On separable non-cooperative zero-sum games

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Non-cooperative zero-sum games which can be converted to linear programs are considered. The obtained linear programs have very large restriction matrices but the rank of these matrices is significantly lower than their size. An important example of such games is the resource allocation problem. A method for constructing an equilibrium point in such games is developed.

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1 Introduction

We consider non-cooperative n-person games which can be converted to linear programs. Usually the size of the corresponding program is large. However the rank of its restriction matrix is often not so large. For this reason it is possible to construct an efficient algorithm for solving such games. An important example of such games is the resource allocation problem. The two-person games of such type are the well known Blotto games [1], sums of matrix games [2]. A special case of such n-person game (polymatrix game) is considered in [3]. Our aims are to construct an algorithm for efficient solving of linear programming problems with small rank of restriction matrices and to describe the class of non-cooperative games for which such an approach is admissible. The algorithm is based on the column generation technique in the simplex method developed for linear programs with very large numbers of variables, but with a relatively small number of constraints (see [4, 5]). The considered problems have very large numbers of both: variables and constraints, therefore we need some generalization of the column generation technique. An alternative method for solving very large linear programs with small rank matrices was considered in [6, 7]. In section 2 we introduce a class of non-cooperative games: separable games, which can be reduced to linear programs with non-full rank matrices. In sections 3 – 5 we consider important particular cases of separable games, which can be converted to linear programs with the rank of restriction matrix significantly smaller than in the general case. Sections 6 – 7 are devoted to the method description.

2 Definitions and main properties

We shall use the following denotations. The set of integers from i up to j is denoted by $i : j$; we use brackets for vector and matrix components, $x[S]$ is a vector with components $x[s],\ s \in S$; $A[S,T]$ is a matrix with components $A[s,t],\ s \in S,\ t \in T$.

Let Γ be a non-cooperative n-person game in which each player has a finite number of strategies. Let X^i be a finite set of strategies of the i-th player. By a situation an n-vector composed of the strategies of each player is meant. Let X be a set of situations in Γ, i.e. $X = \prod_{i=1}^n X^i = \{x = (x^1, \ldots, x^n),\ x^i \in X^i,\ i \in 1 : n\}$. We say Γ is a separable game if the payoff function of each player is the sum of his payoffs from the other players, i.e. if $H^i(x)$ is the payoff function of the i-th player, then

$$H^i(x) = \sum_{j=1}^{n} H_{ij}(x^i, x^j), \tag{1}$$

where $x = (x^1, x^2, \ldots, x^n),\ x^i \in X^i$. Game Γ is a zero-sum game if $\sum_{i=1}^n H^i(x) = 0$ for each $x \in X$.

Let $s = (s^1, s^2, \ldots, s^n)$ be a situation in mixed strategies in Γ, i.e. s^i is a probability distribution on X^i:

$$s^i[x^i] \geq 0, \sum_{x^i \in X^i} s^i[x^i] = 1.$$

We denote the set of mixed strategies of player i by S^i, the set of all situations in the mixed strategies by S. The situation $\bar{s}$ is a Nash equilibrium point if the expected value of the payoff function of player i

$$H_i(s) = \sum_{x \in X} H^i(x) \prod_{j=1}^{n} s^j[x^j] \tag{2}$$

has the following property:

$$H_i(\bar{s} \| s^i) \leq H_i(\bar{s}),\ i \in 1 : n,\ s \in S,$$

where $\bar{s} \| s^i$ is the situation $(\bar{s}^1, \bar{s}^2, \ldots, \bar{s}^{i-1}, s^i, \bar{s}^{i+1}, \ldots, \bar{s}^n)$.

For the separable games the payoff functions (2) can be represented in a simpler form.

Lemma 1: *If* Γ *is a separable zero-sum non-cooperative game, then*

$$H_i(s) = \sum_{j=1}^{n} \sum_{x^i \in X^i} \sum_{x^j \in X^j} H_{ij}(x) s^i[x^i] s^j[x^j]. \tag{3}$$

Proof: Replacing $H_i(x)$ in (2) in accordance with (1), we obtain

$$H_i(s) = \sum_{x \in X} (\sum_{j=1}^{n} H_{ij}(x^i, x^j)) \prod_{k=1}^{n} s^k[x^k] = \sum_{j=1}^{n} \sum_{x \in X} H_{ij}(x^i, x^j) \prod_{k=1}^{n} s^k[x^k] =$$
$$= \sum_{j=1}^{n} \sum_{x^i \in X^i} \sum_{x^j \in X^j} H_{ij}(x^i, x^j) s^i[x^i] s^j[x^j] \sum_{x^{ij} \in X^{ij}} (\prod_{k \neq i, k \neq j} s^k[x^k]),$$

where x^{ij} is an $(n-2)$-vector of all the components of vector x except x^i and x^j, and X^{ij} is $\prod_{k \neq i, k \neq j} X^k$. Because

$$\sum_{x^{ij} \in X^{ij}} (\prod_{k \neq i, k \neq j} s^k[x^k]) = \prod_{k \neq i, k \neq j} (\sum_{x^k \in X^k} s^k[x^k]) = 1,$$

we obtain (3). The lemma is proved.

Thus we have obtained a game $\Gamma^{'}$ with the set of strategies of the i-th player S^i and with payoff functions

$$H_i(s) = \sum_{j=1}^{n} \sum_{x^i \in X^i} \sum_{x^j \in X^j} H_{ij}(x^i, x^j) s^i[x^i] s^j[x^j].$$

It is clear that $\Gamma^{'}$ is a zero-sum game. Game $\Gamma^{'}$ is equivalent to Γ, i.e. if s is an equilibrium point in $\Gamma^{'}$, then it is an equilibrium point in mixed strategies in Γ and vice versa. Let us consider now the following two-person zero-sum game Δ. The set of strategies of both players in Δ is the set of all situations in $\Gamma^{'}$, i.e. the set of all vectors $s \in S$, and the payoff function of the first player in Δ is

$$\Phi(s,t) = \sum_{i=1}^{n} H_i(t \| s^i). \tag{4}$$

Lemma 2: (i) *Game* Δ *is "fair" (i.e. its value is 0 and the sets of optimal strategies of the first and the second player are the same).*

(ii) *Game* Δ *is equivalent to* $\Gamma^{'}$, *i.e. if* s *is an optimal strategy of the first (second) player in* Δ, *then* s *is an equilibrium point in* $\Gamma^{'}$ *and vice versa.*

Proof: Payoff functions $H_i(s)$ in $\Gamma^{'}$ are bilinear functions of s^i , hence $H_i(s)$ is a concave function of variable s^i and a convex function of the set of variables $(s^1, \ldots, s^{i-1}, s^{i+1}, \ldots, s^n)$. Therefore Δ is a convex game in the sense of [8] and it satisfies all the conditions of lemma 10.3 in [8]. Hence the lemma's statement is true.

Thus we should get a solution of the zero-sum two-person game Δ in order to solve games Γ and $\Gamma^{'}$. The game Δ is a polyhedral game, and we need to determine

$$\max_{s\in S}\min_{t\in S}\Phi(s,t),$$

where S is a polyhedron:

$$S=\{s=(s^1[X^1],s^2[X^2],\ldots,s^n[X^n]):s^i[X^i]\geq 0,\sum_{x^i\in X^i}s^i[x^i]=1\}.$$

A polyhedral game can be converted to a matrix game. For instance, let G be a polyhedral game with bilinear payoff function $K(x,y)$ and with compact polyhedral sets of strategies of the first and the second players M and N respectively. Let $ex(M)$ and $ex(N)$ be the sets of extreme points of M and N. We consider the matrix game $G^{'}$ with the matrix A whose rows are the elements of $ex(M)$, columns are the elements of $ex(N)$. We define $A[s,t]=K(s,t)$ for $s\in ex(M),\ t\in ex(N)$. The solution of G is obtained from the optimal mixed strategies of $G^{'}$ in the following way. If $\sigma[ex(M)]$ is an optimal mixed strategy of the first player in the game $G^{'}$, then $x=\sum_{s\in ex(M)}\sigma[s]s$ is the optimal strategy of the first player in the game G. The same statement holds for the second player.

When we convert the game Δ to the matrix game in such a way, we obtain the matrix game $\Delta^{'}$ with a square matrix A of order $|ex(S)|=\prod_{i=1}^{n}|X^i|$, where $|Y|$ is the number of elements in set Y. However, the rank of matrix A is significantly smaller than its size. This statement is a corollary of the next theorem. At first we introduce the following notations. We consider a matrix B consisting of zeros and ones. Each row of B corresponds to some situation of game Γ, that is B has $\prod_{i=1}^{n}|X^i|$ rows. The columns of B are divided into n blocks. The i-th block corresponds to set X^i and consists of $|X^i|$ columns. Matrix B has n ones in each row: one 1 in each block, namely if some row corresponds to situation $x=(x^1,x^2,\ldots,x^n)$, then it has 1 in the i-th block in column corresponding to strategy x^i.

Then we consider a matrix H consisting of blocks $H_{ij}[X^i,X^j]$:

$$H=\begin{pmatrix} H_{11} & H_{12} & \ldots & H_{1n}\\ . & . & \ldots & .\\ . & . & \ldots & .\\ H_{n1} & H_{n2} & \ldots & H_{nn}\end{pmatrix}$$

Matrix H is a square matrix of order $\sum_{i=1}^{n}|X^i|$.

Theorem 1: *Matrix A has the following representation:*

$$A=BHB^T. \tag{5}$$

Proof: The matrices in both sides of (5) are square matrices of order $\prod_{i=1}^n |X^i|$. Each of their rows and columns corresponds to some situation in Γ. Let $y = (y^1, \dots, y^n)$ and $z = (z^1, \dots, z^n)$ be two situations in Γ. An element of the matrix BHB^T in row y and in column z is $\sum_{i=1}^n \sum_{j=1}^n H_{ij}[y^i, z^j]$.

Element $A[y, z]$ is the value of function $\Phi(s, t)$, where s and t are extreme points of polyhedron S which represent y and z respectively. According to (4) and to lemma 1, we have

$$\Phi(s,t) = \sum_{i=1}^n H_i(t \| s^i) = \sum_{i=1}^n \sum_{j=1}^n \sum_{x^i \in X^i} \sum_{x^j \in X^j} H_{ij}[x^i, x^j] s^i[x^i] t^j[x^j].$$

Because $s^i[y^i] = 1,\ t^j[z^j] = 1$ and the other components of s and t are zeros, we obtain

$$A[y, z] = \sum_{i=1}^n \sum_{j=1}^n H_{ij}[y^i, z^j],$$

that is the same expression as element of BHB^T. The theorem is proved.

Formula (5) implies that the rank of matrix A is not greater than the sum of the numbers of pure strategies of all the players. This is significantly smaller than the number of rows (or columns) of A, which is equal to the product of the numbers of pure strategies of all the players.

In the following sections we consider some important classes of separable games, in which matrix H can be represented as a product of some matrices with small rank. Therefore the rank of matrix A will be significantly smaller for these classes.

3 Games with degenerated payoff functions

Let us define degenerated games as separable games whose payoff functions have the following representation:

$$H^{ij}(x^i, x^j) = \sum_{k=1}^{m_i} \sum_{l=1}^{m_j} G^{ij}_{kl} r^i_k(x^i) r^j_l(x^j).$$

We assume that the numbers m_i are small compared with the numbers of pure strategies of the players. Matrices G^{ij} are $(m_i \times m_j)$-matrices.

Let us consider matrices R^i with $|X^i|$ rows and m_i columns. Each row of R^i corresponds to some pure strategy of player i. Let $x^i \in X^i,\ k \in 1 : m_i$, we define $R^i[x^i, k] = r^i_k(x^i)$. It is easy to verify that matrix H_{ij} has the following representation: $H_{ij} = R^i G^{ij} (R^j)^T$.

Let us consider the block-angular matrix R with blocks R^i:

$$R = \begin{pmatrix} R^1 & & & \\ & R^2 & & \\ & & \dots & \\ & & & R^n \end{pmatrix}$$

and matrix G composed of blocks G^{ij}:

$$G = \begin{pmatrix} G^{11} & G^{12} & \dots & G^{1n} \\ . & . & \dots & . \\ . & . & \dots & . \\ G^{n1} & G^{n2} & \dots & G^{nn} \end{pmatrix}.$$

It is clear that $H = RGR^T$.

Therefore matrix A can be represented as a product of five matrices: $A = BRGR^TB^T$. The order of the matrix G is $m = \sum_{i=1}^n m_i$, hence the rank of A is lower than or equal to m.

4 Polymatrix games

Let us consider an n-person game in which each pair of players (i, j) plays the matrix game with $(m_{ij} \times m_{ji})$-matrix G^{ij}, and the total payoff of each player is the sum of his payoffs in all games in which he participates. We assume $G^{ji} = -(G^{ij})^T$. Obviously, this game is a separable zero-sum game. Any pure strategy of player i is determined by the choice of one row in each matrix G^{ij} $(j \in 1 : n)$. Thus the pure strategy of each player is an n-vector with integer components. Let us put down the product representation for matrices H_{ij} in this case. Let us consider zero-one matrices R^{ij} in which each row corresponds to a pure strategy of player i and the number of the columns is m_{ij}. Element $R^{ij}[x^i, k]$ is equal to 1 if player i selects row k of matrix G^{ij} in the game with j, using strategy x^i. The other elements of R^{ij} are equal to zero. Then matrix H_{ij} has the following representation:

$$H_{ij} = R^{ij}G^{ij}(R^{ji})^T.$$

In order to obtain the representation for matrices H and A, we construct the following matrices G, R and $\underline{R}$. Matrix G is a block-angular matrix composed of blocks G^{ij}:

$$G = \begin{pmatrix} G^{11} & & & & & & \\ & \dots & & & & & \\ & & G^{1n} & & & & \\ & & & \dots & & & \\ & & & & G^{n1} & & \\ & & & & & \dots & \\ & & & & & & G^{nn} \end{pmatrix}.$$

Matrices R and $\underline{R}$ have the following structure:

$$R = \begin{pmatrix} R^{11} & \dots & R^{1n} & & & \\ & & & \dots & & \\ & & & & R^{n1} & \dots & R^{nn} \end{pmatrix},$$

$$\underline{R} = \begin{pmatrix} (R^{11})^T & & & \\ & (R^{21})^T & & \\ & & \dots & \\ & & & (R^{n1})^T \\ \dots & \dots & \dots & \dots \\ (R^{1n})^T & & & \\ & (R^{2n})^T & & \\ & & \dots & \\ & & & (R^{nn})^T \end{pmatrix}.$$

Then matrices H and A have the following representations:

$$H = RG\underline{R},\ A = BRG\underline{R}B^T.$$

Since the order of matrix G is equal to $m = \sum_{ij} m_{ij}$, the rank of matrix A is not greater than m.

5 Resources allocation problems

Let us consider a non-cooperative n-person game in which each player has some amount of resources (player i has K_i units of resources). The players allocate their resources to t places. The pure strategy of player i is a t-vector $x^i = (x^i_1, \dots, x^i_t)$, $x^i_\nu \in 0 : K_i,\ \sum_{\nu=1}^t x^i_\nu = K_i$. The component x^i_ν is the amount of the resources which player i allocates to place ν. The game for players i and j on place ν is the matrix game with $(K_i + 1) \times (K_j + 1)$ payoff matrix $p^{ij}_\nu = \{p^{ij}_\nu(x^i_\nu, x^j_\nu)\}$. The total payoff of player i is the sum of his payoffs on all places in games with the other players.

This game is a zero-sum separable game. Let us introduce a block-angular matrix G^{ij}, which is composed of blocks p^{ij}_ν:

$$G^{ij} = \begin{pmatrix} p^{ij}_1 & & & \\ & p^{ij}_2 & & \\ & & \dots & \\ & & & p^{ij}_t \end{pmatrix}.$$

Let us consider a zero-one matrix R^i_ν. Each row of R^i_ν corresponds to some pure strategy of player i. The number of columns in this matrix is equal to $K_i + 1$. The element in the row corresponding to strategy x^i and in column k ($k \in 0 : K_i$) is equal to one, if player i in his strategy x^i allocates k units of the resource to place ν. Compose a matrix R^i of matrices R^i_ν: $R^i = (R^i_1, R^i_2, ..., R^i_t)$.

Then matrix H_{ij} can be represented in the following form:

$$H_{ij} = R^i G^{ij} (R^j)^T,$$

and matrix H can be represented in the form $H = RGR^T$, where matrices R and G are composed of R^i and G^{ij} as in section 3.

In all these cases matrix A can be represented as a product of five matrices. The rank of A is not greater than the minimal rank of these matrices. Therefore the rank of A is significantly lower than its order.

6 Method

We consider now a method for solving matrix games with matrices which have rank significantly lower than their order.

Let A be a $(m \times n)$-matrix whose rank is not greater than μ. We assume that we know a representation of A in product form $A = PQ$, where P is a $(m \times \mu)$-matrix and Q is a $(\mu \times n)$-matrix (m and n are significantly greater than μ). Note that for the games above such a representation is known. For instance, we can define $P = BRG$, $Q = R^T B^T$ in sections 3 and 5, $P = BRG$, $Q = \underline{R}B^T$ in section 4.

Thus we have to consider a matrix game with the matrix $A = PQ$. This game is equivalent to the following linear program:

$$\begin{array}{c} \min v \\ Ay \leq ve, \\ yeS_n, \end{array} \tag{6}$$

where S_n is a simplex

$$S_n = \{y \in R^n : \sum_{j=1}^{n} y_j = 1, \ y_j \geq 0\}$$

and $e \in R^n$ is a vector of ones. The solution of the problem (6) is an optimal strategy of the second player.

We define a set Ω as $\Omega = \{z \in R^\mu : \ z = Qy, \ y \in S_n\}$. Then problem (6) can be written as

$$\begin{array}{c} \min v \\ Pz \leq ve, \\ z \in \Omega. \end{array} \tag{7}$$

Since Ω is a polyhedron, there exist a matrix S and a vector s such that $\Omega =$ $= \{z \in R^\mu : \ Sz \leq s\}$. We do not know matrix S and vector s explicitly, but we can write down the following problem:

$$\begin{array}{c} \max(-v) \\ Pz \leq ve, \\ Sz \leq s, \end{array} \tag{8}$$

which is equivalent to (7). Let us consider a dual problem to (8). By u we denote the vector of dual variables for the first group of restrictions of (8) and by w the vector of dual variables for the second group of restrictions of (8). Then the dual problem is

$$\begin{gathered} \min\ (s, w) \\ P^T u + S^T w = 0, \\ \sum_{j=1}^{m} u_j = 1, \\ u \geq 0, w \geq 0. \end{gathered} \tag{9}$$

Problem (9) has $(\mu + 1)$ restrictions, as we have assumed this number is not very large. But the number of variables in (9) is very large. Moreover, we do not know this number exactly. We shall use the simplex method with a column generation for the solution of problem (9). Here we need to store only matrices and vectors of order not greater than $(\mu + 1)$.

Consider a basic solution of (9). It consists of at most $(\mu + 1)$ non-zero basic variables which correspond to the basic vectors. We can determine the dual variables for this basic solution as usually. Those are the variables of problem (8): z and v. If z and v satisfy the restrictions of (8), then they constitute a solution of (8), and we can construct vector y – the solution of (6), as it is described later. If z and v do not satisfy the restrictions of (8), then we determine the vector to introduce into the basis of problem (9).

Thus having vector z and number v, we have to verify whether they satisfy the restrictions of (8). At first we verify the first group of restrictions of (8), then, if they are satisfied, we verify the second group. We assume that we know how to solve the following problem: for a given vector z to find

$$\tau = max_{j \in 1:m}(P_j, z) = (P_q, z), \tag{10}$$

where P_j is the j-th row of matrix P. In the following section it will be shown how to solve this problem for the games of sections 3-5. If $\tau \leq v$, then the restrictions $Pz \leq v$ are satisfied and we have to verify the second group of restrictions. Otherwise we introduce the vector $\begin{pmatrix} P_q^T \\ 1 \end{pmatrix}$ into the basis of problem (9).

The verification of the second group of restrictions of (8) is a more difficult problem because we do not know matrix S and vector s. These restrictions are satisfied if vector z belongs to set Ω. To verify that, we consider the following linear programming problem:

$$\begin{gathered} \min\ \sum_{i=1}^{\mu}(\omega_i^+ + \omega_i^-) \\ Qy + \omega^+ - \omega^- = z \\ \sum_{j=1}^{n} y_j = 1, \\ y_j, \omega_i^+, \omega_i^- \geq 0. \end{gathered} \tag{11}$$

If the minimum in (11) is equal to zero, then $z \in \Omega$, and the second group of restrictions of (8) is satisfied. Otherwise $z \bar{\in} \Omega$, and we should choose a vector to introduce into the basis in (9). Problem (11) has $(\mu+1)$ restrictions, $(n+2\mu)$ variables and can be solved with the column generation method as well. Thus consider a basic solution of (11); let $(\pi; \rho)$ be the vector of dual variables. The current basic solution is an optimal one if the following inequalities are true:

$$\begin{array}{c} \pi Q_j + \rho \leq 0 \ (j \in 1:n), \\ -1 \leq \pi_i \leq 1 \ (i \in 1:\mu), \end{array} \tag{12}$$

where Q_j is the j-th column of matrix Q. We assume that we know how to solve the following problem: for a given vector π to find

$$\sigma = \max_{j \in 1:n} (\pi, Q_j). \tag{13}$$

Problem (13) is analogous to problem (10). If some component π_i is greater than 1 or less than -1, we introduce a unit vector corresponding to that π_i into the basis of problem (11). Let all π_i lie between -1 and 1. To verify the first group of conditions of (12) we find σ by solving problem (13). If $\sigma + \rho \leq 0$, then we already have an optimal solution of (11). Otherwise we have found the vector to introduce into the basis in (11): the vector Q_j, which realizes the maximum (13), supplemented by 1.

Let y, ω^+, ω^- give a solution of (11), and the minimum in (11) is equal to zero. Then vector z is a solution of (7) and vector y is a solution of (6). Hence y is an optimal strategy of the second player in the matrix game with matrix A.

If the minimum in (11) is positive, then $z \bar{\in} \Omega$ and we have to define the vector to introduce into the basis in problem (9). We have obtained also the optimal basic solution of the dual problem to (11). It is a vector $(\pi; \rho)$, satisfying conditions (12) and also the condition

$$(\pi, z) + \rho > 0, \tag{14}$$

because the minimum in (11) is positive.

The following theorem establishes that vector π is a row in matrix S and number $-\rho$ is a component of the vector s, which define a violated restriction of $Sz \leq s$ in (8). We denote the set of the vectors $(\pi; \rho)$ which satisfy (12) by M. Let $ex(M)$ be the set of all extreme points of M. We consider a set $\Omega' = \{z : (\pi, z) + \rho \leq 0, \ (\pi; \rho) \in ex(M)\}$.

Theorem 2: *The following equality is true:* $\Omega = \Omega'$.

Proof: Let $z \in \Omega$. Then the minimal value of the objective function in (11) is zero. Therefore the maximal value of the function $(\pi, z) + \rho$ for all $(\pi; \rho) \in M$ is zero as well according to the duality theorem. Hence $(\pi, z) + \rho \leq 0$ for all $(\pi; \rho) \in M$, and $z \in \Omega'$.

Let $z \in \Omega'$. Then $(\pi, z) + \rho \leq 0$ for all $(\pi; \rho) \in ex(M)$, hence the inequality holds for all $(\pi; \rho) \in M$. Therefore the maximal value of function $(\pi, z) + \rho$ on M is less than or equal to zero. That implies that the minimum in (11) is zero and $z \in \Omega$.

The theorem is proved.

Hence we may introduce the vector consisting of vector π – the optimal solution of the dual problem to (11), supplemented by zero and having $(-\rho)$ as a component in the objective function, into the basis in (9).

We described only methods to determine the vectors to be introduced into the basis in all linear programs, because all other operations of the simplex method can be performed in the usual way.

7 Computation of τ and σ

Let us consider now a method of determining the values τ and σ in (10) and (13) for the games which were described in sections 3 – 5. As already mentioned, the main difficulty is the large value of m and n, therefore we cannot enumerate all the vectors P_j or Q_j to calculate τ and σ. We convert problems (10) and (13) for the games above to the problem of the maximization of some separable function. The possibility of solving that problem depends on the structure of the set of strategies in the corresponding game. In particular, in the resources allocation problem, the corresponding maximum can be found with the dynamic programming method. As we have seen, in the games above the payoff matrix A is represented as a product of five matrices: $A = BRGSB^T$. In sections 3 and 5 matrix S is R^T. We denote $P = BRG$, $Q = SB^T$. Then $A = PQ$. Let us consider problem (10) for the game with degenerated payoff functions. In this case vector z has $\sum_{i=1}^{n} m_i$ components; let z be

$$(z_1^1, \ldots, z_{m_1}^1, \ldots, z_1^n, \ldots, z_{m_n}^n).$$

Let us introduce numbers ρ_k^i $(i \in 1:n,\ k \in 1:m)$:

$$\rho_k^i = \sum_{j=1}^{n} \sum_{l=1}^{m_j} G_{kl}^{ij} z_l^j.$$

Then

$$(P_j, z) = \sum_{i=1}^{n} \sum_{k=1}^{m_i} r_k^i(x^i) \rho_k^i. \tag{15}$$

Thus, to find τ we have to maximize the separable function on set X of all the situations.

Problem (13) is analogous. Vector π has the same structure as vector z, i.e.

$$\pi = (\pi_1^1, \ldots, \pi_{m_1}^1, \ldots, \pi_1^n, \ldots, \pi_{m_n}^n)$$

and

$$(\pi, Q_j) = \sum_{i=1}^{n} \sum_{k=1}^{m_i} r_k^i(x^i)\pi_k^i.$$

Let us consider now problem (10) for the resources allocation problem. Vector z has $t\sum_{i=1}^{n}(K_i+1)$ components in this case in accordance with the size of matrix G. We denote these components as $z^{j\nu}(l)$ ($j \in 1:n,\ \nu \in 1:t,\ l \in 0:K_j$). Then it is easy to verify that

$$(P_q, z) = \sum_{i=1}^{n} \sum_{\nu=1}^{t} \sum_{j=1}^{n} \sum_{l=0}^{K_i} p_\nu^{ij}(x_\nu^i, l) z^{j\nu}(l).$$

We can calculate values

$$r_\nu^i(k) = \sum_{j=1}^{n} \sum_{l=0}^{K_i} p_\nu^{ij}(k, l) z^{j\nu}(l)$$

for each $i \in 1:n$, $\nu \in 1:t$ and $k \in 0:K_i$. Then $(P_q, z) = \sum_{i=1}^{n}\sum_{\nu=1}^{t} r_\nu^i(x_\nu^i)$. Hence in this case the solution of problem (10) can be obtained from the following n problems: for each $i \in 1:n$ to find

$$\max \sum_{\nu=1}^{t} r_\nu^i(x_\nu^i)$$

subject to conditions

$$\sum_{\nu=1}^{t} x_\nu^i = K_i,$$

x_ν^i are integer and non-negative.

All these problems can be solved by the dynamic programming method. Problem (13) for this game can be solved in the same way.

The analogous approach can be used to solve problems (10) and (13) for polymatrix games. There vector z has $m = \sum_{ij} m_{ij}$ components and, in accordance with the structure of matrix G, vector z consists of vectors z^{ij} with m_{ij} components. We calculate a vector $\rho = Gz$, which consists of vectors $\rho^{ij} = G^{ij}z^{ij}$. If x_j^i is the row number in matrix G^{ij}, which player i chooses in his strategy x^i, then we have to maximize the function:

$$\sum_{ij} \rho^{ij}[x_j^i]$$

in (10). This problem can be solved separately for each i. The method for maximizing function $\sum_{ij} \rho^{ij}[x_j^i]$ with given i depends on the structure of the set of pure strategies of player i. The approach to problem (13) is the same in this case.

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