A PARALLEL ALGORITHM FOR NON-COOPERATIVE RESOURCE ALLOCATION GAMES

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Non-cooperative resource allocation games which generalize the Blotto game are considered. These games can be reduced to matrix games, but the size of the obtained matrices is huge. We establish special properties of these matrices which make it possible to develop algorithms operating with relatively small arrays for the construction of the Nash equilibrium in the considered games. The computations in the algorithms can be performed in parallel.

1. INTRODUCTION

Let us consider a non-cooperative game (game Γ) with a finite set I of players. Player $i \in I$ owns integer capacity $K_i > 0$ of some resource. The players allocate their resources among a finite set N of terrains. A pure strategy of player i is defined as a vector

$$x_i := (x_{i\nu}),\ \nu \in N,\ x_{i\nu} \text{ are non-negative integers, and } \sum_{\nu \in N} x_{i\nu} = K_i.$$

The component $x_{i\nu}$ is interpreted as the number of resource units which player i allocates to terrain ν.

The payoff of each player is assumed to be additive, that is, if all players use pure strategies (player i uses a strategy x_i), and $x := (x_i),\ i \in I$, is a strategy profile, then player i obtains payoff

$$\pi_i(x) := \sum_{j \in I} \pi_{ij}(x_i, x_j),$$

and his payoff against rival j is assumed to be additive subject to the terrains:

$$\pi_{ij}(x_i, x_j) := \sum_{\nu \in N} \pi_{ij\nu}(x_{i\nu}, x_{j\nu}). \tag{1}$$

Here $\pi_{ii}(x_i, x_i)$ is assumed to be 0, for all x_i. Moreover, any pairwise interaction is assumed to be zero-sum, i.e.

$$\pi_{ij}(x_i, x_j) + \pi_{ji}(x_j, x_i) = 0. \tag{2}$$

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Evidently, (2) implies that $\sum_{i \in I} \pi_i(x) = 0$, for all profiles of strategies x, so Γ is a zero-sum game.

Note that player i has

$$m_i := \left(\begin{array}{c} K_i + |N| - 1 \\ |N| - 1 \end{array} \right)$$

pure strategies, it is the number of ordered partitions of integer $K_i + |N|$ in $|N|$ parts (see [1]).

The game Γ is a generalization of the well-known Blotto game (see, for example, [2,3]).

We are interested to construct a method for finding a Nash equilibrium in Γ. As shown in section 2, the Nash equilibrium can be obtained from the solution of some fair matrix game, that is, from the solution of some feasibility problem. The order of the game matrix is huge, it is $\prod_{i \in I} m_i$. However, the rank of the matrix is much smaller than its order. Because of this a method can be constructed which operates with arrays and matrices of relatively small size.

The resource allocation game considered in the present paper is a particular case of the separable non-cooperative game introduced in [4]. The method for the construction of the Nash equilibrium described there operates with vectors and matrices with size equal to the rank of the game matrix.

Special properties of the resource allocation game make it possible to reduce the magnitude of arrays required by the method and to organize the computations so that they can be performed in parallel.

2. NASH EQUILIBRIUM AND EQUIVALENT MATRIX GAME

Let X_i be the set of pure strategies of player i in game Γ. We denote $\prod_{i \in I} X_i$ by X. For the resource allocation game,

$$|X_i| = m_i = \left(\begin{array}{c} K_i + |N| - 1 \\ |N| - 1 \end{array} \right).$$

Let s_i be a mixed strategy for player i, i.e. s_i is a probability distribution on X^i:

$$s_i(x_i) \geq 0, \sum_{x_i \in X_i} s_i(x_i) = 1.$$

Let S_i be the set of mixed strategies for player i, and $S := \prod_{i \in I} S_i$ the set of randomized-strategy profiles. For $s \in S$, the expected value of the payoff function for player i is

$$\pi_i(s) = \sum_{x \in X} \pi_i(x) \prod_{j \in I} s_j(x_j). \tag{3}$$

However, taking into account that player interactions are pairwise, we can represent $\pi_i(s)$ in a simpler form:

$$\pi_i(s) = \sum_{j \in I} \pi_{ij}(s_i, s_j),$$

where

$$\pi_{ij}(s_i, s_j) := \sum_{x_i \in X_i, x_j \in X_j} \pi_{ij}(x_i, x_j) s_i(x_i) s_j(x_j).$$

For any $t_i \in S_i$, we let (s_{-i}, t_i) denote the randomized-strategy profile in which the i-th component is t_i and all other components are as in s. Thus,

$$\pi(s_{-i}, t_i) = \sum_{x \in X} \left(\prod_{j \in I \setminus i} s_j(x_j) \right) t_i(x_i) \pi_i(x),$$

or

$$\pi(s_{-i}, t_i) = \sum_{j \in I} \sum_{x_i \in X_i, x_j \in X_j} \pi_{ij}(x_i, x_j) t_i(x_i) s_j(x_j). \tag{4}$$

Definition 2.1 *We say that $s \in S$ is a Nash equilibrium if*

$$\pi_i(s) \geq \pi_i(s_{-i}, t_i), \text{ for each } i \in I, t_i \in S_i.$$

Now we construct a fair matrix game Δ which is equivalent to game Γ, that is, each randomized-strategy profile in Γ corresponds to some mixed strategy for a player in Δ and vice versa. The strategy corresponding to the Nash equilibrium is an optimal mixed strategy in Δ. Since game Δ is fair (i.e. its value is 0 and the sets of optimal strategies for the first and second player are the same), an optimal mixed strategy can be found by the solution of some linear feasibility problem.

Construct $|X_i| \times |X_j|$-matrices H_{ij}: an element in the row corresponding to the strategy $x_i \in X_i$ and in the column corresponding to the strategy $x_j \in X_j$ is defined as $H_{ij}[x_i, x_j] := \pi_{ij}(x_i, x_j)$. H_{ii} is defined as $|X_i| \times |X_i|$-matrix of zeros. It is clear that $H_{ij} = -H_{ji}^T$.

Then we consider a matrix H consisting of blocks H_{ij}:

$$H := \begin{pmatrix} H_{11} & H_{12} & \dots & H_{1\,|I|} \\ . & . & \dots & . \\ . & . & \dots & . \\ H_{|I|\,1} & H_{|I|\,2} & \dots & H_{|I|\,|I|}. \end{pmatrix}$$

Matrix H is a square matrix of order $\sum_{i \in I} |X_i|$. It is a skew-symmetric matrix, that is, $H = -H^T$.

Consider a matrix B consisting of zeros and ones. Each row of B corresponds to some $x \in X$, that is, B has $m := \prod_{i \in I} |X_i|$ rows. The columns of B are divided into $|I|$ blocks. The i-th block B_i corresponds to set X_i and consists of $|X_i|$ columns, each column corresponding to some $x_i \in X_i$. Matrix B has $|I|$ ones in each row: one 1 in each block, namely, if some row corresponds to $x = (x_i)$, $i \in I$, then it has 1 in the i-th block in the column corresponding to strategy x_i.

Let us consider now a matrix game Δ with $(m \times m)$-matrix

$$A = BHB^T. \tag{5}$$

It is clear that matrix A is skew-symmetric. It means that game Δ is fair, its value is 0 and the sets of optimal strategies for the first and second player are the same.

Lemma 2.2 *If a vector y is a mixed strategy for the second player in Δ, then $s = B^T y$ is a randomized-strategy profile in Γ. The reverse: if s is a randomized-strategy profile in Γ, then there exists a vector y satisfying $B^T y = s$ which is a mixed strategy for the second player in Δ.*

Proof. Let y be a mixed strategy for the second player in Δ, and $s = B^T y$. Then $s = (s_i)$, $i \in I$, where $s_i = B_i^T y$. Since B_i has one unity in each row, and $\sum_{x \in X} y(x) = 1$, we have $\sum_{x_i \in X_i} s_i(x_i) = 1$, for all $i \in I$. It is clear that $s_i \geq 0$, for all i and x^i, because B_i and y are non-negative. So s_i is a mixed strategy for player i in Γ, and s is a randomized-strategy profile.

Let $s = (s_i)$, $i \in I$, be a randomized-strategy profile in Γ.

Consider a vector y with components $y(x)$, $x \in X$, defined as

$$y(x) := \prod_{i \in I} s_i(x_i).$$

It is clear that y is a probability vector, that is, y is a mixed strategy for the second player in Δ. Show that $B_i^T y = s_i$, for all $i \in I$.

Let t be a component of vector $B_i^T y$ corresponding to some $z \in X_i$. Then

$$t = \sum_{\{x \in X:\ x_i = z\}} \prod_{j \in I} s_j(x_j) = s_i(z) \prod_{j \neq i} \sum_{x_j \in X_j} s_j(x_j).$$

Since $\sum_{x_j \in X_j} s_j(x_j) = 1$, we have $B_i^T y = s_i$, that is, $B^T y = s$. ∎

Obviously, the lemma's statements also relate to the first player strategies.

Theorem 2.3 *Game Δ is equivalent to Γ, i.e. if y is an optimal strategy for the second player in Δ, then $s = B^T y$ is a Nash equilibrium in Γ, and if $s \in S$ is a Nash equilibrium in Γ, then each y satisfying $s = B^T y$ is an optimal strategy for the second player in Δ.*

Proof. Let y be an optimal strategy for the second player in Δ. Since Δ is fair, this means that

$$\begin{aligned} & Ay \leq 0, \\ & \textstyle\sum_{x \in X} y(x) = 1, \\ & y \geq 0. \end{aligned} \tag{6}$$

Let $s = B^T y$, $s = (s_i)$, $i \in I$, s_i is a mixed strategy for the i-th player. Let t_i be another mixed strategy for player i, and $u = (s_{-i}, t_i)$.

By lemma 2.2, there exists a mixed strategy for the first player z such that $zB = u$. By (6), we have

$$zAy \leq 0,$$

or taking into account representation (5),

$$uHs \leq 0.$$

The latter inequality is equivalent to

$$\sum_{k \in I, j \in I} u_k H_{kj} s_j \leq 0.$$

By definition of H_{kj},

$$u_k H_{kj} s_j = \sum_{x_k \in X_k, x_j \in X_j} \pi(x_k, x_j) u_k(x_k) s_j(x_j)$$

For $k \neq i$,

$$\sum_{j\in I} u_k H_{kj} s_j = \sum_{j\in I} s_k H_{kj} s_j = \sum_{j\in I} \sum_{x_k\in X_k, x_j\in X_j} \pi(x_k, x_j) s_k(x_k) s_j(x_j) = \pi_k(s),$$

and for $k = i$, by (4),

$$\sum_{j\in I} u_i H_{ij} s_j = \sum_{j\in I} t_i H_{ij} s_j = \sum_{j\in I} \sum_{x_i\in X_i, x_j\in X_j} \pi(x_i, x_j) t_i(x_i) s_j(x_j) = \pi_i(s_{-i}, t_i).$$

These relations imply that

$$\sum_{k\neq i} \pi_k(s) + \pi_i(s_{-i}, t_i) \leq 0,$$

and since Γ is a zero sum game, i.e. $\sum_{k\in I} \pi_k(s) = 0$, we have

$$\pi_i(s_{-i}, t_i) \leq \pi_i(s).$$

Since $i \in I$ and $t_i \in S_i$ was chosen arbitrarily, this means that s is a Nash equilibrium in Γ.

Let now s be a Nash equilibrium in Γ. Take $x \in X$: $x = (x_i)$, $i \in I$. Let u_i be the mixed strategy for the i-th player which chooses the strategy x_i with the probability 1, and $u := (u_i)$, $i \in I$. Then $u = b_x$, where b_x is the row of matrix B corresponding to the strategy profile x. For each $i \in I$, we have

$$\pi_i(s_{-i}, u_i) \leq \pi_i(s).$$

Summing these inequalities, we obtain

$$\sum_{i\in I} \pi_i(s_{-i}, u_i) \leq \sum_{i\in I} \pi_i(s) = 0.$$

Using (4), we have

$$uHs \leq 0.$$

Let y be a mixed strategy for the second player in Δ such that $s = B^T y$ (such y exists by lemma 2.2). Then $b_x H B^T y \leq 0$. Since x was chosen arbitrarily, we have

$$BHB^T y \leq 0,$$

that is, y is an optimal strategy for the second player in Δ. ■

Thus, the problem of finding a Nash equilibrium in Γ has been reduced to the solution of a system of linear inequalities (6). However, the number of variables and constraints in (6) is huge because the order of matrix A is

$$\prod_{i\in I} |X_i| = \prod_{i\in I} \binom{K_i + |N| - 1}{|N| - 1}.$$

We need to exploit specific properties of matrix A to construct an efficient method for the solution of system (6). The basic property is that the rank of matrix A is significantly smaller than its order, and we can represent this matrix as a product of matrices with relatively small size.

Representation (5) gives some factorization for matrix A, and we can see that the rank of A is not greater than the number of rows in H which is equal to the sum of the numbers of pure strategies of all the players. This is significantly smaller than the number of rows (or columns) in A which is equal to the product of these numbers, but it is still a large number. We construct a factorization for matrix H which shows that in fact the rank of matrix A is much smaller than that of H. This factorization makes it possible to construct an efficient method for the solution of (6) which operates with arrays of relatively small size.

We begin with a factorization of matrices H_{ij}. The factorization reflects the additive form of the elements of these matrices expressed by (1).

Let us introduce $(K_i+1)\times(K_j+1)$- matrices $p_{ij\nu}$, $\nu \in N$, whose elements are payoffs for player i from player j on terrain ν, i.e. $p_{ij\nu}[k_i,k_j] := \pi_{ij\nu}(k_i,k_j)$, $0 \le k_i \le K_i$, $0 \le k_j \le K_j$. Construct a block-angular matrix G_{ij}, which is composed of blocks $p_{ij\nu}$:

$$G_{ij} := \begin{pmatrix} p_{ij1} & & & \\ & p_{ij2} & & \\ & & \cdots & \\ & & & p_{ij|N|} \end{pmatrix}.$$

G_{ij} is an $(|N|(K_i+1)\times|N|(K_j+1))$-matrix.

Let us consider zero-one matrices $R_{i\nu}$, $i \in I, \nu \in N$. Each row of $R_{i\nu}$ corresponds to some pure strategy of player i. The number of columns in this matrix is equal to K_i+1. The element in the row corresponding to strategy x_i and in column k $(0 \le k \le K_i)$ is equal to one, if player i allocates k units of the resource to terrain ν in his strategy x^i. $R_{i\nu}$ are $(|X_i|\times(K_i+1))$-matrices. Compose a matrix R_i from matrices $R_{i\nu}$ merging their columns for all $\nu \in N$: $R_i := (R_{i1}, R_{i2}, ..., R_{i|N|})$, R_i is an $(|X_i|\times|N|(K_i+1))$-matrix.

Then matrix H_{ij} can be represented in the following form:

$$H_{ij} = R_i G_{ij} R_j^T,$$

and matrix H can be represented in the form $H = RGR^T$, where matrices R and G are composed of R_i and G_{ij}:

$$R := \begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & \cdots & \\ & & & R_{|I|} \end{pmatrix},$$

$$G := \begin{pmatrix} G_{11} & G_{12} & \dots & G_{1\,|I|} \\ . & . & \dots & . \\ . & . & \dots & . \\ G_{|I|\,1} & G_{|I|\,2} & \dots & G_{|I|\,|I|} \end{pmatrix}.$$

R is a $(\sum_{i\in I}|X_i|\times|N|(\sum_{i\in I}K_i+|I|))$-matrix, and G is a square matrix of the order $|N|(\sum_{i\in I}K_i+|I|)$.

Hence matrix A can be represented as a product of five matrices: $A = BRGR^TB^T$. The rank of A is not greater than the minimal rank of these matrices. Therefore the rank of A is significantly lower than its order. The order of A is

$$\prod_{i=1}^{n} \binom{K_i + |N| - 1}{|N| - 1},$$

and the rank of A is less than or equal to the rank of matrix G, that is,

$$\text{rank } A \leq |N|(\sum_{i \in I} K_i + |I|).$$

3. METHODS FOR FINDING A NASH EQUILIBRIUM

We should find a solution of a matrix game with an $(m \times m)$-matrix A, where

$$m = \prod_{i=1}^{n} \binom{K_i + |N| - 1}{|N| - 1},$$

and $A = BRGR^TB^T$.

The value of the game is 0, therefore the game is equivalent to the following feasibility problem:

$$\begin{aligned} &Ay \leq 0, \\ &y \in S_m, \end{aligned} \tag{7}$$

where S_m is a simplex

$$S_m = \{y \in E^m : \sum_{j=1}^{m} y_j = 1, \; y_j \geq 0\}.$$

The solution of problem (7) is an optimal strategy for the second player in Δ. By theorem 2.3, $s = B^Ty = (s_i)$, $i \in I$, is a Nash equilibrium in game Γ. Therefore the problem of finding a Nash equilibrium can be reformulated as follows:

$$\begin{aligned} &BRGR^Ts \leq 0, \\ &\textstyle\sum_{x_i \in X_i} s_i(x^i) = 1, \; s_i \geq 0, \text{ for } i \in I. \end{aligned} \tag{8}$$

We develop a method for the solution of (8) which allows parallel computations. The method operates with arrays and matrices of the size which is not greater than

$$\mu := |N|(\sum_{i \in I} K_i + |I|),$$

the order of matrix G. We suppose that μ is a relatively small number, that is, we can store vectors and matrices of order μ.

Define sets Ω_i, for $i \in I$, as

$$\Omega_i = \{z_i \in E^{|N|(K_i+1)} : \; z_i = R_i^T s_i, \; \sum_{x_i \in X_i} s_i(x_i) = 1, \; s_i \geq 0\}.$$

Then, taking into consideration the structure of matrices B, G and R, problem (8) can be rewritten as

$$\begin{array}{l}\sum_{i\in I} B_i R_i \sum_{j\in I} G_{ij} z_j \le 0,\\ z_i \in \Omega_i, \text{ for } i \in I.\end{array} \tag{9}$$

Since Ω_i are polyhedra, they can be represented as a finite intersection of half-spaces. This means that there exist matrices Q_i and vectors q_i such that

$$\Omega_i = \{z_i \in E^{|N|(K_i+1)} : \ Q_i z_i \le q_i\}.$$

Denoting $\sum_{i\in I} B_i R_i G_{ij}$ by P_j, we rewrite (9) in the following form:

$$\begin{array}{l}\sum_{j\in I} P_j z_j \le 0,\\ Q_i z_i \le q_i, \text{ for } i \in I.\end{array} \tag{10}$$

Thus, the process of obtaining a Nash equilibrium consists of the following stages:

1. Calculation of z_i, $i \in I$, from system (10).
2. Calculation of a Nash equilibrium $s = (s_i)$ from systems

$$z_i = R_i^T s_i, \ \sum_{x_i \in X_i} s_i(x_i) = 1, \ s_i \ge 0.$$

But first of all we fix the representation of Ω_i as the intersection of half-spaces because such representation is not unique.

3.1. Representation of Ω_i

Consider a polyhedron in E^{p+1}, where $p := |N|(K_i + 1)$, consisting of vectors (π, ρ) (π is a p-vector, and ρ is a number) satisfying the following conditions:

$$\begin{array}{l}R_i \pi + \rho e \le 0\\ -1 \le \pi_j \le 1 \ (j = 1, 2, \ldots, p).\end{array} \tag{11}$$

Here e is an $|X_i|$-vector of ones.

Let $\{(\pi_k, \rho_k), \ k \in K\}$ be a (finite) set of all extreme points of this polyhedron. Define Q_i as a $(|K| \times p)$-matrix whose k-th row is π_k, and q_i as a $|K|$-vector with $q_i[k] := -\rho_k$. Show that Ω_i can be represented by the system of inequalities $Q_i z \le q_i$.

Theorem 3.1 $\Omega_i = \{z \in E^{|N|(K_i+1)} : \ Q_i z \le q_i\}$.

Proof. Take $z \in \Omega_i$ and show that $Q_i z \le q_i$.

Consider the linear programming problem:

$$\max \ (\langle \pi, z\rangle + \rho) \tag{12}$$

subject to conditions (11).

The dual problem to (12) is

$$\begin{array}{l}\min \ \sum_{j=1}^{p} (\omega_j^+ + \omega_j^-)\\ R_i^T s_i + \omega^+ - \omega^- = z\\ \sum_{x_i \in X_i} s_i(x_i) = 1,\\ s_i, \omega^+, \omega^- \ge 0.\end{array} \tag{13}$$

Since $z \in \Omega_i$, the minimal value of the target function in (13) is zero. Therefore the maximal value in (12) is also zero, that is,

$$\langle \pi_k, z\rangle + \rho_k \leq 0, \text{ for all } k \in K.$$

This means that $Q_i z \leq q_i$.

Take now z satisfying $Q_i z \leq q_i$ and show that $z \in \Omega_i$.

Suppose $z \bar{\in} \Omega_i$. Then the minimal value of the target function in (13) is positive. Hence the maximal value in (12) is also positive, that is, there exists an extreme point (π_k, ρ_k) such that

$$\langle \pi_k, z\rangle + \rho_k > 0.$$

This means that z does not satisfy conditions $Q_i z \leq q_i$. ■

3.2. Solution of system (10)

System (10) is a system of linear inequalities with a huge number of constraints: the first group of them $\sum_{j\in I} P_j z_j \leq 0$ has $m = |X|$ inequalities, and the number of inequalities in other groups is equal to the number of extreme points in the polyhedra described in section 3.1. However, the number of variables in (10) is relatively small, it is equal to $\mu = |N|(\sum_{i\in I} K_i + |I|)$.

We consider two approaches to the solution of (10). The first is the simplex method with the column generation technique (see [5,6]). In this method, we have a current basic solution z_i, $i \in I$, together with the current basis, i.e. a set of no more than μ constraints of (10) which are satisfied as equalities for the current z_i. The next steps are to find the most violated restriction in (10), introduce it into the basis, define the basic constraint to be removed from the basis and recalculate z_i according to the new basis. The process stops if for the current z_i all conditions (10) are satisfied.

The other approach is the row action method (see [7]). In this method, we have a current approximation z_i^k for the solution of (10). Then we find the most violated restriction in (10) for z_i^k and define the next approximation z_i^{k+1} as the projection (orthogonal or non-orthogonal) of z_i^k onto the corresponding hyperplane. The sequence z_i^k converges to the solution of (10) (see [8]).

The common part of both approaches is the following problem: for given z_i, $i \in I$, find the most violated restriction in (10). We concentrate on this problem because other actions (projections onto a hyperplane and reconstructions of a basis in the simplex method) are performed as usual and operate with vectors and matrices with the order no more than μ.

Let us consider a method for determining the most violated restriction for the first group:

$$\sum_{j\in I} P_j z_j \leq 0. \tag{14}$$

Denote $v_i := \sum_{j\in I} G_{ij} z_j$, v_i can be calculated easily because z_j are known vectors. Since $P_j = \sum_{i\in I} B_i R_i G_{ij}$, we need to find the maximal element in the vector

$$\sum_{i\in I} B_i R_i v_i.$$

We may search for the maximal element in $B_iR_iv_i$ separately for each i. Indeed, let x_i be a pure strategy for the i-th player in Γ which corresponds to the maximal element of $B_iR_iv_i$. Then the row of B corresponding to $x \in X$, where $x = (x_i)$, $i \in I$, is the row with the maximal element of $\sum_{i\in I} B_iR_iv_i$.

Matrix B_i contains one unity in each row, so the product of a row of B_i by R_i is a row of R_i, that is, an $(|N|(K_i+1))$-vector containing $|N|$ blocks with one unity in each block. The row corresponds to some pure strategy for the i-th player, namely, to the strategy $x_i = (x_{i\nu})$ such that a unity in block $\nu \in N$ is on the position $x_{i\nu}$ (positions are numbered from 0 to K_i).

Vector v_i is also an $(|N|(K_i+1))$-vector, its components also regarded as divided into $|N|$ blocks with (K_i+1) elements in each block. Let the element in block ν at position x_ν be denoted by $v_{i\nu}[x_\nu]$.

Hence the maximal element in $B_iR_iv_i$ can be calculated from the following problem:

$$\max \sum_{\nu\in N} v_{i\nu}[x_\nu] \tag{15}$$

subject to conditions

$$\begin{array}{l} \sum_{\nu\in N} x_\nu = K_i, \\ x_\nu \text{ are integer and non-negative.} \end{array} \tag{16}$$

This problem can be solved by the dynamic programming technique. The solution of this problem defines x_i a pure strategy for the i-th player. The set (x_i), $i \in I$, corresponds to the most violated constraint in (14).

Problems (15)-(16) can be solved in parallel, for each i.

Consider now how to define the most violated restriction for the system

$$Q_iz_i \le q_i, \tag{17}$$

where z_i is a given $(|N|(K_i+1))$-vector, and Q_i and q_i are defined as in section 3.1. According to the representation of Ω_i, we need to find a p-vector π and a number ρ which form the solution of the linear programming problem:

$$\max\ (\langle \pi, z\rangle + \rho) \tag{18}$$

subject to conditions (11).

If the maximal value in (18) is positive, then the solution of this problem is an extreme point of polyhedron (11) corresponding to the most violated restriction in (17), otherwise conditions (17) are satisfied.

We solve (18) together with its dual problem (13) by the simplex method with the column generation technique. The number of constraints in (13) is $p+1$, it is less than μ, so we can store and operate vectors and matrices of this order.

Let us have a feasible basic solution for (13). Corresponding dual variables π_j, $1 \le j \le p$, and ρ are defined from the following conditions:

$$\begin{array}{l} \langle R_i[x_i], \pi\rangle + \rho = 0, \text{ if } s_i(x_i) \text{ is a basic variable,} \\ \pi_j = 1, \text{ if } \omega^+ \text{ is a basic variable,} \\ \pi_j = -1, \text{ if } \omega^- \text{ is a basic variable.} \end{array}$$

Here $R_i[x_i]$ is a row of matrix R_i corresponding to $x_i \in X_i$.

If these π and ρ satisfy (11), then they are a solution of (18). Otherwise, we need to change the current basis and basic solution in (13).

If $\pi_j > 1$, for some j, introduce variable ω^+ into the basis. If $\pi_j < -1$, for some j, introduce variable ω^- into the basis.

Let all π_j lie between -1 and 1. Verify the conditions

$$R_i\pi + \rho e \leq 0. \tag{19}$$

Find

$$\tau := \max_{x_i \in X_i} \langle R_i[x_i], \pi \rangle \tag{20}$$

and the row $R_i[x_i^*]$ on which this maximum is attained. If $\tau + \rho \leq 0$, then conditions (19) are satisfied. Otherwise, we introduce variable $s_i(x_i^*)$ into the basis in (13).

We define the variable to be removed from the basis and reconstruct the basis as in the usual simplex method. It requires processing vectors and matrices of order no more than $p+1$.

The process of changing the basis is repeated till we get the solution of problem (18). At that moment, the solution of (18) (π, ρ) defines the most violated restriction in (17).

It remains to describe how to find τ and x_i^* in (20). Row $R_i[x_i]$ consists of $|N|$ blocks with a unique unity in each block: block ν contains 1 at position $x_{i\nu}$ (positions are numbered from 0 to K_i). The components of vector π are also regarded as divided into $|N|$ blocks with $K_i + 1$ components in each block. Let $\pi_\nu[x_\nu]$ be the component in block ν at position x_ν. Then (20) can be rewritten as

$$\tau = \max \sum_{\nu \in N} \pi_\nu[x_\nu] \tag{21}$$

subject to conditions

$$\begin{array}{l} \sum_{\nu \in N} x_\nu = K_i, \\ x_\nu \text{ are integer and non-negative.} \end{array} \tag{22}$$

Problem (21)-(22) is similar to problem (15)-(16), and can also be solved by the dynamic programming technique.

Note that problems (17) may be solved in parallel, for each i.

3.3. Calculation of a Nash equilibrium

After we have solved the inequality system (10) and obtained its solution z_i, $i \in I$, we can calculate mixed strategies s_i forming a Nash equilibrium. We can obtain them as a solution of linear programming problems (13) with $z = z_i$, for all $i \in I$. A method for the solution of these problems is described in the previous section. Note that although s_i has $|X_i|$ components, the basic solution of (13) has no more than $|N|(K_i + 1) + 1$ non-zero components, so we obtain a Nash equilibrium with a relative small number of non-zero components by the described method.

In fact, we need not solve problems (13) as a separate stage because we have already obtained their solution while we verified inequalities (17) for the last time.

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