Goal Programming Solutions Generated by Utility Functions*

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Abstract. Goal programming problem is determined by a feasible set of vectors and a goal point; its aim is to select a feasible point the most close to the goal point in certain sense. A particular case of the goal programming problem is the rationing problem, when a given amount of resource must be distributed among agents with unequal claims on the resource. We impose axioms on a selection rule that guarantee the solution of the rationing problem under the selection rule of the goal programming problem to be equal sacrifice with respect to a certain family of utility functions of agents. For problems with convex relatively closed feasible sets, the general selection rule minimizes the proximity measure to the goal point given by the sum of antiderivatives of differences between utility functions. The least square solution and maximal weighted entropy solution are obtained under additional axioms.

Keywords: Goal programming, claim point, bargaining problem, taxation problem, bankruptcy problem, entropy.

1 Introduction

Goal programming problem is determined by a set of feasible vectors (allocations) and a goal point (claims of agents). The problem is to select a feasible point the most close to the goal point in certain sense. Usually the proximity concept is defined by some set of axioms. An example of goal programming problem is the bargaining problem with claims, considered by Chun and Thomson (1992). The goal point is also called a claim point by Chun and Thomson (1992), an ideal point by Naumova (1983).

Another example of goal programming problem is a rationing problem. Here a given amount of resource must be divided among agents with unequal claims on the resource. Bankruptcy and taxation problems are important examples of rationing problems. Literature on the rationing problem is surveyed by Moulin (1999).

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The concept of goal programming problem is similar to the well known bargaining problem defined by a feasible set and status quo point, determining the lower bounds for coordinates of the solution. Bargaining problem may be regarded as a goal programming problem with the goal point being equal to $(+\infty, \ldots, +\infty)$. The first axiomatization for the bargaining problem was introduced by Nash (1950). Numerous studies on bargaining problem are surveyed in Roth (1979), Thomson (1990).

The most papers on the bargaining problem propose an axiomatic justification of utility function that is maximized at the solution point. Similarly, the solutions of goal programming problem provide an axiomatic justification for the proximity measure between vectors. Goal programming problem can be considered as a problem of constructing objective functions depending on goal points as well.

In this paper we introduce the following axioms to justify a solution of the goal programming problems with convex relatively closed feasible sets: compatibility; independence from irrelevant alternatives with fixed goal point; independence from intermediate "projection" of goal point to hyperplane, containing feasible set; independence from unconstrained variables; optimality for unconstrained variables; optimality on line; some type of continuity.

Under the set of axioms, we establish interconnection between solutions of the goal programming problem and restrictions of these solutions to the rationing problem.

First, we show that the solution of the rationing problem by the selection rule of the goal programming problem satisfies consistency and path independence conditions, studied in numerous papers on rationing problems. This allows to apply the next result about the representation of strictly monotonic, consistent, and path independent solution of the rationing problem, obtained by Young (1988) and generalized by Naumova (2000). There exists a system of utility functions of agents such that all agents have equal differences between values of their utility functions at the solution point and at the goal point. These solutions of the rationing problem are called equal sacrifice solutions with respect to the system of utility functions. The positive homogeneity and anonymity assumptions imply that utility functions are logarithmic, i.e., the solution of the rationing problem is proportional to the goal point.

Next, we show that equal sacrifice solutions of the rationing problem are uniquely extended to solutions of general goal programming problem. For the convex relatively closed feasible set, the solution of the goal programming problem is the point minimizing the measure of proximity to the goal point. Here the measure of proximity is defined as the sum of antiderivatives of differences between utility functions.

For example, if utility functions are logarithms of coordinates, then the solution of the rationing problem is proportional to the goal point. In this case the solution of the goal programming problem maximizes weighted entropy. If utility functions are coordinates, then the solution of the goal programming problem minimizes the Euclidean distance.

Thus, the considered system of axioms generates a class of solutions for goal programming problems with convex relatively closed feasible sets based on utility functions of agents and including least square solution and maximal weighted entropy solution.

The first reasoning to select the point maximizing weighted entropy with weights generated by the goal point appears in Bregman and Romanovskij (1975). Their feasible sets are intersections of finite number of hyperplanes with positive orthant. The reasoning is based on the convergence of iterative process for minimization of convex continuously differentiable function on the intersection of finite number of hyperplanes proved by Bregman (1967).

Naumova (1983) formalizes their ideas and proposes an axiomatic justification of weighted entropy for goal programming problems with convex relatively closed feasible sets in positive orthant. She proves that proportional solution of rationing problem is uniquely extended to weighted entropy solution of goal programming problem under the set of axioms considered in this paper.

This result is generalized in Bregman and Naumova (1984) with proofs omitted. They show that non-symmetric equal sacrifice solution for rationing problem (called symmetric solution with respect to a system of utility functions) is uniquely extended to goal programming solution. The other problem is to provide the existence of utility functions determining the solution of the rationing problem. The existence of these utility functions is stated in Bregman and Naumova (1984) under additional assumption of existence some continuous derivatives of the results of these sharings.

Csiszár (1991) considers axioms both for solution of bargaining problem and for solution of goal programming problem with feasible sets being intersections of finite number of hyperplanes. He uses the same set of axioms for goal programming problem as Bregman and Naumova (1984) minus the axiom of "optimality on a line" and additional supposition that he calls "distinctness". ("Distinctness" means that projections of goal point to different hyperplanes must differ unless both hyperplanes contain the goal point.) Note that Csiszár does not consider connections between solutions of goal programming and rationing problems. His technics is based on solving the goal programming problem with hyperplane feasible sets. One of the classes of solutions he considers, is the class of solutions minimizing proximity measure the same as in Bregman and Naumova; these proximity measures are called Bregman's distances.

The main concern of this paper is to provide a new proof for the same axiomatic justification of goal programming solutions minimizing Bregman's distances not involving either existence of derivatives or "distinctness". We consider both existence of derivatives and "distinctness" to be unnatural conditions for this class of problems.

The structure of the paper is the following. In Sect. 2, we describe the goal programming problem and introduce axioms to characterize its solution. In Sect. 3, we give basic definitions for the rationing problem and discuss the results about equal sacrifice solutions for it both in symmetric and non-symmetric cases. In Sect. 4, we show that the solution of the rationing problem reduced from the solution of the goal programming problem, satisfying our axioms, is equal sacrifice with respect to a family of utility functions of agents. In Sect. 5 we extend uniquely equal sacrifice solution of the rationing problem (without symmetry assumptions) to solution of the goal programming problem. Axiomatic justifications of weighted entropy solution and of Euclidean metrics are obtained as corollaries of this result.

2 The model

We consider families of problems Γ such that each problem $P \in \Gamma$ is determined by a finite set $N = N(P)$ (the *set of coordinates* or the *set of agents*) with $N \subset \mathcal{N} = \{1, 2, \ldots\}$ (here $\mathcal{N}$ is the set of potential agents), a nonempty set $S = S(P) \subset R^N$ (the *feasible set*) and a vector $x^0 = x^0(P) \in R^N$ (the *goal point*), i.e. a problem P is a triple $P = \{N, S, x^0\}$.

A *choice function* defined on the family of problems Γ is a mapping f such that $f(P) \in S(P)$ for all $P \in \Gamma$.

Our aim is to determine f in some "natural" way so as $f(P)$ be close to $x^0(P)$ in a definite sense. The mapping f will be determined from a system of conditions (axioms) as it is common in the theory of bargaining problems (see Nash (1950), Roth (1979), Yanovskaya (1981), Naumova (1983)).

In what follows, we consider families Γ of problems entirely determined by intervals $X^i = (\alpha_i, \beta_i)$, $i \in \mathcal{N}$, where $-\infty \le \alpha_i < \beta_i \le +\infty$, namely, Γ consists of all the problems $P = \{N, S, x^0\}$, where $x^0 \in X^N = \prod_{i \in N} X^i$, $S \subset X^N$, and S is nonempty, convex, and relatively closed in X^N.

If $\alpha_i = -\infty$ and $\beta_i = +\infty$ for all $i \in \mathcal{N}$, then the family of problems is denoted by Γ_R. If $\alpha_i = 0$ and $\beta_i = +\infty$ for all $i \in \mathcal{N}$, then the family of problems is denoted by Γ_+.

We now consider properties of choice function f defined on Γ, assuming everywhere that all the problems mentioning in the statements belong to Γ.

A1. (Compatibility.) If $x^0(P) \in S(P)$, then $f(P) = x^0(P)$.

If the goal point x^0 is feasible, then it must be chosen.

A2. (Independence from intermediate projection the goal point to hyperplane.) If $P_1 = \{N, S, x^0\}$, $P_2 = \{N, Q, x^0\}$, $S \subset Q$ and $Q = \{x \in X^N : \sum_{i \in N} a_i x_i = b\}$, then $f(P_1) = f(\{N, S, f(P_2)\})$.

If $f(P)$ is taken as the result of adapting the goal point $x^0(P)$ to the set $S(P)$, then A2 can be interpreted as saying that the result is independent of an intermediate adaptation on the hyperplane.

A3. (Optimality for unconstrained variables.) If $S(P) = Q \times X^i$ for some $i \in N(P)$, then $f_i(P) = x_i^0(P)$.

If the variable i is not involved in the constraints, then it coincides in the solution with its goal value.

A4. (Independence from unconstrained variables.) If $P_1 = \{N, Q, x^0\}$, $P_2 = \{N \cup \{i\}, Q \times X^i, y^0\}$ and $y^0_j = x^0_j$ for all $j \in N$, then $f_j(P_1) = f_j(P_2)$ for all $j \in N$.

Addition of an unconstrained variable does not change the previous components of the solution.

A5. (Optimality on line.) If $S \subset Q = \{y \in X^N : y = x^0 + \lambda q,\ \lambda \in R^1\}$, where $q \in R^N$, then $f(\{N, S, x^0\}) = x^0 + \mu q$, where $|\mu| = \min\{|\lambda| : x^0 + \lambda q \in S\}$.

This property requires choosing the point in S on the line which is closest to the goal point. Such a point exists and is unique for the Γ under consideration.

A6. (Independence from irrelevant alternatives with fixed goal point.) If $P_1 = \{N, S, x^0\}$, $P_2 = \{N, Q, x^0\}$, $S \subset Q$ and $f(P_2) \in S$, then $f(P_1) = f(P_2)$.

If the best point (with respect to the goal point) in the larger set is feasible for the smaller set, then it is the best point in the smaller set. Similar condition for bargaining problems without goal point was introduced by Nash (1950) as the axiom of independence from irrelevant alternatives.

A choice function f is *LC-continuous* if for problems $P = \{N, S, x^0\}$ and $P^m = \{N, S^m, x^0\}$, where $S = \{x \in X^N : \sum_{i \in N} a_i x_i = b\}$ and $S^m = \{x \in X^N : \sum_{i \in N} a_{im} x_i = b\}$ with $a_{im} = 0$ as $a_i = 0$, the condition $a_{im} \to a_i$ as $m \to \infty$ for all $i \in N$ implies that $f(P^m) \to f(P)$ as $m \to \infty$.

We use the following notation in what follows:

$$f^{ij}(b, x^0) = f(\{\{ij\}, S^{ij}(b), x^0\}),$$

where $S^{ij}(b) = \{x \in X^{ij} : x_i + x_j = b\}$.

A choice function f is *symmetric with respect to a system of functions* $\{c_i\}$ $(i \in \mathcal{N})$ defined on (α_i, β_i), if

$$c_i(f^{ij}_i(b, x^0)) - c_i(x^0_i) = c_j(f^{ij}_j(b, x^0)) - c_j(x^0_j)$$

for all $b \in (\alpha_i + \alpha_j, \beta_i + \beta_j)$.

If $c_i(x_i)$ is the utility of the variable x_i for agent i, then symmetry with respect to $\{c_i\}$ means that utility increments are equal on $S^{ij}(b)$ for agents i and j with respect to the goal point.

Sometimes, the following additional properties of choice function will be used.

A7. (Weak symmetry). If $X^i(\Gamma) = X^j(\Gamma)$ and $x^0_i = x^0_j$ for all $i, j \in N$, then $f_i(N, S(N, b), x^0) = f_j(N, S(N, b), x^0)$. (Here $S(N, b) = \{x \in X^N : \sum_{i \in N} x_i = b\}$.)

A8. (Invariance under coordinate translation). $\Gamma = \Gamma_R$ and for any $y \in R^N$, $f(\{N, S + y, x^0 + y\}) = f(\{N, S, x^0\}) + y$, where $S + y = \{x + y : x \in S\}$.

A9. (Positive homogeneity). $\Gamma = \Gamma_+$ and $f(N, \alpha S, \alpha x^0) = \alpha f(N, S, x^0)$ for any $\alpha > 0$, where $\alpha S = \{\alpha x : x \in S\}$.

3 Rationing problems and equal sacrifice solutions

A special case of the goal programming problem is a rationing problem. A *rationing problem* is a triple $\{N, b, x^0\}$, where N is a finite set of agents, b is a real number representing the amount of resources to be divided, $x^0 = \{x_i^0\}_{i \in N}$ is a vector of claims of agents.

Thus, rationing problem $\{N, b, x^0\}$ is a goal programming problem $\{N, S(N, b), x^0\}$, where $S(N, b) = \{x \in X^N : \sum_{i \in N} x_i = b\}$.

We consider two families of rationing problems. The family $\Gamma_R^0 \subset \Gamma_R$ consists of all triples $\{N, b, x^0\}$ with $b, x_i^0 \in R^1$, and the family $\Gamma_+^0 \subset \Gamma_+$ consists of all triples $\{N, b, x^0\}$ with $b, x_i^0 > 0$.

Remark that these families of rationing problems are wider than most of the usually considered ones, since we suppose neither $b \leq \sum_{\in N} x_i^0$, nor $b \geq \sum_{i \in N} x_i^0$. The family Γ_R^0 coincides with the set of distributing problems considered by Herrero et al. (1999).

A *rationing method* is a choice function r defined on a family of rationing problems.

Any choice function f on Γ determines the rationing method $r = r(f)$ by $r(f)(N, b, x^0) = f(N, S(N, b), x^0)$.

Now we describe some properties of rationing methods to be used below.

Denote $x_{[Q]} = \{x_i\}_{i \in Q}$ for $x \in R^N$, $Q \subset N$.

Let r be a rationing method.

r is *consistent* if

$$r(N \setminus i, b - r_i(N, b, x^0), x^0_{[N \setminus i]}) = r(N, b, x^0)_{[N \setminus i]}$$

for all $i \in N$.

r is *path independent* if

$$r(N, b, x^0) = r(N, b, r(N, b_1, x^0))$$

for all admissible b, b_1.

(This definition of path independence is stronger than the usually considered ones, since it contains no interrelations between b, b_1, and $\sum_{i \in N} x_i^0$.)

r is *strictly monotonic* if

$$b < b_1 \text{ implies } r_i(N, b, x^0) < r_i(N, b_1, x^0) \text{ for all } i \in N.$$

r is *strictly order-preserving* if

$$x_i^0 < x_j^0 \text{ implies } r_i(N, b, x^0) < r_j(N, b, x^0) \text{ for all } i, j \in N.$$

r is *weakly symmetric* if

$$x_i^0 = x_j^0 \text{ implies } r_i(N, b, x^0) = r_j(N, b, x^0).$$

(This property is also called *equal treatment of equals*.)

A rationing method r defined on Γ_R^0 (or on Γ_+^0) is an *equal sacrifice method* if there exists a strictly increasing function u defined on R^1 (or on $(0, +\infty)$) such that

$$u(x_i^0) - u(r_i(N, b, x^0)) = u(x_j^0) - u(r_j(N, b, x^0)) \text{ for all } i, j \in N.$$

Thus, equal sacrifice method satisfies symmetry condition. Its generalization for non-symmetric case is the following.

Let $\{u_i\}$ $(i \in \mathcal{N})$ be a family of strictly increasing functions defined on R^1 (or on $(0, +\infty)$). A rationing method r defined on Γ_R^0 (or on Γ_+^0) is *equal sacrifice with respect to the family of functions* $\{u_i\}_{i \in \mathcal{N}}$ if

$$u_i(x_i^0) - u_i(r_i(N, b, x^0)) = u_j(x_j^0) - u_j(r_j(N, b, x^0)) \text{ for all } i, j \in N.$$

If the rationing method $r(f)$, generated by the choice function f, is equal sacrifice with respect to a family of functions $\{u_i\}$, then f is symmetric with respect to $\{u_i\}_{i \in \mathcal{N}}$. The converse statement holds if $r(f)$ is consistent.

It can be easily seen that equal sacrifice with respect to a family of strictly increasing functions rationing method is consistent and path-independent.

The converse result is valid under some additional restrictions.

Young (1988) considered the family $\Gamma_+^1 \subset \Gamma_+^0$ of rationing problems with $0 < b \leq \sum_{i \in N} x_i^0$. The definition of path-independence is changed by the following.

A rationing method r satisfies *upper composition* condition if

$$r(N, b, x^0) = r(N, b, r(N, b_1, x^0))$$

for all b, b_1 with $0 < b \leq b_1 \leq \sum_{i \in N} x_i^0$.

Young (1988) proved the following theorem which we cite in reformulation of Moulin (1999).

The Young Theorem. *1) A rationing method satisfying consistency, upper composition, strict monotonicity and strict order-preserving conditions must be an equal sacrifice method;*

2) A rationing method satisfying consistency, upper composition, strict monotonicity, strict order-preserving and positive homogeneity conditions must be an equal sacrifice method derived either from utility function $u_{(p)}$ with $p < 0$ (where $u_{(p)}(x) = x^p$), or from utility function $u(x) = \ln(x)$.

It is easily seen that Young's strict monotonicity and strict order-preserving conditions imply the weak symmetry condition. Non-symmetric generalization of Young's Theorem is the following.

Theorem (Naumova (2000)). *Let r be a rationing method defined on Γ_R^0 or on Γ_+^0, satisfy consistency, path-independence, strict monotonicity conditions, and*

$$r_i(\{i, j\}, b, x^0) \to +\infty \text{ as } b \to +\infty,$$

$$r_i(\{i, j\}, b, x^0) \to -\infty \text{ as } b \to -\infty.$$

Then r is equal sacrifice with respect to a family of strictly increasing continuous functions.

4 Equal sacrifice for rationing method induced by choice function

In this section we study the rationing method $r(f)$, induced by a choice function f, and satisfying the following axioms: compatibility; independence from unconstrained variables; optimality for unconstrained variables, and independence from intermediate projection the goal point on hyperplane containing the feasible set. We show that $r(f)$ is equal sacrifice for weakly symmetric f and equal sacrifice w.r.t. a family of functions in non-symmetric case (under additional monotonicity conditions). First, we prove that $r(f)$ is consistent and path-independent and then we apply the results about the representation of such rationing methods, cited in Sect. 3.

The proofs of Lemmas 4.1, 4.3, and 4.4 are in Appendix.

Lemma 1. *Suppose that A1-A4 hold for a choice function f defined on $\Gamma = \Gamma_R$ or on $\Gamma = \Gamma_+$, $b, d, (d-b) \in R^1$ and $b, d, (d-b) > 0$ for $\Gamma = \Gamma_+$.*

Then for any finite $N \subset \mathcal{N}$, $k \in N$, $x^0 \in X^N$,

$$r(f)(N \setminus k, b, x^0_{N\setminus k}) = r(f)(N \setminus k, b, r(f)(N, d, x^0)_{[N\setminus k]}) \text{ for all } k \in N. \quad (1)$$

Lemma 2. *Suppose that A1-A4 hold for a choice function f defined on $\Gamma = \Gamma_R$ or on $\Gamma = \Gamma_+$.*

Then $r(f)$ is a consistent rationing method.

Proof. Consider the equation (1) for $b = d - r(f)_k(N, d, x^0)$ and apply A1. Then we have

$$\begin{aligned}
&r(f)(N \setminus k, d - r(f)_k(N, d, x^0), x^0_{[N\setminus k]}) \\
&= r(f)(N \setminus k, d - r_k(N, d, x^0), r(f)(N, d, x^0)_{[N\setminus k]}) \\
&= r(f)(N, d, x^0)_{[N\setminus k]}.
\end{aligned}$$

□

Lemma 3. *Suppose that A1-A4 hold for a choice function f defined on Γ_R or on Γ_+, and $f_i^{ij}(b, x^0)$ are strictly increasing in b for all $i, j \in \mathcal{N}$.*

Then $r_i(f)(N, d, x^0)$ are strictly increasing in d for all $i \in N$.

Lemma 4. *Suppose that A1-A4 hold for a choice function f defined on Γ_R or on Γ_+, $f_i^{ij}(b, x^0)$ are strictly increasing in b for all $i, j \in \mathcal{N}$ and*

$$f_i^{ij}(b, x^0) \to +\infty \text{ as } b \to +\infty, \quad (2)$$

$$f_i^{ij}(b, x^0) \to -\infty \text{ as } b \to -\infty. \quad (3)$$

Then $r(f)$ is path-independent.

Theorem 1. *Suppose that A1-A4 and A7 hold for a choice function f defined on Γ_R or on Γ_+, $f_i^{ij}(b, x^0)$ are strictly increasing in b for all $i, j \in \mathcal{N}$ and*

$$f_i^{ij}(b, x^0) \to +\infty \text{ as } b \to +\infty, \tag{4}$$

$$f_i^{ij}(b, x^0) \to -\infty \text{ as } b \to -\infty. \tag{5}$$

Then $r(f)$ is equal sacrifice.

If, moreover, A9 holds for f, then $r(f)$ is equal sacrifice method derived from the utility function $u(x) = \ln(x)$.

Proof. First we prove that $r(f)$ is strictly order-preserving. Suppose that there exists a rationing problem $\{N, b, x^0\}$ with $x_i^0 < x_j^0$ and $r(f)_i(N, b, x^0) \geq r(f)_j(N, b, x^0)$. Since $r(f)$ is strictly increasing in b, it is continuous in b, therefore there exists b_1 such that $r(f)_i(N, b_1, x^0) = r(f)_j(N, b_1, x^0)$. By Lemma 4, $r(f)(N, b, x^0) = r(f)(N, b_1, y)$, where $y = r(f)(N, b_1, x^0)$ and $y_i = y_j$. This contradicts weak symmetry condition.

The path-independence implies upper composition, hence all conditions of Young's Theorem are fulfilled and $r(f)$ is equal sacrifice.

If, in addition, the choice function f satisfies A9, then $r(f)$ is equal sacrifice either with respect to $u_i(x_i) = \ln(x_i)$ or with respect to $u_i(x_i) = (x_i)^p$, where $p < 0$. The latter case contradicts to $f_i^{ij}(b, x^0) \to +\infty$ as $b \to +\infty$. □

Theorem 2. *Suppose that A1-A4 hold for a choice function f defined on Γ_R or on Γ_+, $f_i^{ij}(b, x^0)$ are strictly increasing in b for all $i, j \in \mathcal{N}$ and*

$$f_i^{ij}(b, x^0) \to +\infty \text{ as } b \to +\infty, \tag{6}$$

$$f_i^{ij}(b, x^0) \to -\infty \text{ as } b \to -\infty. \tag{7}$$

Then $r(f)$ is equal sacrifice with respect to a family of strictly increasing continuous functions.

Proof. Due to Lemmas 2 – 4, $r(f)$ satisfies all conditions of Naumova's Theorem in Naumova (2000). □

5 Extension of equal sacrifice solution to goal programming problem

In this section we extend equal sacrifice solution of the rationing problem to the choice function of the goal programming problem. Due to results of Sect. 4, it permits to obtain general solution of the goal programming problem under the considered set of axioms. Additional restrictions on choice function brings us the least square solution and maximal weighted entropy solution.

The proofs of Theorems 5.1 and 5.4 are in Appendix.

Theorem 3. *Suppose that c_i $(i \in \mathcal{N})$ are strictly increasing continuous functions on $X^i = (\alpha_i, \beta_i)$ and $\{c_i(x_i) : x_i \in X^i\} = (-\infty, +\infty)$.*

Then there exists a unique choice function f on Γ which satisfies A1–A6, is LC-continuous, is continuous with respect to the goal point for fixed $S(P)$ and $N(P)$, and is symmetric with respect to the system $\{c_i\}_{i \in \mathcal{N}}$; moreover,

$$f(P) = \operatorname{argmin}_{x \in S(P)} \sum_{i \in N(P)} \int_{x_i^0(P)}^{x_i} [c_i(t) - c_i(x_i^0(P))] dt. \tag{8}$$

Theorem 4. *There exists a unique function f on Γ_R which satisfies A1-A8, is LC-continuous and continuous with respect to the goal point, moreover,*

$$f(P) = \operatorname{argmin}_{x \in S(P)} \sum_{i \in N(P)} (x_i - x_i^0(P))^2.$$

Proof. Consider $f^{ij}(b, x^0)$. If $x_i^0 = x_j^0$, then $f^{ij}(b, x^0) - x^0$ is a normal to $S(\{i,j\}, b)$ by A7, if x^0 is arbitrary, then $f^{ij}(b, x^0) - x^0$ is also a normal to $S(\{i,j\}, b)$ by A8, hence f is symmetric with respect to the system $\{c_i\}$ of functions, where $c_i(x_i) = x_i$ for all $i \in \mathcal{N}$. Then apply Theorem 3. □

Theorem 5. *Suppose that the following conditions hold for a choice function f on Γ_R or on Γ_+:*

(C1.) A1-A6 hold.

(C2.) f is continuous with respect to the goal point and is LC-continuous.

(C3.) $f_i^{ij}(b, x^0) \to +\infty_{b \to +\infty}$, $f_i^{ij}(b, x^0) \to -\infty_{b \to -\infty}$ for all i, j and x^0.

(C4.) $f_i^{ij}(b, x^0)$ is strictly increasing in b.

Then there exists a system $\{c_i\}_{i \in \mathcal{N}}$ of functions such that

$$f(P) = \operatorname{argmin}_{x \in S(P)} \sum_{i \in N(P)} \int_{x_i^0(P)}^{x_i} (c_i(t) - c_i(x_i^0(P))) dt.$$

Theorem 6. *Let a choice function f defined on Γ_+ satisfies the following conditions.*

(C1.) A1-A6, A7, A9 hold.

(C2.) f is continuous with respect to the goal point and is LC-continuous.

(C3.) $f_i^{ij}(b, x^0) \to +\infty_{b \to +\infty}$, $f_i^{ij}(b, x^0) \to -\infty_{b \to -\infty}$ for all i, j and x^0.

(C4.) $f_i^{ij}(b, x^0)$ is strictly increasing in b.

Then

$$f(P) = \operatorname{argmin}_{x \in S(P)} \sum_{i \in N(P)} x_i (\ln(x_i / x_i^0(P)) - 1).$$

Proof. Apply Theorems 1 and 3. □

6 Conclusions

In this paper we have considered solutions for goal programming problems with convex relatively closed feasible sets under the following set of axioms: compatibility; independence from irrelevant alternatives with fixed goal point; independence from intermediate "projection" of goal point to hyperplane, containing feasible set; independence from unconstrained variables; optimality for unconstrained variables; optimality on line; some type of continuity.

Under additional monotonicity assumptions, we have completely described this class of solutions (Theorem 5.3). These are solutions, minimizing "Bregman's distances" to the goal point.

The least square solution has been obtained under additional axioms of anonymity and invariance under coordinate translations (Theorem 5.2).

The weighted entropy solution has been obtained under additional axioms of anonymity and positive homogeneity (Theorem 5.4).

Previous papers on this subject involved either existence of derivatives for solutions or "distinctness" assumptions. We consider both existence of derivatives and "distinctness" to be unnatural conditions for this class of problems and did not use them in proofs.

7 Appendix: Proofs

Proof (of Lemma 1). Denote $Q = \{x \in X^N : \sum_{i \in N \setminus k} x_i = b\}$, $T = \{x \in X^N : \sum_{i \in N \setminus k} x_i = b, x_k = d - b\}$, $U = \{x \in X^N : x_k = d - b\}$, $S(N,d) = \{x \in X^N : \sum_{i \in N} x_i = d\}$, $y = r(f)(N,d,x^0) = f(N, S(N,d), x^0)$.

Since $T \subset S(N,d)$ and $T \subset Q$, it follows from A2 that

$$f(N,T,x^0) = f(N,T,y) = f(N,T,f(N,Q,y)).$$

By A3 and A4,

$$f(N,Q,y) = \{r(f)(N \setminus k, b, y_{[N\setminus k]}), y_k\} = z.$$

Since $T \subset U$, it follows from A2, A3, A4, and A1 that
$f(N,T,z) = f(N,T,f(N,U,z)) = f(N,T,\{r(f)(N \setminus k, b, y_{[N\setminus k]}), d-b\}) = \{r(f)(N \setminus k, b, y_{[N\setminus k]}), d-b\}$.

Thus, $f(N,T,x^0) = \{r(f)(N \setminus k, b, y_{[N\setminus k]}), d-b\}$.

Moreover, it follows from A2 that $f(N,T,x^0) = f(N,T,f(N,Q,x^0)$. By A3 and A4, $w = f(N,Q,x^0) = \{r(f)(N \setminus k, b, x^0_{[N\setminus k]}), x^0_k\}$. Since $T \subset U$, it follows as above from A2, A3, A4, and A1 that $f(N,T,w) = \{r(f)(N \setminus k, b, x^0_{[N\setminus k]}), d-b\}$. Hence $r(f)(N \setminus k, b, x^0_{[N\setminus k]}) = r(f)(N \setminus k, b, y_{[N\setminus k]})$, and the statement (1) is proved. □

Proof (of Lemma 3). Let $d_1 > d_2$, then $r(f)_k(N,d_1,x^0) > r(f)_k(N,d_2,x^0)$ for some $k \in N$. Let $i \neq k$. Denote

$$b_j = r(f)_i(N,d_j,x^0) + r(f)_k(N,d_j,x^0) \text{ for } j = 1,2.$$

Then consistency of $r(f)$ implies that

$$f^{ik}(b_j, x^0_{[\{ik\}]}) = r(f)(N, d_j, x^0)_{[\{ik\}]} \text{ for } j = 1, 2. \tag{9}$$

Then it follows from (9) and strict increasing of $f^{ik}_k(b, x^0_{[\{ik\}]})$ in b that $b_1 > b_2$.

As $f^{ik}_i(b, x^0_{[\{ik\}]})$ is strictly increasing in b, the equation (9) imply that $r(f)_i(N, d_1, x^0) > r(f)_i(N, d_2, x^0)$. □

Proof (of Lemma 4). Let a finite $N \subset \mathcal{N}$, $k \in \mathcal{N} \setminus N$.

First we prove that for any $x^0 = \{x^0_{[N]}, x^0_k\}$,

$$\{\sum_{i \in N} r(f)_i(N \cup k, d, x^0) : d \in D\} = D, \tag{10}$$

where $D = (-\infty, +\infty)$ for $\Gamma = \Gamma_R$, $D = (0, +\infty)$ for $\Gamma = \Gamma_+$.

By Lemma 3, $r(f)_i(N \cup \{k\}, d, x^0)$ $(i \in N)$ are strictly increasing in d, so they are continuous in d, therefore $\sum_{i \in N} r(f)_i(N \cup \{k\}, d, x^0)$ is strictly increasing and continuous in d. It suffices to prove that it is not bounded from above in d, not bounded from below for $\Gamma = \Gamma_R$, and

$$\sum_{i \in N} r(f)_i(N \cup \{k\}, d, x^0) \to 0 \text{ as } d \to 0. \tag{11}$$

Suppose that $\sum_{i \in N} r(f)_i(N \cup \{k\}, d, x^0)$ is bounded from above in d, then $r(f)_k(N \cup \{k\}, d, x^0) \to +\infty$ as $d \to +\infty$, and, by Lemma 3, $r(f)_i(N \cup \{k\}, d, x^0)$ are bounded from above in d for all $i \in N$. If $d \to +\infty$ then, due to monotonicity in d, for any $i \in N$, $b_{ik}(d) = r(f)_i(N \cup \{k\}, d, x^0) + r(f)_k(N \cup \{k\}, d, x^0) \to +\infty$.

Lemma 2 and condition (2) imply that

$$r(f)_i(N \cup \{k\}, d, x^0) = f^{ik}_i(b_{ik}(d), x^0) \to +\infty$$

, as $d \to +\infty$, but it was proved above that it is bounded from above.

It can be proved similarly, that if $\Gamma = \Gamma_R$, then $\sum_{i \in N} r(f)_i(N \cup \{k\}, d, x^0)$ is not bounded from below. The condition (11) follows immediately from Lemma 3.

Thus the condition (10) is proved.

Therefore, at a fixed x^0, for any $c \in D$, there exists $d(c)$ such that $\sum_{i \in N} r(f)_i(N \cup \{k\}, d(c), x^0) = c$ and, moreover, $d(c) = d(c, x^0)$ is a continuous and strictly increasing in d function defined on $D \times X^{N \cup \{k\}}$. As $r(f)$ is consistent,

$$r(f)(N \cup \{k\}, d(c), x^0)_{[N]} = r(f)(N, c, x^0_{[N]}). \tag{12}$$

We shall prove now that for any $a, b \in R^1$ with $a, b > 0$ for $\Gamma = \Gamma_+$, for any $x^0_{[N]} \in X^N$,

$$r(f)(N, b, x^0_{[N]}) = r(f)(N, b, r(f)(N, a, x^0_{[N]})). \tag{13}$$

If $(d(a) - b) \in X^k$, then Lemma 1 and equation (12) imply that

$$\begin{aligned} r(f)(N, b, x^0_{[N]}) &= r(f)(N, b, r(f)(N \cup k, d(a), x^0)_{[N]}) \\ &= r(f)(N, b, r(f)(N, a, x^0_{[N]})). \end{aligned}$$

Hence, if either $\Gamma = \Gamma_R$ or $\Gamma = \Gamma_+$ and $d(a) > b$, then the equation (13) holds and the lemma is proved.

Consider the case when $\Gamma = \Gamma_+$. We shall prove the fulfillment of (13). We consider three subcases.

1). $a > \sum_{i\in N} x_i^0$. Fix $x_k^0 > b$. Then $\bar{d} = \sum_{i\in N\cup\{k\}} x_i^0 > x_k^0 > b$. By A1, $\bar{d} = d(\sum_{i\in N} x_i^0, x^0)$, where $x^0 = \{x_{[N]}^0, x_k^0\}$. As $d(c)$ is strictly increasing in c, in this subcase $d(a) > \bar{d} > b$, so (13) holds.

2). $a = \sum_{i\in N} x_i^0$, then $r(f)(N, a, x_{[N]}^0) = x_N^0$ by A1, so the statement of this lemma is true.

3). $a < \sum_{i\in N} x_i^0$. Let $c \in [a, \sum_{i\in N} x_i^0]$. Denote $x(c) = r(f)(N, c, x_{[N]}^0)$. As $\Gamma = \Gamma_+$, $d(c) > c$. By the continuity of $d(c)$, there exists an open set $V(c)$ such that $c \in V(c)$ and $d(c_1) > c_2$ for any $c_1, c_2 \in V(c)$.

Let $c_1, c_2 \in V(c)$, $c_1 < c_2$. Then, by the proved above in subcase 1),

$$r(f)(N, b, x(c_1)) = r(f)(N, b, r(f)(N, c_2, x(c_1)))$$

.

Since $c_2 < d(c_1)$, $x(c_2) = r(f)(N, c_2, x^0) = r(f)(N, c_2, x(c_1))$. Therefore, $r(f)(N, b, x(c_1)) = r(f)(N, b, x(c_2))$, and so $r(f)(N, b, x(c))$ is locally constant in c. This implies that $r(f)(N, b, x(c))$ is constant in c for $c \in [a, \sum_{i\in N} x_i^0]$. By A1, $x(\sum_{i\in N} x_i^0) = x_{[N]}^0$, so $r(f)(N, b, x(a)) = r(f)(N, b, \sum_{i\in N} x_i^0)$ implies (13). $\square$

Proof (of Theorem 3). Let $f(P)$ be defined in (8). Denote $\psi_i(x_i, x_i^0) = \int_{x_i^0}^{x_i} (c_i(t) - c_i(x_i^0))dt$, $\psi_N(x, x^0) = \sum_{i\in N} \psi_i(x_i, x_i^0)$ for finite $N \subset \mathcal{N}$, $x^0, x \in X^N$. Then $\psi_N(x, x^0)$ is a strictly convex function with respect to x, hence since $\{c_i(x_i) : x_i \in X^i\} = R^1$, $f(P)$ exists and is unique.

A1 and A3 hold since $\psi_i(x_i^0, x_i^0) \le \psi_i(x_i, x_i^0)$ for any $x_i \in X^i$. A4 holds since $\psi_N(x, x^0)$ is a function with separating variables. A5 follows from the convexity of $\psi_N(x, x^0)$ with respect to x and the fulfillment of A1. A6 is realized as $f(P)$ is a minimum point of a function.

Let us check A2. Denote $g_N(x, x^0) = grad_x \psi_N(x, x^0)$, then

$$\psi_N(x, y) = \psi_N(x, x^0) - \psi_N(y, x^0) - (g_N(y, x^0), (x - y)) \tag{14}$$

Let $P_1 = \{N, S, x^0\}$, $P_2 = \{N, Q, x^0\}$, $Q = \{x \in X^N : \sum_{i\in N} a_i x_i = b\}$, $S \subset Q$, $y = f(P_2)$. Then by the Lagrange multipliers rule there exists $\lambda \in R^1$ such that $g_N(y, x^0) = \lambda\{a_i\}_{i\in N}$. It follows from (14) that if $z \in Q$, then $\psi_N(z, x^0) - \psi_N(y, x^0) = \psi_N(z, y) + (g_N(y, x^0), (z - y)) = \psi_N(z, y)$, as $y \in Q$. Thus $\psi_N(z, x^0) = \psi_N(z, y) + \psi_N(y, x^0)$ for any $z \in Q$, hence $\operatorname{argmin}_{z\in S} \psi_N(z, x^0) = \operatorname{argmin}_{z\in S} \psi_N(z, y)$ and A2 is checked.

It follows from the Lagrange multipliers rule that f is symmetric with respect to $\{c_i\}_{i\in\mathcal{N}}$. The LC-continuity and the continuity with respect to the goal point follow from theorem 3 in Barabash and Busigin (1980).

Thus if f is defined in (8), then all conditions hold.

Suppose now that a choice function f satisfies A1-A6 along with continuity conditions and f is symmetric with respect to a system $\{c_i\}_{i\in\mathcal{N}}$ of strictly increasing continuous functions with $\{c_i(x_i) : x_i \in X^i\} = (-\infty, +\infty)$.

Let $\overline{f}(P)$ be defined in (8). We shall prove that $f = \overline{f}$.

Let $S(P) = \{x \in X^N : \sum_{i\in N} a_i x_i = b\}$, $N(P) = N$. Then it follows from the Lagrange multipliers rule that $c_i(\overline{f}_i(p)) - c_i(x_i^0) = \mu a_i$ for $x^0 = x^0(P)$ and all $i \in N$. Since c_i are strictly increasing and A3 holds, it is sufficient to prove that

$$c_i(f_i(P)) - c_i(x_i^0) = \lambda a_i \tag{15}$$

for some $\lambda \in R^1$.

Let $a_i = 1$ for any $i \in N$. If $|N| = 2$ then (15) is supposed, and the case of arbitrary $|N|$ follows from the consistency property of $r(f)$ proved in Lemma 2. If either $a_i = 0$ or $a_i = 1$, then the coincidence of f and $\overline{f}$ follows from A3 and A4.

Let all $a_i \neq 0$ be integer. In view of A3 and A4, we can assume that $a_i \neq 0$ for all $i \in N$. For simplicity, it is assumed that $N = \{1, \ldots, n\}$, $0 < a_1 \leq \ldots \leq a_k$, and $0 > a_{k+1} \geq \ldots \geq a_n$ for some k with $0 < k \leq n$.

Let $t = a_k$ for $k = n$, and $t = a_k - a_n$ for $k < n$. Denote $\bar{S} = S \times X^{\{n+1,\ldots,n+t+1\}}$, $\bar{N} = N \cup \{n+1, \ldots, n+t+1\}$. Fix $\bar{x}^0 \in X^{n+t+1}$ with $\bar{x}_i^0 = x_i^0$ for $i \leq n$. Let $z = f(\bar{N}, \bar{S}, \bar{x}^0)$ then, in view of A4, it suffices to prove that

$$c_i(z_i) - c_i(\bar{x}_i^0) = \lambda a_i \tag{16}$$

for some $\lambda \in R^1$ and all $i \leq n$.

Consider the following system of $t + 2$ equations:

$$\left.\begin{array}{c}
\sum_{j=1}^{n} \delta_{ij} x_j + \beta_i \gamma x_{n+1} + x_{n+i+1} = \sum_{j=1}^{n} \delta_{ij} z_j + \beta_i \gamma z_{n+1} + z_{n+i+1} \\
(i = 1, \ldots, t), \\
\sum_{i=1}^{a_k} x_{n+i+1} = \sum_{i=1}^{a_k} z_{i+n+1}, \\
\sum_{i=a_k+1}^{t} x_{n+i+1} = \sum_{i=a_k+1}^{t} z_{i+n+1},
\end{array}\right\} \tag{17}$$

where $\gamma = 0$ for $k = n$, $\gamma = 1$ for $k < n$,
$\beta_i = 1$ for $i = 1$ and $i = a_k + 1$, otherwise $\beta_i = 0$,
$\delta_{ij} = 1$ for $j \leq k$ and $i \leq a_j$, $\delta_{ij} = 1$ for $j > k$ and $a_k < i \leq a_k - a_j$, $\delta_{ij} = 0$ in the remaining cases.

Let $Q = \{x \in X^{n+t+1} : x \text{ satisfies } (17)\}$. It is directly verified that if we sum first a_k equations in (17) with the sign +, add $(t+1-a_k)$ next equations in (17) with the sign -, and add the last equation in (17) with the sign +, then we get

$$\sum_{j=1}^{n} a_j x_j = \sum_{j=1}^{n} a_j z_j = b.$$

Therefore, $Q \subset \bar{S}$. Hence, by A2,

$$f(\bar{N}, Q, \bar{x}^0) = f(\bar{N}, Q, z), \tag{18}$$

and,by A1,

$$f(\bar{N}, Q, z) = z = f(\bar{N}, \bar{S}, \bar{x}^0). \tag{19}$$

Let y^1 be the f–projection of $\bar{x}^0$ on the first constraint in (17) then, by the proved above and A4, y^1 is the $\overline{f}$–y^1 on the second constraint in (17),..., y^{t+3} be the f–projection of y^{t+2} on the first constraint in (17), etc. In this way we get a cyclic sequence of f–projections coinciding with the $\overline{f}$–projections on all the constraints in (17). It was shown by Bregman (1967) that $y^m \to \overline{f}(\bar{N}, Q, \bar{x}^0) = y$ as $m \to \infty$. By A2, $f(\bar{N}, Q, \bar{x}^0) = f(\bar{N}, Q, y^m)$ for all m. It follows from the continuity with respect to the goal point that $f(\bar{N}, Q, \bar{x}^0) = f(\bar{N}, Q, y)$. A1 implies $f(\bar{N}, Q, y) = y$. Then (18) and (19) imply $z = y$, i.e. $f(\bar{N}, \bar{S}, \bar{x}^0) = \overline{f}(\bar{N}, Q, \bar{x}^0)$.

The application of the Lagrange multipliers rule to the evaluation of $\overline{f}(\bar{N}, Q, \bar{x}^0)$ brings us, due to (17), the condition $c_i(y_i) - c_i(\bar{x}_i^0) = \lambda a_i$ for some $\lambda \in R^1$ and all $i \leq n$, hence (16) is verified for integer a_i.

The case of rational a_i is reduced to the previous case by multiplying all a_i and b by the same number and using A3, A4 for $a_i = 0$.

The coincidence of f and $\overline{f}$ for the case of arbitrary a_i follows from their LC-continuity.

Now let the set $S = S(P)$ be arbitrary. Suppose that $x = f(P) \neq z = \overline{f}(P)$. Denote $q = x - z$, $Q = \{y \in X^N : y = z + \lambda q, \lambda \in R^1\}$. Then $Q = \cap_{i=1}^m Q^i$, where Q^i are hyperplanes in X^N. Let $x^n = f(N, Q^j, x^{n-1})$ for $n = 1, 2, \ldots$, where $j \leq m$, $n \equiv j(\text{mod.}m)$. It was proved above that $f(N, Q^j, x^n) = \overline{f}(N, Q^j, x^n)$ for all $n = 0, 1, \ldots$. Here the key theorem in Bregman (1967) on the convergence of a relaxation method is suitable for $\overline{f}$, so $x^n \to y = \overline{f}(N, Q, x^0)$.

It follows from A2 that $f(N, S \cap Q, x^0) = f(N, S \cap Q, x^n)$ for any n, so $f(N, S \cap Q, x^0) = f(N, S \cap Q, y)$, since f is continuous with respect to the goal point. Moreover $f(N, S \cap Q, x^0) = x$ by A6, hence $f(N, S \cap Q, y) = x$. It can be proved similarly that $\overline{f}(N, S \cap Q, y) = z$. A5 holds both for f and $\overline{f}$, $y \in Q$ and $S \cap Q$ is convex, hence $x = z$. This contradicts the supposition and completes the proof. □

Proof (of Theorem 5). It follows from Theorem 2 that there exists a system of strictly increasing continuous functions $\{c_i\}_{i \in \mathcal{N}}$ such that f is symmetric with respect to it.

In order to apply Theorem 3, it suffices to prove that $\{c_i(t) : t \in X^i\} = R^1$.

Suppose that $M_j = \sup\{c_j(t) : t \in X^j\} < +\infty$ for some $j \in N$. Let $i \neq j$. Fix $x_i^0 \in X^i$ and $\delta > 0$. Then since c_i is strictly increasing, there exists $\epsilon > 0$ such that $x_i > x_i^0$ and $c_i(x_i) - c_i(x_i^0) < \epsilon$ imply $x_i - x_i^0 < \delta$. Let us take x_j^0 so that $c_j(x_j^0) > M_j - \epsilon$. Then $b > x_i^0 + x_j^0$ implies $f^{ij}(x^0) > x^0$ and $c_i(f_i^{ij}(b, x^0)) - c_i(x_i^0) < \epsilon$ as f is symmetric with respect to $\{c_k\}$. Hence $f_i^{ij}(b, x^0) \leq x_i^0 + \delta$, and this contradicts the condition C3.

It can be proved similarly that if $\Gamma = \Gamma_R$, then $c_i(t) \to -\infty$ as $t \to -\infty$.

Let $\Gamma = \Gamma_+$. Suppose that $c_j(t)$ is bounded from below. Fix $b > 0$, $i \neq j$, $x_i^0 > b$, $\epsilon_i < x_i^0 - b$. Since c_i is strictly increasing, there exists $\delta > 0$ such that $|c_i(x_i) - c_i(x_i^0)| < \delta|$ implies $|x_i - x_i^0| < \epsilon_i$. Since c_j is bounded from below, there exists $\alpha > 0$ such that $|c_j(x_j) - c_j(y_j)| < \delta$ as $x_j, y_j < \alpha$. Let $x_j^0 < \alpha$, then $|c_i(f_i^{ij}(b, x^0)) - c_i(x_i^0)| < \delta$ since $f^{ij}(b, x^0) < x^0$ and f is symmetric with respect to $\{c_k\}$. Hence $|f_i^{ij}(b, x^0) - x_i^0| < \epsilon_i$ and $(f_i^{ij} + f_j^{ij})(b, x^0) > x_i^0 - \epsilon_i > b$. This contradiction proves the theorem. □

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