ω^{Gr}).

For $Y = A$ an abelian variety over k, the semisimplicity conjecture holds by [Fal83, Satz 3], so we may apply Definitions 4.1-4.4. Note that $h_\bullet(A)$ is also the exterior algebra on $h_1(A)$, so

$$\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A)$$

is the Tannakian subcategory of $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)$ generated by $h_1(A)$.

Definition 5.5. Let $\mathrm{Irr}(\mathbb{G})$ denote the set of irreducible (finite-dimensional algebraic) representations of $\mathbb{G}$. Then $\mathrm{Irr}(\mathbb{G}) = \mathrm{Irr}(\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, Y))$ in the sense of Definition 4.2.

Remark 5.6. The Tannakian fundamental group of $\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A)$ with respect to the fiber functor sending a Galois representation to its underlying vector space is $G_p(A)$, the Zariski closure of the image of G_k in $\mathrm{GL}(h_1(A))$, also known as its *p-adic monodromy group.* As mentioned in §5.4, any two fiber functors are isomorphic, so $\mathbb{G}(A) \cong G_p(A)$.

Remark 5.7. The connected component of the identity of $\mathbb{G}(A)$, denoted $\mathbb{G}(A)^0$, contains $\mathbf{MT}(A)_{\mathbb{Q}_p}$ (where $\mathbf{MT}(A)$ is the Mumford-Tate group of A) by work of Deligne. The Mumford-Tate Conjecture ([Mum66, Conjecture 4]) states that $\mathbb{G}(A)^0 = \mathbf{MT}(A)_{\mathbb{Q}_p}$. We refer to [Far16, §5] for a list of cases where this conjecture is known.

Remark 5.8. It follows from Serre's Open Image Theorem ([Ser98]) that if A is a non-CM elliptic curve, then $\mathbb{G} = \mathrm{GL}_2$. More generally, for any abelian variety A, there is some extension l of k for which $\mathbb{G}(A_l)$ is connected. In such a case, for a finite extension l/k, the functor $\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A) \to \mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_l, A)$ induced by restriction along $G_l \hookrightarrow G_k$ is an equivalence.

5.1.2. *Mixed Abelian Galois Groups.* Fix an abelian variety A.

Definition 5.9. We define

$$\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A)$$

to be the Tannakian fundamental group of the category

$$\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, A),$$

defined in Definition 4.7, with respect to ω.

We define

$$\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}}$$

to be the Tannakian fundamental group of the same category with respect to ω^{Gr}.

The two fiber functors are isomorphic (in a manner compatible with associated graded) by [SR72, IV.2.2.2] and [Zie15, Main Theorem 1.2]. We describe the issue more in §5.4 below.

5.2. **Structure of the Graded Tannakian Galois Group.** The inclusion

$$\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A) \hookrightarrow \mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, A)$$

induces a map

$$\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}} \to \pi_1(\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A)) \cong \mathbb{G}$$

that is surjective, with kernel the unipotent radical $U(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}}$ of $\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}}$. We denote the Lie algebra of $U(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}}$ by $\mathfrak{n}(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}}$, its universal envelopping algebra by $\mathcal{U}(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}}$, and its coordinate ring by $A(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}}$.

The functor $\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, A) \to \mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A)$ defined by $V \mapsto \mathrm{Gr}^W_\bullet V$ induces a canonical section

$$s \colon \mathbb{G} \to \pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}}. \tag{11}$$

We therefore have a canonical semidirect product decomposition

$$\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}} \cong \mathbb{G} \ltimes U(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}}, \tag{12}$$

inducing actions of $\mathbb{G}$ on $U(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}}$, $\Pi^{\mathrm{Gr\,dR}}$, and their associated algebras.

§A.2-A.3 describe the non-abelian cohomology of such semidirect products. We thus turn to the topic of non-abelian cohomology, which is the basis for our eventual Tannakian definition of Selmer varieties.

5.3. **Tannakian Non-Abelian Cohomology.** For a unipotent group Π in the Tannakian category $\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, A)$, we denote by Π^{dR} the unipotent group over $\mathbb{Q}_p$ with $\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A)$-action associated to the Lie algebra $(\mathrm{Lie}\,\Pi)^{\mathrm{dR}}$, with its induced Lie algebra structure.

For $V \in \mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, A)$, we have

$$\mathrm{Ext}^1_{\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k,A)}(\mathbb{Q}_p(0), V) = H^1(\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A); V^{\mathrm{dR}}) = H^1(\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}}; V^{\mathrm{Gr\,dR}}),$$

and

$$H^1(\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, A); \Pi) = H^1(\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A); \Pi^{\mathrm{dR}}) = H^1(\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}}; \Pi^{\mathrm{Gr\,dR}}).$$

By Theorem A.4, the semidirect product decomposition (12) gives us:

Corollary 5.10. *We have a natural bijection*

$$H^1_{f,S}(G_k; \Pi) \cong Z^1(U(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}}; \Pi^{\mathrm{Gr\,dR}})^{\mathbb{G}}.$$

Proof. By Theorem 4.11, we have $H^1_{f,S}(G_k; \Pi) \cong H^1(\mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, A); \Pi)$, which as stated above is isomorphic to $H^1(\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}}; \Pi^{\mathrm{Gr\,dR}})$. The result then follows from Theorem A.4, with $G = \pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}}$, $\mathbb{G} = \mathbb{G}(A)$, $U = U(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}}$, and $\Pi = \Pi^{\mathrm{Gr\,dR}}$ as in the previous sections. $\square$

Note that by Proposition A.1, we also have

$$H^1_{f,S}(G_k; \Pi) \cong Z^1(\mathfrak{n}(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}}; \mathrm{Lie}\,\Pi^{\mathrm{Gr\,dR}})^{\mathbb{G}}.$$

5.4. **Graded vs. Ungraded.** We would like to extend the semidirect product decomposition (12) and the description of cohomology (Corollary 5.10) to the usual (ungraded) de Rham fiber functor. For this, we need the notion of a *splitting* of the weight filtration on the de Rham fiber functor.

As in §5.2, the inclusion

$$\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A) \hookrightarrow \mathbf{Rep}^{\mathrm{sf,S}}_{\mathbb{Q}_p}(G_k, A)$$

induces a projection

$$\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A) \to \pi_1(\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A)) \cong \mathbb{G},$$

with kernel denoted $U(\mathcal{O}_{k,S}, A)$, but it is not canonically split. We denote the associated objects by $\mathfrak{n}(\mathcal{O}_{k,S}, A)$, $\mathcal{U}(\mathcal{O}_{k,S}, A)$, and $A(\mathcal{O}_{k,S}, A)$, respectively.

The scheme

$$\underline{\mathrm{Isom}}^{\otimes, \mathrm{Gr}\, W}(\mathrm{dR}, \mathrm{Gr}\, \mathrm{dR})$$

of isomorphisms from the de Rham fiber functor to the graded de Rham fiber functor inducing the identity on associated graded is a $U(\mathcal{O}_{k,S}, A)$-$U(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}}$ bitorsor in the fpqc topology over $\mathbb{Q}_p$ by [Zie15, Main Theorem 1.2]. As already mentioned, it is trivial by [SR72, IV.2.2.2]. In particular, the fundamental exact sequence

$$1 \to U(\mathcal{O}_{k,S}, A) \to \pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A) \to \pi_1(\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A)) \to 1$$

splits, although not canonically. In fact, we have

Proposition 5.11. *The set*

$$\underline{\mathrm{Isom}}^{\otimes, \mathrm{Gr}\, W}(\mathrm{dR}, \mathrm{Gr}\, \mathrm{dR})$$

is naturally in bijection with the set

$$\mathrm{Sec}(\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A))$$

of sections of the fundamental exact sequence. The $U(\mathcal{O}_{k,S}, A)$-torsor structure on $\underline{\mathrm{Isom}}^{\otimes, \mathrm{Gr}\, W}(\mathrm{dR}, \mathrm{Gr}\, \mathrm{dR})$ corresponds to the action of $U(\mathcal{O}_{k,S}, A)$ on $\mathrm{Sec}(\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A))$ by conjugation.

Proof. Any $\alpha \in \underline{\mathrm{Isom}}^{\otimes, \mathrm{Gr}\, W}(\mathrm{dR}, \mathrm{Gr}\, \mathrm{dR})$ induces an isomorphism $\beta \colon \pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A)^{\mathrm{Gr}} \to \pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A)$ over $\pi_1(\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A))$ defined by $\beta(g) = \alpha^{-1} g \alpha$. The composition

$$\beta \circ s \colon \pi_1(\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A)) \to \pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A),$$

with s as in (11), is an element of $\mathrm{Sec}(\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A))$. In other words, we have a map:

$$\underline{\mathrm{Isom}}^{\otimes, \mathrm{Gr}\, W}(\mathrm{dR}, \mathrm{Gr}\, \mathrm{dR}) \to \mathrm{Sec}(\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A)).$$

If we replace α by $\alpha \circ u^{-1}$ for $u \in U(\mathcal{O}_{k,S}, A)$, we replace $\beta(g)$ by $u\beta(g)u^{-1}$. In particular, this map intertwines the (left) action of $U(\mathcal{O}_{k,S}, A)$ on the bitorsor with its (left) conjugation action on $\mathrm{Sec}(\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A))$.

It therefore suffices to show that the latter is a torsor under $U(\mathcal{O}_{k,S}, A)$. This is true because $\mathbb{G}$ has trivial higher cohomology and because $U(\mathcal{O}_{k,S}, A)^{\mathbb{G}} = 0$ (for any choice of splitting). □

Remark 5.12. Given a section $\mathbb{G} \cong \pi_1(\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A)) \to \pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A)$, we may associate a point of $\underline{\mathrm{Isom}}^{\otimes, \mathrm{Gr}\, W}(\mathrm{dR}, \mathrm{Gr}\, \mathrm{dR})$ as follows. For any object M of $\mathbf{Rep}^{\mathrm{f,S}}_{\mathbb{Q}_p}(G_k, A)$, we get from the section a $\mathbb{G}$-action on M^{dR}, which then induces an isomorphism

$$M^{\mathrm{Gr}\, \mathrm{dR}} \xrightarrow{\sim} M^{\mathrm{dR}}$$

sending $\mathrm{D}_{\mathrm{dR}}\,\mathrm{Gr}^W_w V$ to

$$\bigoplus_{\substack{V\in\operatorname{Irr}\mathbb{G}\\ w(V)=-w}} (M^{\mathrm{dR}})^V,$$

where $(M^{\mathrm{dR}})^V := V\otimes_{\mathbb{Q}_p}\mathrm{Hom}_{\mathbb{G}}(V, M^{\mathrm{dR}})$ denotes the V-isotypic component of M^{dR}.

Remark 5.13. A choice of $\alpha\in\underline{\mathrm{Isom}}^{\otimes,\mathrm{Gr}\,W}(\mathrm{dR},\mathrm{Gr}\,\mathrm{dR})$ induces an isomorphism

$$H^1(\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S},A);\Pi^{\mathrm{dR}})\cong Z^1(U(\mathcal{O}_{k,S},A);\Pi^{\mathrm{dR}})^{\mathbb{G}}.$$

By Theorem A.4, this determines a section of the surjection

$$Z^1(\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S},A);\Pi^{\mathrm{dR}})\twoheadrightarrow H^1(\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S},A);\Pi^{\mathrm{dR}}),$$

whose image is

$$\ker(Z^1(\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S},A);\Pi^{\mathrm{dR}})\to Z^1(\mathbb{G};\Pi^{\mathrm{dR}}))$$

and whose projection to $Z^1(U(\mathcal{O}_{k,S},A);\Pi^{\mathrm{dR}})$ is $Z^1(U(\mathcal{O}_{k,S},A);\Pi^{\mathrm{dR}})^{\mathbb{G}}$.

Let's see what happens if we change α. If $c\in Z^1(U(\mathcal{O}_{k,S},A);\Pi^{\mathrm{dR}})$ is $\mathbb{G}$-equivariant with respect to α, one may check that

$$v\mapsto u(c(u^{-1}vu))$$

is $\mathbb{G}$-equivariant with respect to $\alpha\circ u^{-1}$.

Remark 5.14. Given $M\in\mathbf{Rep}^{\mathrm{f,S}}_{\mathbb{Q}_p}(G_k,A)$ and an extension

$$1\to M\to E\to\mathbb{Q}_p(0)\to 1$$

representing an element of $\mathrm{Ext}^1_{\mathbf{Rep}^{\mathrm{f,S}}_{\mathbb{Q}_p}(G_k,A)}(\mathbb{Q}_p(0),M) = H^1(\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S},A);M^{\mathrm{dR}})$, we may write down a cocycle representing this cohomology class by choosing a lift $1_E\in E$ of $1\in\mathbb{Q}_p(0)$ and then considering the cocycle

$$u\mapsto u(1_E)-1_E.$$

It is easy to see that this cocycle is $\mathbb{G}$-equivariant if and only if 1_E is $\mathbb{G}$-invariant. Assuming that M contains no subquotient isomorphic to $\mathbb{Q}_p(0)$, there is a unique lift 1_E of 1 that is $\mathbb{G}$-equivariant. The notion of $\mathbb{G}$-equivariance of course depends on the choice of α, and this 1_E is precisely the lift corresponding to the splitting of the weight filtration determined by α.

We fix once and for all a point of

$$\underline{\mathrm{Isom}}^{\otimes,\mathrm{Gr}\,W}(\mathrm{dR},\mathrm{Gr}\,\mathrm{dR}).$$

We thus fix an identification $U(\mathcal{O}_{k,S},A)\cong U(\mathcal{O}_{k,S},A)^{\mathrm{Gr}}$ and thus an identification

$$H^1_{f,S}(G_k;\Pi)\cong Z^1(U(\mathcal{O}_{k,S},A);\mathrm{Lie}\,\Pi^{\mathrm{dR}})^{\mathbb{G}}.$$

We therefore may mostly ignore the distinction between dR and Gr dR, although we revisit it briefly in §6.2.

From now on, we also *often suppress the superscript* dR *or* Gr dR *on* Π and therefore write, for example,

$$Z^1(U(\mathcal{O}_{k,S},A);\Pi)^{\mathbb{G}}.$$

Remark 5.15. In fact, if Π is semisimple, then $U(\mathcal{O}_{k,S}, A)$ acts trivially on Π^{dR}, so we get a $\mathbb{G}$-action on Π^{dR}. This induces an isomorphism

$$\Pi^{\mathrm{dR}} \cong \Pi^{\mathrm{Gr\,dR}},$$

independent of choice of point of $\underline{\mathrm{Isom}}^{\otimes, \mathrm{Gr}\,W}(\mathrm{dR}, \mathrm{Gr}\,\mathrm{dR})$. In the notation of Remark 5.13, $u(c(u^{-1}vu)) = c(v)$ in this case.

5.5. **Structure of the Unipotent Radical.** We have

$$\mathfrak{n}(\mathcal{O}_{k,S}, A)^{ab} = U(\mathcal{O}_{k,S}, A)^{ab} \cong \prod_{M \in \mathrm{Irr}\,\mathbb{G}} H^1_{f,S}(G_k; M)^\vee \otimes_{\mathbb{Q}_p} M^{\mathrm{dR}} \tag{13}$$

as an object of $\mathrm{Pro}\,\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A)$.

Dually, we have a canonical isomorphism

$$\ker(\Delta' \colon A(\mathcal{O}_{k,S}, A) \to A(\mathcal{O}_{k,S}, A) \otimes A(\mathcal{O}_{k,S}, A)) \cong \bigoplus_{M \in \mathrm{Irr}\,\mathbb{G}} H^1_{f,S}(G_k; M) \otimes_{\mathbb{Q}_p} M^{\vee\mathrm{dR}}$$

in $\mathrm{Ind}\,\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A)$.

Let us briefly describe this isomorphism explicitly. Let $M \in \mathrm{Irr}(\mathbb{G})$, and let $c \in \mathrm{Ext}^1(\mathbb{Q}_p, M)$. Then c is described by an extension

$$0 \to M \to E_c \to \mathbb{Q}_p \to 0.$$

Choose a lift $1_E \in E_c^{\mathrm{dR}}$ of $1 \in \mathbb{Q}_p$. Given $v \in (M^\vee)^{\mathrm{dR}}$, let $p_v \colon E_c{}^{\mathrm{dR}} \to \mathbb{Q}_p$ be the functional given by the projection defined by 1_E followed by $v \colon M^{\mathrm{dR}} \to \mathbb{Q}_p$. Then the element $c \otimes v$ of $A(\mathcal{O}_{k,S}, A)$ is the Tannakian matrix coefficient ([Bro17b, §2.2])

$$[E_c, 1_E, p_v].$$

Letting $A(\mathcal{O}_{k,S}, A)_{>0}$ denote the augmentation ideal, $A(\mathcal{O}_{k,S}, A)_{>0} \cdot A(\mathcal{O}_{k,S}, A)_{>0}$ is the space of *decomposables*, and

$$\mathfrak{n}(\mathcal{O}_{k,S}, A)^\vee := A(\mathcal{O}_{k,S}, A)_{>0} / A(\mathcal{O}_{k,S}, A)_{>0} \cdot A(\mathcal{O}_{k,S}, A)_{>0}$$

is the Lie coalgebra. Then Δ' induces the cobracket on $\mathfrak{n}(\mathcal{O}_{k,S}, A)^\vee$, and

$$\ker(\Delta' \colon A(\mathcal{O}_{k,S}, A) \to A(\mathcal{O}_{k,S}, A) \otimes A(\mathcal{O}_{k,S}, A)) \cong \bigoplus_{M \in \mathrm{Irr}\,\mathbb{G}} H^1_{f,S}(G_k; M) \otimes_{\mathbb{Q}_p} M^{\vee\mathrm{dR}}$$

is the kernel of the cobracket.

5.6. **A Free Unipotent Group.** The description (13) implies that $U(\mathcal{O}_{k,S}, A)$ is the quotient of a free unipotent group defined by Bloch-Kato Selmer groups. We explain that while this quotient map is conjecturally an isomorphism, one may nonetheless use this free unipotent group for the purposes of the Chabauty-Kim method.

As $\mathbb{G}$ is reductive, we may choose a $\mathbb{G}$-equivariant splitting of the projection

$$\mathfrak{n}(\mathcal{O}_{k,S}, A) \twoheadrightarrow \mathfrak{n}(\mathcal{O}_{k,S}, A)^{ab}. \tag{14}$$

Let $F_{k,S,A}$ denote the image of $\mathfrak{n}(\mathcal{O}_{k,S}, A)^{ab}$ under this splitting, and set

$$\mathfrak{n}(F_{k,S,A}) := \mathrm{FreeLie}\, F_{k,S,A},$$

and let $U(F_{k,S,A}), \mathcal{U}F_{k,S,A}$, and $A(F_{k,S,A})$ denote the associated pro-unipotent group, universal enveloping algebra, and coordinate ring, respectively.

We have a map

$$\theta\colon \mathfrak{n}(F_{k,S,A}) \to \mathfrak{n}(\mathcal{O}_{k,S}, A)$$

of Lie algebra objects in $\operatorname{Pro}\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A)$ that is an isomorphism on abelianizations. In particular, θ is surjective.

If we knew the following conjecture, then we would know that θ is an isomorphism:

Conjecture 5.16 ([Fon92, Conjecture 12.6])**.** *The category $\mathbf{Rep}^{\mathrm{g}}_{\mathbb{Q}_p}(G_k)$ has cohomological dimension* 1*.*

This conjecture is closely related to [FPR94, Conjecture II.3.2.2], which states that if $V \in \mathbf{Rep}^{\mathrm{g}}_{\mathbb{Q}_p}(G_k)$, and V'' is a quotient of V, then

$$H^1_{f,S}(G_k; V) \to H^1_{f,S}(G_k; V'')$$

is surjective.

Indeed, Conjecture 5.16 would imply that $\operatorname{Ext}^i_{\mathbf{Rep}^{\mathrm{f,S}}_{\mathbb{Q}_p}(G_k,A)}$ vanishes for $i \geq 2$, and hence that $U(\mathcal{O}_{k,S}, A)$ is a free pro-unipotent group over $\mathbb{Q}_p$. Then it follows that θ is an isomorphism.

We nonetheless have an embedding

$$H^1_{f,S}(G_k; \Pi) \cong Z^1(U(\mathcal{O}_{k,S}, A); \Pi)^{\mathbb{G}} \hookrightarrow Z^1(U(F_{k,S,A}); \Pi)^{\mathbb{G}},$$

conjectured to be an isomorphism. In the notation of §2, we may therefore define a Kummer map

$$\mathcal{X}(\mathcal{O}_{k,S})_{\alpha_i} \xrightarrow{\kappa} Z^1(U(F_{k,S,A}); \Pi^{\alpha_i})^{\mathbb{G}}$$

defined by composition of the embedding above with the usual Kummer map. While its representability does not follow from Theorem 4.11, we prove representability in Proposition 8.1. In §6.3, we explain how to define the map $\log_{\mathrm{BK}}$ on all of $Z^1(U(F_{k,S,A}); \Pi^{\alpha_i})^{\mathbb{G}}$ and thus in Remark 6.4 describe a version of the Chabauty-Kim diagram (4) with

$$Z^1(U(F_{k,S,A}); \Pi)^{\mathbb{G}}$$

in place of $H^1_{f,S}(G_k; \Pi)$.

As an associative algebra in $\operatorname{Pro}\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A)$, the universal enveloping algebra $\mathcal{U}F_{k,S,A}$ is the tensor algebra on $F_{k,S,A}$.

Dually, $A(F_{k,S,A})$ becomes the free shuffle algebra on the dual object

$$F^\vee_{k,S,A} = \bigoplus_{M \in \operatorname{Irr}\mathbb{G}} H^1_{f,S}(G_k; V) \otimes_{\mathbb{Q}_p} M^{\vee\mathrm{dR}}$$

of $\operatorname{Ind}\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A)$, with coproduct given by deconcatenation and $\mathbb{G}$-action given by its action on this vector space. Note that it also has the structure of a non-semisimple motive by §A.2.

5.7. **Quotients of the Unipotent Radical.** The algebra $A(F_{k,S,A})$ has a weight filtration. However, it is not necessarily finite-dimensional in each degree, even assuming Conjecture 3.5. That's because given w, there are infinitely many $M \in \operatorname{Irr}\mathbb{G}$ with $w(M) = w$.

Let $\operatorname{Irr}^{\mathrm{ae}}\mathbb{G}$ denote the set of irreducible representations of $\mathbb{G}$ appearing as direct factors of non-negative tensor powers of $h_1(A)$. Let $F^{\mathrm{ae}}_{k,S,A}$ be the quotient of $F_{k,S,A}$ corresponding to

$$\prod_{M \in \operatorname{Irr}^{\mathrm{ae}}\mathbb{G}} H^1_{f,S}(G_k; M)^\vee \otimes_{\mathbb{Q}_p} M^{\mathrm{dR}}.$$

Then the quotient $\mathfrak{n}(F_{k,S,A})^{\mathrm{ae}} := \operatorname{FreeLie} F_{k,S,A}^{\mathrm{ae}}$ of $\mathfrak{n}(F_{k,S,A})$ is finite-dimensional in each degree. More generally, for a subset

$$I \subseteq \operatorname{Irr} \mathbb{G},$$

we define

$$F_{k,S,A}^I := \bigoplus_{M \in I} H^1_{f,S}(G_k; M)^\vee \otimes_{\mathbb{Q}_p} M^{\mathrm{dR}},$$
$$\mathfrak{n}(F_{k,S,A})^I := \operatorname{FreeLie} F_{k,S,A}^I$$

as a quotient of $\mathfrak{n}(F_{k,S,A})$, and

$$U(F_{k,S,A})^I$$
$$A(F_{k,S,A})^I$$

the corresponding pro-unipotent group and coordinate ring, respectively. Assuming $I \subseteq \operatorname{Irr}^{\mathrm{ae}} \mathbb{G}$,[xxii] $\mathfrak{n}(F_{k,S,A})^I$ and $A(F_{k,S,A})^I$ are finite-dimensional in each degree and are strictly-negatively and positively graded, respectively.

Proposition 5.17. *If $I \subseteq \operatorname{Irr} \mathbb{G}$ contains all graded pieces of Π, and the action of $U(F_{k,S,A})$ on Π factors through $U(F_{k,S,A})^I$, then*

$$Z^1(U(F_{k,S,A})^I; \Pi)^{\mathbb{G}} = Z^1(U(F_{k,S,A}); \Pi)^{\mathbb{G}}$$

Proof. Set

$$F'_{k,S,A} := \operatorname{Ker}(F_{k,S,A} \twoheadrightarrow F^I_{k,S,A}).$$

Then the kernel of

$$\mathfrak{n}(F_{k,S,A}) \to \mathfrak{n}(F_{k,S,A})^I$$

corresponds under θ to the Lie ideal $\mathfrak{n}(F_{k,S,A})'$ generated by $F'_{k,S,A}$. This Lie ideal corresponds to the normal subgroup scheme

$$U(F_{k,S,A})' = \operatorname{Ker}(U(F_{k,S,A}) \to U(F_{k,S,A})').$$

Let $c \in Z^1(\mathfrak{n}(F_{k,S,A}); \operatorname{Lie} \Pi)^{\mathbb{G}} = Z^1(U(F_{k,S,A}); \Pi)^{\mathbb{G}}$. Note that c vanishes on $F'_{k,S,A}$, since $\operatorname{Hom}_{\mathbb{G}}(F'_{k,S,A}, \operatorname{Lie} \Pi) = 0$.

It then suffices to show that

$$\operatorname{Ker} c \cap \mathfrak{n}(F_{k,S,A})'$$

is a Lie ideal in $\mathfrak{n}(F_{k,S,A})$. For this, suppose $u \in \mathfrak{n}(F_{k,S,A})$ and $w \in \operatorname{Ker} c \cap \mathfrak{n}(F_{k,S,A})'$. Then

$$c([u,w]) = [c(u), c(w)] + u(c(w)) - w(c(v)) = [c(u), 0] + u(0) - w(c(v)) = 0$$

because $w \in \mathfrak{n}(F_{k,S,A})'$, which acts trivially on $\operatorname{Lie} \Pi$.

It follows that

$$\operatorname{Ker} c = \mathfrak{n}(F_{k,S,A})',$$

so that $c \in Z^1(U(F_{k,S,A})^I; \Pi)^{\mathbb{G}} \subseteq Z^1(U(F_{k,S,A}); \Pi)^{\mathbb{G}}$. As c is arbitrary, we are done. □

Remark 5.18. Given Π, one may ensure the hypothesis of Proposition 5.17 by taking I to include the set of irreducible components of $\operatorname{End}(\operatorname{Lie} \Pi)$ as a $\mathbb{G}$-representation.

[xxii] More generally, as long as I has only representations of negative weight and contains finitely many of each given weight.

6. Localization Maps for Selmer Varieties

Let X, U, Π, $\mathfrak{p}$, and S be as in §2.3 and A as in §4.4. The goal of this section is to explicitly understand the map

$$\log_{\mathrm{BK}} \circ \mathrm{loc}_\Pi \colon H^1_{f,S}(G_k;\Pi) \to \Pi/F^0\Pi$$

via the description of $H^1_{f,S}(G_k;\Pi)$ in §5. The results of this section will allow us to find an element of $\mathcal{O}(\Pi/F^0)$ vanishing on the image of loc_Π by computing a function vanishing on the image of a more explicit map

$$\mathfrak{ev}_{\Pi,F_{k,S,A}}/F^0 \colon Z^1(U(F_{k,S,A});\Pi)^{\mathbb{G}} \times U(F_{k,S,A}) \to \Pi/F^0 \times U(F_{k,S,A}).$$

The material of this section corresponds to [CDC20, §2.3-2.4].

6.1. **Universal Cocycle Evaluation Maps.** For any Π, we have a *universal cocycle evaluation map* (c.f. [CDC20, Definition 2.20])

$$\mathfrak{ev}_\Pi \colon Z^1(U(\mathcal{O}_{k,S},A);\Pi)^{\mathbb{G}} \times U(\mathcal{O}_{k,S},A) \to \Pi \times U(\mathcal{O}_{k,S},A).$$

defined by

$$(c,u) \mapsto (c(u),u).$$

Its pullback along $U(F_{k,S,A}) \twoheadrightarrow U(\mathcal{O}_{k,S},A)$ factors through a map

$$Z^1(U(\mathcal{O}_{k,S},A);\Pi)^{\mathbb{G}} \times U(F_{k,S,A}) \to Z^1(U(F_{k,S,A});\Pi)^{\mathbb{G}} \times U(F_{k,S,A}) \xrightarrow{\mathfrak{ev}_{\Pi,F_{k,S,A}}} \Pi \times U(F_{k,S,A}).$$

Remark 6.1. A key part of applying Chabauty–Kim to a particular variety X is to compute a function on $\Pi/F^0 \times U(F_{k,S,A})$ vanishing on the image of the composition $\mathfrak{ev}_{\Pi,F_{k,S,A}}/F^0$ of $\mathfrak{ev}_{\Pi,F_{k,S,A}}$ with projection from Π to Π/F^0:

$$\mathfrak{ev}_{\Pi,F_{k,S,A}}/F^0 \colon Z^1(U(F_{k,S,A});\Pi)^{\mathbb{G}} \times U(F_{k,S,A}) \xrightarrow{\mathfrak{ev}_{\Pi,F_{k,S,A}}} \Pi \times U(F_{k,S,A}) \to \Pi/F^0 \times U(F_{k,S,A}).$$

This is known as the *geometric step*, following [CDC20, §1.3.4].

In §6.2-6.3, we show how such a function on $\Pi/F^0 \times U(F_{k,S,A})$ specializes to an element of the Chabauty-Kim ideal.

Suppose that the graded pieces of Π are contained in $I \subseteq \operatorname{Irr}\mathbb{G}$ and that the action of $U(F_{k,S,A})$ on Π factors through $U(F_{k,S,A})^I$, as in the hypotheses of Proposition 5.17. Then $\mathfrak{ev}_{\Pi,F_{k,S,A}}/F^0$ is just the pullback along $U(F_{k,S,A}) \to U(F_{k,S,A})^I$ of a map

$$\mathfrak{ev}^I_{\Pi,F_{k,S,A}} \colon Z^1(U(F_{k,S,A})^I;\Pi)^{\mathbb{G}} \times U(F_{k,S,A})^I \to \Pi \times U(F_{k,S,A})^I.$$

In particular, an element of $\mathcal{O}(\Pi/F^0 \times U(F_{k,S,A})^I) = \mathcal{O}(\Pi/F^0) \otimes A(F_{k,S,A})^I$ vanishing on the image of $\mathfrak{ev}^I_{\Pi,F_{k,S,A}}/F^0$ (the composition of $\mathfrak{ev}^I_{\Pi,F_{k,S,A}}$ with the projection $\Pi \times U(F_{k,S,A})^I \twoheadrightarrow \Pi/F^0 \times U(F_{k,S,A})^I$) also vanishes on the image of $\mathfrak{ev}_{\Pi,F_{k,S,A}}/F^0$.

6.2. **p-adic Periods and Localization.** In this section, we describe a p-adic period map contained in forthcoming work of the author and I. Dan-Cohen. This is a $\mathbb{Q}_p$-algebra homomorphism $A(\mathcal{O}_{k,S},A) \to \mathbb{Q}_p$ for $\mathfrak{p} \in \operatorname{Spec}\mathcal{O}_{k,S}$ with residue characteristic p and $\mathcal{O}_\mathfrak{p} \cong \mathbb{Z}_p$, compatible with the non-abelian Bloch-Kato map of [Kim09].

Such a map for mixed Tate motives was defined by [CU13] and used in [CDC20, 2.4.1]. A different p-adic period map for mixed Tate motives was defined in [DG05, 5.28] (c.f. also [Yam10, §3.2] and [Bro17a, 3.4.3]).

Remark 6.2. This map is not logically necessary for the definition of the universal cocycle evaluation map in §6.1 or the geometric step (Remark 6.1). However, the period map motivates these constructions and calculations because it implies that the result of the geometric step is a function that specializes to an element of the Chabauty-Kim ideal.

In forthcoming work with I. Dan-Cohen, we prove the following:

Theorem 6.3. *For $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_{k,S}$ with residue characteristic p and $\mathcal{O}_{\mathfrak{p}} \cong \mathbb{Z}_p$, there is a point*

$$\mathrm{per}_{\mathfrak{p}} \colon \operatorname{Spec} \mathbb{Q}_p \to U(\mathcal{O}_{k,S}, A),$$

satisfying the following property:

Let Π be a unipotent group with negative-weight action of $\pi_1^{\mathbf{MA}}(\mathcal{O}_{k,S}, A)$ and

$$c \in Z^1(U(\mathcal{O}_{k,S}, A), \Pi)^{\mathbb{G}}.$$

Then the composition

$$c/F^0 \circ \mathrm{per}_{\mathfrak{p}} \colon \operatorname{Spec} \mathbb{Q}_p \xrightarrow{\mathrm{per}_{\mathfrak{p}}} U(\mathcal{O}_{k,S}, A) \xrightarrow{c} \Pi \twoheadrightarrow \Pi/F^0$$

is $\log_{\mathrm{BK}}(\mathrm{loc}_{\Pi}(c)) \in \Pi/F^0(\mathbb{Q}_p)$.

Proof Sketch 1. Let ω denote the dR fiber functor and ω^{gr} the $\mathrm{Gr}\,\mathrm{dR}$ fiber functor. For any $M \in \mathbf{Rep}^{\mathrm{f,S}}_{\mathbb{Q}_p}(G_k, A)$, we have a Frobenius map $\phi_M \colon \omega(M) \to \omega(M)$ coming from the Frobenius map on B_{cris}. This map respects the weight filtration, and we denote by $\mathrm{gr}_W \phi$ the action of ϕ on the associated graded.

We prove that there exists a unique path $\mathfrak{p}^{\mathrm{cr}} \in \underline{\mathrm{Isom}}^{\otimes, \mathrm{Gr}\, F_{k,S,A}}(\mathrm{Gr}\,\mathrm{dR}, \mathrm{dR})$ such that for any $M \in \mathbf{Rep}^{\mathrm{f,S}}_{\mathbb{Q}_p}(G_k, A)$,

$$\begin{tikzcd} \omega^{\mathrm{gr}}(M) \arrow[r, "\mathfrak{p}^{\mathrm{cr}}"'] \arrow[d, "\mathrm{gr}_W \phi"'] & \omega(M) \arrow[d, "\phi"'] \\ \omega^{\mathrm{gr}}(M) \arrow[r, "\mathfrak{p}^{\mathrm{cr}}"] & \omega \end{tikzcd}$$

by an analogue of the proof of [Bes02].

A modification of arguments of Paul Ziegler show that there is a (non-unique) path $\mathfrak{p}^H \colon \omega^{\mathrm{gr}} \to \omega$ respecting the Hodge filtration.

We then take $\mathrm{per}_{\mathfrak{p}}$ to be $\mathfrak{p}^H \circ (\mathfrak{p}^{\mathrm{cr}})^{-1}$.

6.3. **p-adic Periods and Universal Cocycle Evaluation.** Since the map $U(F_{k,S,A}) \twoheadrightarrow U(\mathcal{O}_{k,S}, A)$ is a surjection of vector spaces, we may lift $\mathrm{per}_{\mathfrak{p}} \in U(\mathcal{O}_{k,S}, A)$ arbitrarily to an element of $U(F_{k,S,A})$. The choice of lift will not matter, so we denote it, by abuse of notation, by $\mathrm{per}_{\mathfrak{p}}$ as well.

We then have the following diagram of schemes over $\mathbb{Q}_p$:

$$\begin{tikzcd}
H^1_{f,S}(G_k;\Pi) \arrow[rr, "\log_{\mathrm{BK}} \circ \mathrm{loc}_\Pi"'] \arrow[d, equal] & & \Pi/F^0 \arrow[d, equal] \\
Z^1(U(\mathcal{O}_{k,S};A);\Pi)^{\mathbb{G}} \arrow[rr, "\log_{\mathrm{BK}} \circ \mathrm{loc}_\Pi"'] \arrow[d, equal] & & \Pi/F^0 \arrow[d, equal] \\
Z^1(U(\mathcal{O}_{k,S};A);\Pi)^{\mathbb{G}} \times \operatorname{Spec}\mathbb{Q}_p \arrow[rr] \arrow[d, "\mathrm{per}_{\mathfrak{p}}"'] & & \Pi/F^0 \times \operatorname{Spec}\mathbb{Q}_p \arrow[d, "\mathrm{per}_{\mathfrak{p}}"'] \\
Z^1(U(\mathcal{O}_{k,S},A);\Pi)^{\mathbb{G}} \times U(F_{k,S,A}) \arrow[rr, "\mathfrak{ev}_\Pi/F^0"] \arrow[d, hook] & & \Pi/F^0 \times U(F_{k,S,A}) \arrow[d, equal] \\
Z^1(U(F_{k,S,A});\Pi)^{\mathbb{G}} \times U(F_{k,S,A}) \arrow[rr, "\mathfrak{ev}_{\Pi,F_{k,S,A}}/F^0"] \arrow[d, two heads] & & \Pi/F^0 \times U(F_{k,S,A}) \arrow[d, two heads] \\
Z^1(U(F_{k,S,A});A);\Pi)^{\mathbb{G}} \times U(F_{k,S,A})^I \arrow[rr, "\mathfrak{ev}^I_{\Pi,F_{k,S,A}}/F^0"] & & \Pi/F^0 \times U(F_{k,S,A})^I
\end{tikzcd}$$

The commutativity of the second and third rows with the fourth row follows from Theorem 6.3. The commutativity of the rest of the diagram follows easily from the definitions.

Therefore, if $f \in \mathcal{O}(\Pi/F^0 \times U(F_{k,S,A}))^I = \mathcal{O}(\Pi/F^0) \otimes A(F_{k,S,A})^I$ vanishes on the image of $\mathfrak{ev}^I_{\Pi,F_{k,S,A}}/F^0$, then

$$\mathrm{per}_{\mathfrak{p}}^{\#}(f) \in \mathcal{O}(\Pi/F^0)$$

vanishes on the image of $\log_{\mathrm{BK}} \circ \mathrm{loc}_\Pi$. Section 7.3 is concerned with verifying that $\mathfrak{ev}^I_{\Pi,F_{k,S,A}}/F^0$ is non-dominant, while Section 8 is concerned with computing its image in terms to which one may apply $\mathrm{per}_{\mathfrak{p}}^{\#}$.

Remark 6.4. One may think of the use of $F_{k,S,A}$ in terms of an alternative version of (4) with

$$\mathrm{Sel}_{S,\Pi}(\mathcal{X})^F := \bigsqcup_{i=1}^{N} Z^1(U(F_{k,S,A});A);\Pi^{\alpha_i})^{\mathbb{G}}$$

in place of $\mathrm{Sel}_{S,\Pi}(\mathcal{X})$, where the disjoint union is as in (6).

7. Motivic Decomposition of Fundamental Groups

Let X, U, Π, $\mathfrak{p}$, S, and A be as in the previous section. To simplify notation, we fix an implicit basepoint $b \in X(k)$ and set

$$U = U(X) := \pi_1^{\text{ét,un}}(X_{\overline{k}}, b)_{\mathbb{Q}_p}$$

and

$$\begin{aligned} U^1 &:= U \\ U^{n+1} &:= [U, U^n] \\ U_n &:= U/U^{n+1} \\ U[n] &:= U^n/U^{n+1}, \end{aligned}$$

where commutator denotes the closure of the group-theoretic commutator. Notice the short exact sequence

$$0 \to U[n] \to U_n \to U_{n-1} \to 0. \tag{15}$$

We let Π be a finite-dimensional Galois-equivariant quotient of U. In particular, Π factors through U_n for some n. Common examples include $\Pi = U_n$, $\Pi = U[n]$, or more generally $\Pi = U^k/U^{n+1}$ for $k \le n$. The main goal of this section is to explain how to find the semisimplification of Π as a $\mathbb{G}$-representation, or equivalently the class $[\Pi]$ of §7.1.

One upshot of this is that it allows one to bound $\dim_{\mathbb{Q}_p} H^1 f, S(G_k; \Pi)$ and compute $\dim_{\mathbb{Q}_p} H^1_f(G_{k_v}; \Pi) \cong \Pi/F^0\Pi$, as described in §7.3. This provides an algorithm for checking when the inequality (5) is satisfied.[xxiii]

Finally, for the purposes of comparing with Quadratic Chabauty, we define

$$U_Q = U_Q(X)$$

to be the quotient of U_2 by the maximal subspace of $U[2]$ with no subrepresentation isomorphic to $\mathbb{Q}_p(1)$.

Remark 7.1. All computations in this section are completely independent of b, essentially because the graded pieces $U[n]$ are homological in nature.

7.1. **Grothendieck Groups of Galois Representations.** We use the Grothendieck ring

$$K_0(\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)),$$

which has a grading with the nth graded piece given by the group[xxiv]

$$K_0(\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)_{w=n}) := K_0(\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)_{w=n}).$$

We let pr_n denote projection onto the nth component. We thus have $\mathrm{pr}_n([V]) = [\mathrm{Gr}^W_n V]$.

The subring $K_0(\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)_{w\le 0})$ is negatively graded, and we denote by

$$\widehat{K}_0(\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)_{w\le 0})$$

its completion with respect to the ideal $K_0(\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)_{w\le -1})$. If V is an object of $\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)_{w\le -1}$, then $[\mathrm{Sym}\, V] = \sum_{n=0}^{\infty} [\mathrm{Sym}^n V]$ and $[TV] = \sum_{n=0}^{\infty} [V^{\otimes n}]$ are in $\widehat{K}_0(\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)_{w\le 0})$.

To a Galois-equivariant subquotient Π of U, we associate a class $[\Pi]$ in the ring

$$K_0(\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k))$$

defined by requiring

$$[\Pi_2] = [\Pi_1] + [\Pi_3]$$

[xxiii]More precisely, in the spirit of §5.6, it allows us to either verify that the inequality is satisfied or that Conjecture 5.16 implies that the inequality is not satisfied.

[xxiv]Not a ring for $n \neq 0$!

when there is a (Galois-equivariant) short exact sequence

$$0 \to \Pi_1 \to \Pi_2 \to \Pi_3 \to 0 \tag{16}$$

and that if Π is abelian, then $[\Pi]$ is the class of the corresponding Galois representation.

The group Π may be identified via the Lie exponential with its Lie algebra $\operatorname{Lie}\Pi$, which has the structure of a p-adic Galois representation. Then $[\Pi] = [\operatorname{Lie}\Pi]$, since $\Pi = \operatorname{Lie}\Pi$, when Π is abelian, and a short exact sequence of the form (16) induces a corresponding short exact sequence of Lie algebras.

By repeated application of (15), we have

$$[U_n] = \sum_{k=1}^{n} [U[k]].$$

Note that if $h_1(X) \in \mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k, A)$ for an abelian variety A, then

$$[\Pi] \in K_0(\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k, A)) = K_0(\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A)) = K_0(\mathbb{G}) := K_0(\mathbf{Rep}(\mathbb{G})),$$

where $\mathbb{G} = \mathbb{G}(A)$. In practice, we will want to compute $[\Pi]$ explicitly as an element of $K_0(\mathbb{G})$. This is done in [Cor21, §7.4-6] in many cases when $\mathbb{G} = \mathrm{GL}_2$.

7.2. **Decomposition of $U[k]$ in terms of U_1.** We outline a general procedure for computing the class of U_n in $K_0(\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k))$ in terms of $h_1(X)$. In particular, as described in Remark 7.2, it may be used to give universal formulas for $[U[k]]$ in terms of λ operations in the λ-ring $K_0(\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k))$.

We suppose k is a positive integer less than or equal to n, until otherwise specified. If X has at most one puncture, then $U[1] = h_1(X)$ is pure of weight -1, so the $-k$th weight-graded piece is

$$\mathrm{Gr}^W_{-k} \operatorname{Lie} U_n = U[k].$$

As U_n is a unipotent group, we have $\mathcal{O}(U_n) \cong \operatorname{Sym} \operatorname{Lie} {U_n}^\vee$ as $\mathbb{Q}_p$-algebras, in a way that respects Galois action on associated graded. We thus have

$$\mathrm{Gr}^W_\bullet \mathcal{O}(U_n)^\vee \cong \mathrm{Gr}^W_\bullet \operatorname{Sym} \operatorname{Lie} U_n.$$

If we know the structure of $\mathrm{Gr}^W_\bullet \mathcal{O}(U_n)$ as a $\mathbb{G}$-represenation, this allows us to inductively compute the structure of $U[k] = U^k/U^{k+1}$, as follows. We have

$$\begin{aligned}
\mathrm{Gr}^W_{-k} \mathcal{O}(U_n)^\vee &= \mathrm{Gr}^W_{-k} \operatorname{Sym} \operatorname{Lie} U_n \\
&= \mathrm{Gr}^W_{-k} \operatorname{Sym} \mathrm{Gr}^W_{\geq -k} \operatorname{Lie} U_n \\
&= \mathrm{Gr}^W_{-k} \operatorname{Sym}(\mathrm{Gr}^W_{-k} \operatorname{Lie} U_n \oplus \mathrm{Gr}^W_{>-k} \operatorname{Lie} U_n) \\
&= \mathrm{Gr}^W_{-k} \operatorname{Lie} U_n \oplus \mathrm{Gr}^W_{-k} \operatorname{Sym} \mathrm{Gr}^W_{>-k} \operatorname{Lie} U_n \\
&= U[k] \oplus \mathrm{Gr}^W_{-k} \operatorname{Sym}(\oplus_{i=1}^{k-1} U[i])
\end{aligned}$$

Note that n is irrelevant here, as long as $n \geq k$. We thus get

$$[U[k]] = [\mathrm{Gr}^W_{-k} \mathcal{O}(U_n)^\vee] - [\mathrm{Gr}^W_{-k} \operatorname{Sym}(\oplus_{i=1}^{k-1} U[i])] = \mathrm{pr}_{-k}([\mathcal{O}(U_n)^\vee] - \mathrm{pr}_{-k}[\operatorname{Sym}(\oplus_{i=1}^{k-1} U[i])]) \tag{17}$$

The term $\mathrm{pr}_{-k}[\operatorname{Sym}(\oplus_{i=1}^{k-1} U[i])]$ decomposes according to nontrivial partitions of k. We represent a partition by a sequence $n_1, \cdots, n_k$ for which $\sum_{i=1}^{k} i n_i = k$, and we call it nontrivial if $n_k = 0$. We then have

$$\mathrm{pr}_{-k}[\mathrm{Sym}(\oplus_{i=1}^{k-1} U[i])] = \sum_{k=\sum_{i=1}^{k-1} in_i} \prod_{j=1}^{k-1} [\mathrm{Sym}^{n_j}(U[j])]. \tag{18}$$

Remark 7.2. Equation 17 is equivalent to the relation

$$\mathrm{HS}_U^{\mathrm{mot}}(t) := \sum_{k=0}^{\infty} \mathrm{pr}_k[\mathcal{O}(U)]t^k = \prod_{k\geq 1} (1-t^k)^{-[V_k^\vee]}.$$

of [BCL22, Proposition 2.2], where $(1-t^k)^{-[V_k^\vee]} := \sum_{i=0}^{\infty} s^i([V_k^\vee])t^i$, and $s^i(x) = (-1)^i \lambda^i(-x)$ is the ith symmetric power operation in the λ-ring $K_0(\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k))$. As described in [BCL22, Remark 2.3], the relation may be combined with a description of $\mathcal{O}(U)$ ([BCL22, Proposition 2.1] or the description just below) to find universal (independent of the curve) formulas for $[U[k]]$ in terms of $[U_2]$ and the λ-operations in $K_0(\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k))$. These relations may be expressed in terms of generating series by the relation ([BCL22, Proposition 2.1])

$$\mathrm{HS}_U^{\mathrm{mot}}(t) = \frac{1}{1 - h^1(\overline{X})t - (h^0(D_X)(-1) - [\mathbb{Q}_p(-1)])t^2}.$$

For a particular A and hence particular $\mathbb{G}$, one may compute these operations in the ring $K_0(\mathbb{G})$.

If X is affine, then U is a free pro-unipotent group, and we have

$$\mathrm{Gr}^W_{-k} \mathcal{O}(U_n)^\vee \cong h_1(X)^{\otimes k}.$$

7.2.1. *Projective Case.* Suppose X is projective, and let X' be the complement of a point in X. We set $U := U(X)$ and $U' := U(X')$. Then

$$\mathrm{Lie}\, U' \cong \mathrm{FreeLie}\, h_1(X)$$

is free on $h_1(X) = h_1(X')$, while

$$U$$

is the quotient of U' by an element of $U'^2 \setminus U'^3$ on which GL_2 acts as $M_{0,1}$. More precisely, this element corresponds to the dual

$$h_2(X) \to \wedge^2 h_1(X) \cong U(X')[2]$$

of the intersection pairing $\wedge^2 h^1(X) \to h^2(X)$.

For $k = 1, 2, 3$, this does nothing more than remove a copy of $h_2(X) \cong \mathbb{Q}_p(1)$ from $U'[2]$ and remove a copy of $h_1(X)(1)$ from $U'[3]$. In the projective case, we prefer to first compute the associated graded of the Lie algebra as if it were affine and then mod out by the appropriate Lie ideal.

7.3. **Computing Dimensions.** We discuss how to use the class $[\Pi]$ of §7.1 to bound $H^1_{f,S}(G_k;\Pi)$ and compute $H^1_f(G_v;\Pi)$, and thus to verify cases of the dimension inequality (5).

We define linear functions

$$d, l, d^S \colon K_0(\mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)) \to \mathbb{Z}$$

by $d([V]) = h^1_f(G_k;V)$ (resp. $l([V]) = h^1_f(G_v;V)$, $d^S([V]) = h^1_{f,s}(G_k;V)$). We set

$$c := l - d$$

and $c^S := l - d^S$.

We use the notation
$$d(\Pi)$$
(resp. $l(\Pi)$, $c(\Pi)$, $d^S(\Pi)$, $c^S(\Pi)$) to refer to $d([\Pi])$ (resp. $l([\Pi])$, $c([\Pi])$, $d^S([\Pi])$, $c^S([\Pi])$).

By the short exact sequence for Galois cohomology, we have
$$\dim H^1_{f,S}(G_k;\Pi) \leq d^S(\Pi) \tag{19}$$
$$\dim H^1_f(G_\mathfrak{p};\Pi) = l(\Pi), \tag{20}$$
the latter by the fact that the weights are non-zero and that crystalline H^2 vanishes.

Remark 7.3. In fact, it would follow from Conjecture 5.16 below that
$$\dim H^1_{f,S}(G_k;\Pi) = d^S(\Pi).$$

In the notation of §5.6, we have
$$d^S(\Pi) = \dim Z^1(U(F_{k,S,A});\Pi)^{\mathbb{G}}.$$

7.3.1. *Computing l and d.* Let $V \in \mathbf{Rep}^{\mathrm{sg}}_{\mathbb{Q}_p}(G_k)_n)$.

One may explicitly compute $l([V])$ for $n \leq -1$ and, assuming Conjecture 3.5, $d([V])$ for $n \leq -3$, as follows. For $n \leq -1$, we have
$$l([V]) = \dim_{\mathbb{Q}_p}(\mathrm{D}_{\mathrm{dR}}\, V/\, \mathrm{D}^+_{\mathrm{dR}}\, V) \tag{21}$$
by Fact 3.9, and for $n \leq -3$, we have
$$\sum_{v \in \{p\}_k} \dim_{\mathbb{Q}_p}(\mathrm{D}_{\mathrm{dR}}\, V_v/\, \mathrm{D}^+_{\mathrm{dR}}\, V_v) - \sum_{v \in \{\infty\}_k} h^0(G_v;V) \tag{22}$$
by Fact 3.8 if we assume Conjecture 3.5.

Suppose V appears as a component of a system of realizations, including a $\mathbb{Q}$-vector space V^{B}_v for every $v \in \{\infty\}_k$ with involution F_v (resp. $F_v = \mathrm{id}$) when v is real (resp. complex), and k-vector space V^{dR} with a Hodge filtration along with isomorphisms
$$\mathrm{D}_{\mathrm{dR}}(V) \otimes_{\mathbb{Q}_p} k_v \cong V^{\mathrm{dR}} \otimes_k k_v$$
for all $v \in \{p\}_k$, and
$$V^{\mathrm{B}}_v \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong V$$
for every $v \in \{\infty\}_k$ intertwining F_v with $c_v \in G_k$ when v is real. This is the case, for example, if $V = U[k]$, or more generally, if V is cut out of $U(X)$ by morphisms of k-schemes and group-theoretic operations. If the Mumford-Tate Conjecture (Remark 5.7) holds, this is true for any V appearing in $U[k]$ for some k. In either case, V^{B} and V^{dR} are defined as corresponding subquotients of the Betti and de Rham unipotent fundamental groups of X.

Then (21) and (22) become
$$l([V]) = \dim_k V^{\mathrm{dR}} - \dim_k F^0 V^{\mathrm{dR}} = \dim_k V^{\mathrm{dR}}/F^0 V^{\mathrm{dR}}$$ [xxv]
$$d([V]) = \dim_{\mathbb{Q}} V^{\mathrm{dR}}/F^0 V^{\mathrm{dR}} - \sum_{v \in \{\infty\}_k} (V^{\mathrm{B}})^{F_v} = [k:\mathbb{Q}]\, l([V]) - \sum_{v \in \{\infty\}_k} (V^{\mathrm{B}})^{F_v}$$

[xxv] More generally, without assuming $k_\mathfrak{p} \cong \mathbb{Q}_p$, we have $l([V]) = [k_\mathfrak{p} : \mathbb{Q}_p] \dim_k V^{\mathrm{dR}}/F^0 V^{\mathrm{dR}}$.

Notice in particular that these expressions depend only the real Hodge structure $V_v^{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{R}$ (with Frobenius if v is real). Therefore, they depend only on the genus g of X and $d := \#D_X(\overline{k})$. This is expressed by the relation ([BCL22, Lemma 2.6])

$$\mathrm{HS}_{\mathrm{loc}}(t) := \prod_{k \geq 1} (1-t^k)^{-h_f^1(G_{\mathfrak{p}};U[k])} = \frac{1-gt}{1-2g-(d-1)t^2}.$$

For $d([V])$ for $n = -1, -2$, we note that $d([U[1]]) = d([h_1(X)])$, which is the p-Selmer rank of J and hence its Mordell-Weil rank r assuming the Tate-Shafarevich Conjecture. We have (by the proof of [BCL22, Lemma 2.10])

$$d(U[2]) = [k:\mathbb{Q}]l([V]) - \sum_{v \in \{\infty\}_k} (V^{\mathrm{B}})^{F_v} + 1 - \rho - d,$$

where $\rho := \operatorname{rank}_{\mathbb{Z}} \operatorname{Hom}(\operatorname{Jac}(X), \operatorname{Jac}(X)^\vee)^+$.

In particular, under Conjecture 3.5, all ranks $h_f^1(G_k; U[k])$ depend only on g, d, ρ, r, and $d_1 := \#D_X(\mathbb{R})$, as expressed in [BCL22, §2] for $k = \mathbb{Q}$ by the relation

$$\mathrm{HS}_{\mathrm{glob}}(t) := \prod_{k \geq 1} (1-t^k)^{-h_f^1(G_k;U[k])}$$

$$= (1-t)^r(1-t^2)^{\rho+d-1} \frac{1-gt}{1-2gt-(d-1)t^2} \prod_{j=0}^{\infty} \left(\frac{1-2gt^{2^{j+1}}-(d-1)t^{2^{j+2}}}{(1-(d_1-1)t^{2^{j+1}})(1-2gt^{2^j}-(d-1)t^{2^{j+1}})} \right)^{-\frac{1}{2^{j+1}}}.$$

Let us briefly discuss d^S. In general, this requires a careful analysis of Frobenius weights as in §3.4. If S contains only places of good reduction for J, then $h_1(X)$ has Frobenius weights -1 at every $v \in S$, so $d^S([V]) = d([V])$ when $n \neq 2$ by Proposition 3.14. If J has bad reduction at some $v \in S$, then one must carefully analyze (the toroidal part of) the reduction of J at v. We carry out an explicit example in [Cor21, §4.1, 5.3], where $d^S([V]) \neq d([V])$ for $n = 3$ and in particular depends on the number of places of split multiplicative reduction in S.

8. Coordinates and Weight Filtrations on Tannakian Selmer Varieties

In this section, we discuss how one might use the theory thus described to explicitly put coordinates on Π, $U(\mathcal{O}_{k,S}, A)$, and $H^1_{f,S}(G_k;\Pi)$ so as to compute the image of $\log_{\mathrm{BK}}$ of (4). We fix k, S, $\mathcal{X}$, $b \in \mathcal{X}(\mathcal{O}_{k,s})$, and Π as in §2. We focus on elements of $\mathcal{X}(\mathcal{O}_{k,s})_\alpha$ for

$$\alpha = \prod_{v \in T_0 \setminus S} \kappa_v(b)$$

in the sense of §2.4.2. We may of course enlarge S to contain all of T_0 so that $\mathcal{X}(\mathcal{O}_{k,s})_\alpha = \mathcal{X}(\mathcal{O}_{k,s})$.

8.1. Abstract Coordinates on $U(\mathcal{O}_{k,S}, A)$.

Let $I \subseteq \operatorname{Irr} \mathbb{G}$ be a finite subset satisfying the conditions of Proposition 5.17. We fix a basis Σ_V of $H^1_{f,S}(G_k;V)^\vee$ for each $V \in I$ and a basis T_v of V (often adapted to the tensor product in $\mathbf{Rep}\,\mathbb{G}$). We let

$$\Sigma := \coprod_{V \in I} \Sigma_V \times T_V.$$

We have a degree function on Σ sending an element of $\Sigma_V \times T_V$ to $w(V)$.

Then Σ is a basis of $F^I_{k,S,A}$. This provides a basis of $\mathcal{U}(F_{k,S,A})^I$ consisting of words in Σ. There is a grading given by the degree function on Σ, for which the increasing filtration is the weight filtration and thus independent of Σ.

We have a dual basis consisting of

$$f_w \in A(F_{k,S,A})^I$$

for w a word in Σ, with shuffle product and deconcatenation coproduct as in [CDC20, §2.1.3]. Note that since Σ is finite, $A(F_{k,S,A})^I$ is finite-dimensional in each degree.

8.2. **Abstract Coordinates on $Z^1(U(\mathcal{O}_{k,S}, A); \Pi)^{\mathbb{G}}$.** We would like to put coordinates on $Z^1(U(F_{k,S,A}); \Pi)^{\mathbb{G}}$. By Proposition 5.17, we have $Z^1(U(F_{k,S,A}); \Pi)^{\mathbb{G}} = Z^1(U(F_{k,S,A})^I; \Pi)^{\mathbb{G}}$, so we may focus on the latter.

In [CDC20, §3.3], we put coordinates on $Z^1(U(F_{k,S,A})^I; \Pi)^{\mathbb{G}}$ for $\mathbb{G} = \mathbb{G}_m$ and Π a quotient of the polylogarithmic quotient of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. In this case, we heavily rely on the fact that $U(\mathcal{O}_{k,S}, A)$ acts trivially on Π.

Nonetheless, it is possible to write down coordinates more generally, based on the following proposition, due in a variant form to M. Lüdtke and appearing in ongoing joint work of the author, I. Dan-Cohen, and M. Lüdtke:

Proposition 8.1. *With I as in Proposition 5.17, there is an isomorphism of functors*

$$Z^1(U(F_{k,S,A}); A); \Pi_R)^{\mathbb{G}} = Z^1(U(F_{k,S,A})^I; A); \Pi_R)^{\mathbb{G}} \cong \prod_{V \in I} \mathrm{Hom}_{\mathbb{G}}(V, \mathrm{Lie}\,\Pi)_R^{\Sigma_V}$$

on $\mathbb{Q}_p$-algebras R.

Proof. The set of cocycles is the set of sections of the projection

$$\Pi_R \rtimes U(F_{k,S,A})^I \twoheadrightarrow U(F_{k,S,A})^I.$$

As $U(F_{k,S,A})^I$ is free on $F^I_{k,S,A}$, this is the set of maps of vector spaces $F^I_{k,S,A} \to \mathrm{Lie}(\Pi_R \rtimes U(F_{k,S,A})^I)$ respecting the projection to $U(F_{k,S,A})^I$, or equivalently, linear maps

$$\mathrm{Hom}_{\mathrm{Mod}_R}((F^I_{k,S,A})_R, \mathrm{Lie}\,\Pi_R).$$

The condition of $\mathbb{G}$-equivariance cuts out the subset

$$\mathrm{Hom}_{\mathbb{G}-\mathrm{Mod}_R}((F^I_{k,S,A})_R, \mathrm{Lie}\,\Pi_R) = \prod_{V \in I} \mathrm{Hom}_{\mathbb{G}-\mathrm{Mod}_R}(H^1_{f,S}(G_k; V)^\vee \otimes_{\mathbb{Q}_p} V_R, \mathrm{Lie}\,\Pi_R).$$

By tensor adjunction, we have

$$\mathrm{Hom}_{\mathbb{G}-\mathrm{Mod}_R}(H^1_{f,S}(G_k; V)^\vee \otimes_{\mathbb{Q}_p} V_R, \mathrm{Lie}\,\Pi_R) = \mathrm{Hom}_{\mathbb{Q}_p}(H^1_{f,S}(G_k; V)^\vee, \mathrm{Hom}_{\mathbb{G}}(V, \mathrm{Lie}\,\Pi)_R)).$$

Finally, since Σ_V is a basis of $H^1_{f,S}(G_k; V)^\vee$, this last set is in bijection with $\mathrm{Hom}_{\mathbb{G}}(V, \mathrm{Lie}\,\Pi_R)^{\Sigma_V}$, so we are done. $\square$

By Schur's Lemma, $\mathrm{Hom}_{\mathbb{G}}(V, \mathrm{Lie}\,\Pi)$ is a vector space of dimension equal to the number of copies of V appearing in Π, i.e., the image of $[\Pi]$ under the projection $K_0(\mathbf{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(G_k, A)) \twoheadrightarrow \mathbb{Q}[V]$. Fixing a basis Υ_V of $\mathrm{Hom}_{\mathbb{G}}(V, \mathrm{Lie}\,\Pi)$ for each $V \in I$, we find that

$$\bigcup_{V \in I} \Upsilon_V \times \Sigma_V \tag{23}$$

is a set of coordinates on the affine space $Z^1(U(F_{k,S,A}); A); \Pi)^{\mathbb{G}}$.

Note that as a $\mathbb{G}$-representation, $\operatorname{Lie}\Pi \cong \prod_{V\in I} \operatorname{Hom}_{\mathbb{G}}(V, \operatorname{Lie}\Pi) \otimes V$. Thus $\bigcup_{V\in I} \Upsilon_V \times T_V$ is a basis of $\operatorname{Lie}\Pi$, thus corresponding to coordinates on the affine space Π via the exponential map. We generally choose Υ_V in such a way that these coordinates are easily expressible in terms of elements of $\mathcal{O}(\Pi)$ given by words in differential forms on X as in Chen's theory.

8.2.1. *Weight Filtration on the Selmer Variety.* Each $V \in \operatorname{Irr}\mathbb{G}$ is a direct factor of $h^1(A)^{\otimes n}$ for some $n \in \mathbb{Z}$ and therefore has weight $w(V) := n$ as a Galois representation. We may thus define a degree function on the basis (23), which in turn defines a grading on the coordinate ring of $Z^1(U(F_{k,S,A}); A); \Pi)^{\mathbb{G}}$. The increasing filtration associated to this grading is the weight filtration of [Bet21, §3.2.1].

8.3. **Galois Action and the Cocycle Evaluation Map.** We have described a way to put coordinates on $Z^1(U(F_{k,S,A}); A); \Pi)^{\mathbb{G}}$. To compute $\mathfrak{ev}_{\Pi,F_{k,S,A}}$, we need, we need to know how to describe a full cocycle in terms of these coordinates. In other words, we need to know how to describe a section of

$$\Pi \rtimes U(F_{k,S,A})^I \twoheadrightarrow U(F_{k,S,A})^I$$

in terms of what it does to the set Σ of generators of $U(F_{k,S,A})^I$. One must therefore understand the group structure on $\Pi \rtimes U(F_{k,S,A})^I$, or in other words, the action of $U(F_{k,S,A})^I$ on Π.

In specific cases such as $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $X = E \setminus \{O\}$, this is given by the Ihara action ([Iha90]) and the Nakamura action ([Nak13]), respectively. In general, we suggest the following method for computing the action. We suppose for simplicity that X is projective, so that $U[n]$ is semisimple for each n.

The idea is to compare coproduct formulae in $\mathcal{O}(\Pi)$ and $A(F_{k,S,A})$. This is because of the simple observation that the cocycle condition $c(g_1g_2) = c(g_1)(g_1c(g_2))$ is equivalent to:

$$g_1c(g_2) = c(g_1)^{-1}c(g_1g_2).$$

The left-hand side gives the action of an arbitrary $g_1 \in U(F_{k,S,A})$ on the image $c(g_2)$ of a cocycle in Π, while the right-hand side depends only on the group law in Π and $U(F_{k,S,A})$. There is a subtlety in that this determines the action only on $\operatorname{Im}(\mathfrak{ev}_\Pi)$, which is not all of Π if the dimension inequality (5) holds. However, we may carefully use this use this observation to our advantage to compute the image of the cocycle evaluation map by inductively computing the action of $U(F_{k,S,A})$ on $\Pi' := U(F_{k,S,A}) \operatorname{Im}(\mathfrak{ev}_\Pi)$ in each degree. We note that $\Pi' = \operatorname{Im}(\mathfrak{ev}_{\Pi'})$, since any cocycle must take values in Π'.

We let Π^{n+1} denote the image of U^{n+1} in Π, $\Pi_n := \Pi/\Pi^{n+1}$, and $\Pi[n] = \Pi^n/\Pi^{n+1} \subseteq \Pi_n$. We set Π'_n to be the preimage in Π_{n+1} of $U(F_{k,S,A}) \operatorname{Im}(\mathfrak{ev}_{\Pi'_n})$.

We now claim that we may inductively compute the action of $U(Z, A)$ on Π'_{n+1}. Supposing we have computed it for n, we let $\pi \in \Pi'_{n+1}$. Then $U(F_{k,S,A}) \operatorname{Im}(\mathfrak{ev}_{\Pi'_{n+1}})$ surjects onto $U(F_{k,S,A}) \operatorname{Im}(\mathfrak{ev}_{\Pi'_n}) = \Pi'_{n+1}/(\Pi'_{n+1} \cap \Pi[n+1])$. That means we can write $\pi = \pi'\pi''$ for $\pi' \in U(F_{k,S,A}) \operatorname{Im}(\mathfrak{ev}_{\Pi'_{n+1}})$ and $\pi'' \in \Pi'_{n+1} \cap \Pi[n+1]$. Now the action on π' may be computed by the above procedure, while the action on π'' is trivial because $\Pi[n+1]$ is semisimple.

To compute the group law in $U(Z, A)$, we will need a version of Goncharov's coproduct formula ([Gon05, Theorem 1.2]) for elements of $\mathcal{O}(U_E)$. One might either use the motivic correlators of [Gon19] and their associated cobracket or the coproduct for tensor products of cycles in [Pat13].

Appendix A. Unipotent Groups and Cohomology

A.1. **Unipotent Groups.** Let U be a pro-unipotent group over a field K. Then we have the Lie algebra $\operatorname{Lie} U$ and its universal enveloping algebra $\mathcal{U}U$, along with the coordinate ring $\mathcal{O}(U)$. The first two are naturally a Lie algebra object and cocommutative Hopf algebra object, respectively, of $\operatorname{Pro}\mathbf{Vect}_K^{\mathrm{fin}}$, while the latter is a commutative Hopf algebra object of $\operatorname{Ind}\mathbf{Vect}_K^{\mathrm{fin}}$. The natural duality between $\operatorname{Pro}\mathbf{Vect}_K^{\mathrm{fin}}$ and $\operatorname{Ind}\mathbf{Vect}_K^{\mathrm{fin}}$ induces a natural isomorphism

$$\mathcal{O}(U) \cong \mathcal{U}U^{\vee}.$$

If Π is a unipotent group with an action of U (equivalently, Π is a unipotent group in the Tannakian category $\mathbf{Rep}_K(U)$ in the sense of [Del89, §5]), then $M := \mathcal{U}\Pi$ has the structure of a module over

$$A := \mathcal{U}U.$$

If $\rho\colon A \otimes M \to M$ denotes the multiplication map, then

$$\rho \circ (\mathrm{id}_A \otimes \mathrm{mult}_M) = \mathrm{mult}_M \circ (\rho \otimes \rho) \circ (\Delta_A \otimes \mathrm{id}_{M\otimes M})\colon A \otimes M \otimes M \to M,$$

where $\rho \otimes \rho$ sends $u_1 \otimes u_2 \otimes \pi_1 \otimes \pi_2$ to $\rho(u_1 \otimes \pi_1) \otimes \rho(u_2 \otimes \pi_2)$. This implies that

$$\operatorname{Lie} U \subseteq \mathcal{U}U = A$$

acts via derivations on M, while

$$U \subseteq \mathcal{U}U = A$$

acts via automorphisms on M. We also have

$$\Delta_M \circ \rho = (\rho \otimes \rho) \circ (\Delta_A \otimes \Delta_M)\colon A \otimes M \to M \otimes M,$$

which implies that the action of $\operatorname{Lie} U \subseteq A$ preserves $\operatorname{Lie}\Pi \subseteq M$, and the action of $U \subseteq A$ preserves both $\operatorname{Lie}\Pi$ and Π in M, acting via derivations and automorphisms,[xxvi] respectively.

For an object W of $\operatorname{Pro}\mathbf{Vect}_K^{\mathrm{fin}}$, we denote by

$$\operatorname{FreeLie} W$$

the free pro-nilpotent Lie algebra on W.

Proposition A.1. *We have*

$$Z^1(U;\Pi) = Z^1(\mathfrak{n}(U);\operatorname{Lie}\Pi),$$

where

$$Z^1(\mathfrak{n}(U);\operatorname{Lie}\Pi)$$

denotes the space of K-linear maps

$$c\colon \mathfrak{n}(U) \to \operatorname{Lie}\Pi$$

satisfying the cocycle condition

$$c([g,h]) = [c(g), c(h)] + g(c(h)) - h(c(g)),$$

where $\mathfrak{n}(U)$ acts on $\operatorname{Lie}\Pi$ by derivations.

[xxvi]In fact, they preserve the coalgebra structure, in that elements of $\operatorname{Lie} U$ act as coderivations, and elements of U act as automorphisms of the Hopf algebra.

Proof. Either can easily be seen to be in bijection with the space of K-linear maps

$$c\colon A \to M$$

satisfying the two conditions

- $(c \otimes c) \circ \Delta_A = \Delta_M \circ c\colon A \to M \otimes M$ (i.e., c is a homomorphism of coalgebras)
- $c \circ \mathrm{mult}_A = \mathrm{mult}_M \circ (\mathrm{id}_M \otimes \rho) \circ (c \otimes \mathrm{id}_A \otimes c) \circ (\Delta_A \otimes \mathrm{id}_A)\colon A \otimes A \to M$.

□

A.2. **Extension of a Reductive Group by a Unipotent Group.** Let $\mathbb{G}$ be a reductive group acting on U, and let

$$G := \mathbb{G} \ltimes U.$$

We suppose that the action of U on Π extends to an action of G on Π. The subgroup $\mathbb{G} \subseteq G$ acts on U by conjugation and on Π via the latter's action of G.

The action of $\mathbb{G}$ on U induces an action on $\mathrm{Lie}\, U$ and on $\mathcal{U}(U)$ by Lie and Hopf algebra automorphisms, respectively. We also let U act on $\mathcal{U}(U)$ by left-translation. This is compatible with the action of $\mathbb{G}$ on $\mathcal{U}(U)$ in that it extends to an action of

$$G$$

on

$$\mathcal{U}(U).$$

Remark A.2. For $g \in \mathrm{GL}_2$, $u \in U(\mathcal{O}_{k,S}, E)$, and $\pi \in \Pi$, we have

$$(g(u))(g\pi) = (gug^{-1})(g\pi) = g(u\pi),$$

implying that the action map

$$U(\mathcal{O}_{k,S}, E) \times \Pi \to \Pi$$

is GL_2-equivariant. The same is true of the associated Lie algebra and universal enveloping algebra actions as in §A.1.

A.3. **Cohomology and Cocycles.** The main goal of this section is to prove a generalization of [DCW16, Proposition 5.2].

Some computations below use the fact (which follows from the definition of semidirect product) that for $g \in \mathbb{G}$, $u \in U$, and $\pi \in \Pi$, we have

$$g(u)(\pi) = g(u(g^{-1}(\pi))).$$

We recall some facts about nonabelian cocycles. Some of our discussion borrows from [Bro17a, §6.1]. Although we do not denote it in the notation, all points are relative to a K-algebra R.

For a cocycle $c\colon G \to \Pi$ and $g \in G$, we denote $c(g)$ by c_g. We recall that for a K-algebra R, we have

$$Z^1(G;\Pi)(R) := \{c\colon G_R \to \Pi_R \mid c_{g_1 g_2} = c_{g_1} g_1(c_{g_2})\, \forall\, g_1, g_2 \in G\},$$

where the cocycle condition is imposed on the functor of points. We similarly define $Z^1(U;\Pi)$. The trivial cocycle $1 \in Z^1(G;\Pi)$ (resp. $Z^1(U;\Pi)$) sends all of G (resp. U) to the identity of Π.

Recall also that Π acts on $Z^1(G;\Pi)$ (resp. $Z^1(U;\Pi)$) by sending $\pi \in \Pi$ and $c \in Z^1(G;\Pi)$ (resp. $c \in Z^1(U;\Pi)$) to

$$g \mapsto \pi c_g g(\pi^{-1})$$

(resp. $u \mapsto \pi c_u u(\pi^{-1})$) and that

$$H^1(G;\Pi) := Z^1(G;\Pi)/\Pi.$$

Lemma A.3. *We have*

$$Z^1(G;\Pi) \cong \ker(\mathrm{Hom}(G,\Pi \rtimes G) \to \mathrm{Hom}(G,G)).$$

given by

$$c \mapsto \rho_c,$$

where $\rho_c(g) := c_g g \in \Pi \rtimes G$. *The action of* Π *is given by conjugation on* $\Pi \rtimes G$.

Proof. Since

$$\rho_c(g_1)\rho_c(g_2) = (c_{g_1}g_1)(c_{g_2}g_2) = c_{g_1}g_1(c_{g_2})g_1g_2 = c_{g_1}g_1(c_{g_2})g_1g_2,$$

while

$$\rho_c(g_1g_2) = c_{g_1g_2}g_1g_2,$$

the homomorphism condition for c is equivalent to the cocycle condition for ρ.

To check the action of Π, note that for $\pi \in \Pi$ and $g \in G$, we have

$$\begin{aligned}
\pi\rho_c(g)\pi^{-1} &= \pi(c_g g)\pi^{-1} \\
&= \pi c_g g\pi^{-1} \\
&= \pi c_g g(\pi^{-1})g \\
&= \pi(c)_g g \\
&= \rho_{\pi(c)}(g),
\end{aligned}$$

as desired. $\square$

Recall that $\mathbb{G}$ acts on $Z^1(U;\Pi)$ by

$$g(c)_u = g(c_{g^{-1}(u)}),$$

and the Π- and $\mathbb{G}$-actions on $Z^1(U;\Pi)$ are compatible in that

$$g(\pi(c)) = g(\pi)(g(c))),$$

so that the $\mathbb{G}$-action induces an action on

$$H^1(U;\Pi).$$

The main result of this section is the following generalization of [DCW16, Proposition 5.2]:

Theorem A.4. *Suppose that* $\Pi^{\mathbb{G}} = 1$. *Then each equivalence class of cocycles* $[c]$ *in* $H^1(G;\Pi)$ *contains a unique representative* c_0 *such that*

$$c_0(g) = 1 \text{ for each } g \in \mathbb{G},$$

and its restriction to U *is a* $\mathbb{G}$*-equivariant cocycle*

$$c_0\big|_U : U \to \Pi.$$

The map

$$[c] \mapsto c_0\big|_U$$

defines a bijection

$$H^1(G;\Pi) \cong Z^1(U;\Pi)^{\mathbb{G}}.$$

We prove this result via a series of lemmas.

Lemma A.5. *The restriction map*

$$Z^1(G;\Pi) \to Z^1(U;\Pi)$$

induces a bijection

$$\ker(Z^1(G;\Pi) \to Z^1(\mathbb{G};\Pi)) \cong Z^1(U;\Pi)^{\mathbb{G}}$$

Proof. Suppose $c \in \ker(Z^1(G;\Pi) \to Z^1(\mathbb{G};\Pi))$. Then for $u \in U$, we have

$$\begin{aligned} g(c)_u &= g(c_{g^{-1}(u)}) \\ &= g(c_{g^{-1}ug}) \\ &= g(c_{g^{-1}u}g^{-1}(u(c_g))) \\ &= g(c_{g^{-1}u}) \\ &= g(c_{g^{-1}}g^{-1}(c_u)) \\ &= g(g^{-1}(c_u)) \\ &= c_u \end{aligned}$$

Conversely, suppose $c \in Z^1(U;\Pi)^{\mathbb{G}}$. We extend c to an element of $Z^1(G;\Pi)$ as follows. For $ug \in G$, we set

$$c_{ug} = c_u.$$

It is clear by construction that it is trivial when restricted to $\mathbb{G}$. To show that it is a cocycle, suppose we have $g_1, g_2 \in \mathbb{G}$ and $u_1, u_2 \in U$. Then

$$\begin{aligned} c_{u_1g_1u_2g_2} &= c_{u_1g_1u_2g_1^{-1}g_1g_2} \\ &= c_{u_1g_1(u_2)g_1g_2} \\ &= c_{u_1g_1(u_2)} \\ &= c_{u_1}u_1(c_{g_1(u_2)}) \\ &= c_{u_1}u_1(g_1(c_{u_2})) \\ &= c_{u_1g_2}u_1(g_1(c_{u_2g_2})), \end{aligned}$$

as desired.

Given $c \in Z^1(U;\Pi)^{\mathbb{G}}$, it is clear that the maps are mutual inverses. Given $c \in \ker(Z^1(G;\Pi) \to Z^1(\mathbb{G};\Pi))$, note that

$$c_{ug} = c_u,$$

so the maps are indeed mutual inverses.

□

Lemma A.6. *The natural map*

$$H^1(G;\Pi) \to H^1(U;\Pi)$$

induces an isomorphism

$$H^1(G;\Pi) \cong H^1(U;\Pi)^{\mathbb{G}}$$

Proof. As $\mathbb{G}$ is reductive, all higher group cohomology of $\mathbb{G}$ vanishes, so the Hochschild-Serre spectral sequence for the inclusion $U \hookrightarrow G$ degenerates. In particular, we get

$$H^1(G;\Pi) = H^0(\mathbb{G}; H^1(U;\Pi)) = H^1(U;\Pi)^{\mathbb{G}}.$$

□

By Lemma A.5 and Lemma A.6, we have a diagram

$$\begin{tikzcd}
\ker(Z^1(G;\Pi) \to Z^1(\mathbb{G};\Pi)) \arrow[r] \arrow[d, "\cong"'] & H^1(G;\Pi) \arrow[d, "\cong"] \\
Z^1(U;\Pi)^{\mathbb{G}} \arrow[r] & H^1(U;\Pi)^{\mathbb{G}}
\end{tikzcd}$$

It now suffices to prove that either horizontal arrow of the diagram is an isomorphism. We prove this for the top horizontal arrow whenever $\Pi^{\mathbb{G}} = 0$.

Lemma A.7. *Suppose $\Pi^{\mathbb{G}} = 0$. Then the set of splittings of $\Pi \rtimes \mathbb{G} \to \mathbb{G}$ forms a pointed torsor under Π acting by conjugation.*

Proof. To check transitivity, note that the set of splittings modulo Π is the cohomology set $H^1(\mathbb{G};\Pi)$. But this vanishes because $\mathbb{G}$ is reductive. The action is simply transitive because the stabilizer of a section is $\Pi^{\mathbb{G}}$, which is trivial by assumption. The point of the torsor is the trivial splitting $\mathbb{G} \hookrightarrow \Pi \rtimes \mathbb{G}$. □

Lemma A.8. *The set*

$$\ker(Z^1(G;\Pi) \to Z^1(\mathbb{G};\Pi)) \subseteq Z^1(G;\Pi)$$

maps bijectively onto $H^1(G;\Pi)$.

Proof. By Lemma A.3, under the identification

$$Z^1(G;\Pi) = \ker(\mathrm{Hom}(G, \Pi \rtimes G) \to \mathrm{Hom}(G,G)),$$

the action of Π is by conjugation. Via the inclusion $\mathbb{G} \subseteq G$, we may consider $\Pi \rtimes \mathbb{G}$ as a subgroup of $\Pi \rtimes G$ (it is not normal, but this does not matter). For any $\rho \in \ker(\mathrm{Hom}(G, \Pi \rtimes G) \to \mathrm{Hom}(G,G))$, $\rho_c\big|_{\mathbb{G}}$ is a splitting of $\Pi \rtimes \mathbb{G} \to \mathbb{G}$.

For $\pi \in \Pi$, we have $\rho_{\pi(c)}\big|_{\mathbb{G}} = \pi\rho_c\big|_{\mathbb{G}}\pi^{-1}$. Therefore, by Lemma A.7, there is a unique $\pi \in \Pi$ for which $\rho_{\pi(c)}\big|_{\mathbb{G}}$ is the trivial splitting.

Now $\rho_{\pi(c)}\big|_{\mathbb{G}}$ is the trivial splitting iff $c\big|_{\mathbb{G}}$ is trivial, i.e. if $c \in \ker(Z^1(G;\Pi) \to Z^1(\mathbb{G};\Pi))$. Therefore, each class $[c] \in H^1(G;U)$ contains a unique representative in $\ker(Z^1(G;\Pi) \to Z^1(\mathbb{G};\Pi))$. □

Remark A.9. One may alternatively show that the bottom horizontal arrow is an isomorphism, as follows. If Π is abelian, this amounts to considering the short exact sequence

$$0 \to \Pi/\Pi^U \to Z^1(U,\Pi) \to H^1(U,\Pi) \to 0$$

and then taking $\mathbb{G}$-fixed points and using the fact that $\mathbb{G}$ is reductive and $\Pi^{\mathbb{G}} = 0$ to get

$$Z^1(U,\Pi)^{\mathbb{G}} \cong H^1(U,\Pi)^{\mathbb{G}}.$$

One may replicate this argument for general Π by showing that if Π is a nonabelian group acting on a pointed set (or variety) Y, and both have a compatible action of $\mathbb{G}$, then there is a long exact sequence of pointed sets

$$\Pi^{\mathbb{G}} \to Y^{\mathbb{G}} \to (Y/\Pi)^{\mathbb{G}} \to H^1(\mathbb{G};\Pi).$$

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