# ON THE AUTONOMOUS NORM ON THE GROUP OF HAMILTONIAN DIFFEOMORPHISMS OF THE TORUS

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Abstract. We prove that the autonomous norm on the group of Hamiltonian diffeomorphisms of the two-dimensional torus is unbounded. We provide explicit examples of Hamiltonian diffeomorphisms with arbitrarily large autonomous norm. For the proofs we construct quasimorphisms on $\mathrm{Ham}(\mathbf{T}^2)$ and some of them are Calabi.



## 1. Introduction

Let $M$ be a smooth manifold and let $X\colon M \to TM$ be a compactly supported vector field with the flow $\Psi_X\colon \mathbf{R} \to \mathrm{Diff}(M)$. The time-one map $\Psi_X(1)$ of the flow is called the *autonomous* diffeomorphism associated with the vector field $X$. The subset $\mathrm{Aut}(M) \subset \mathrm{Diff}_0(M)$ of autonomous diffeomorphisms is conjugation invariant and, since the group of diffeomorphisms isotopic to the identity is simple, it generates $\mathrm{Diff}_0(M)$. In other words, a compactly supported diffeomorphism of $M$ isotopic to the identity is a finite product of autonomous ones. One may ask for a minimal decomposition and this question leads to the concept of the autonomous norm which is defined by

$$\|f\|_{\mathrm{Aut}} := \min\{n \in \mathbf{N} \mid f = a_1 \cdots a_n,\, a_i \in \mathrm{Aut}(M)\}.$$

It is the word norm associated with the generating set $\mathrm{Aut}(M)$. Since this set is conjugation invariant, so is the autonomous norm. It follows

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from the work of Burago-Ivanov-Polterovich [5] and Tsuboi [18, 19] that for many manifolds all conjugation invariant norms on $\mathrm{Diff}_0(M)$ are bounded. Hence the autonomous norm is bounded in those cases.

The situation is different for the groups of area preserving diffeomorphisms of surfaces. For example, the autonomous norm on the group $\mathrm{Diff}_0(\mathbf{D}^2, \mathrm{area})$ of compactly supported area preserving diffeomorphisms of the open disc is unbounded [3]. The same is true for the group $\mathrm{Ham}(\Sigma)$ of Hamiltonian diffeomorphisms of closed oriented surfaces different from the torus [8, 2, 4]. The present paper deals with the remaining case of the torus:

**Theorem 1.1.** *The autonomous norm on the group* $\mathrm{Ham}(\mathbf{T}^2)$ *of Hamiltonian diffeomorphism of the torus is unbounded.*

One way to prove unboundedness of a conjugation invariant norm on a group $G$ is to construct an unbounded quasimorphism $\psi\colon G \to \mathbf{R}$ which is Lipschitz with respect to this norm. If such a norm is a word norm then it suffices to construct a quasimorphism which is bounded on the generating set which implies that it is Lipschitz. If $G = \mathrm{Diff}_0(M, \mathrm{vol})$ is the group of volume preserving diffeomorphisms of a manifold $M$ then nontrivial quasimorphisms on $G$ can be obtained from nontrivial quasimorphisms on the fundamental group of $M$ as follows.

Let $z \in M$ be the basepoint and let $\mathbf{g}$ be an auxiliary Riemannian metric on $M$. For every point $x \in M$ chose a path $\gamma_x\colon [0,1] \to M$ from $z$ to $x$ by choosing a measurable section of the map $\pi\colon P \to M$, where

$$P = \{\gamma\colon [0,1] \to M \mid \gamma(0) = z,\, \gamma(1) = x \text{ and } \gamma \text{ is a geodesic of } \mathbf{g}\}.$$

Let $f \in \mathrm{Diff}_0(M, \mathrm{vol})$ and let $\{f_t\}$ be an isotopy from the identity to $f$. For every $x \in M$ the isotopy $\{f_t\}$ defines a loop based at $x$ by $\gamma(f,x) = \gamma_x f_t(x)\overline{\gamma_{f(x)}}$, where the bar denotes the path in the reverse direction. This loop is well defined up to homotopy of loops based at $z$ provided that evaluating loops of diffeomorphisms of $M$ at the basepoint produces homotopically trivial loops in $M$. This holds, for example, if the center of the fundamental group of $M$ is trivial or if $\{f_t\}$ is a Hamiltonian isotopy in a symplectic manifold.

Let $\psi\colon \pi_1(M,z) \to \mathbf{R}$ be a quasimorphism and let $f \in \mathrm{Diff}_0(M, \mathrm{vol})$ be a compactly supported diffeomorphism isotopic to the identity. Then, given that the volume of $M$ is finite, the map $\Psi\colon\ \mathrm{Diff}_0(M, \mathrm{vol}) \to \mathbf{R}$ defined by

$$\Psi(f) = \int_M \psi(\gamma(f,x))dx$$

is a well defined quasimorphism. This construction and the argument are due to Polterovich [13]. Notice that the construction can be performed for an action $G \to \mathrm{Diff}_0(M, \mathrm{vol})$ of a group $G$ on $M$. For example, if $M$ is simply connected then $G = \mathrm{Diff}_0(M)$ can act on another manifold which is not simply connected. Concretely, if $\Sigma$ is a surface then $\mathrm{Diff}_0(\Sigma, \mathrm{area})$ acts on the configuration space $M = \mathbf{X}_n(\Sigma)$. The fundamental group of this configuration space is (by definition) the pure braid group on $n$-strings on the surface $\Sigma$. Geometrically, this construction generalizes the above one in the sense that an isotopy and a configuration of points defines a pure braid $\gamma(f, x_1, \ldots, x_n)$ rather than a single loop (up to homotopy). We provide more details in Section 2.4. Historically the braid approach was the first original idea due to Gambaudo and Ghys [8] applied to diffeomorphisms of the disc and the sphere. It was later generalized by the first named author to other surfaces [2]. To sum up, the construction gives a linear map

$$\mathcal{G}\colon \mathbf{Q}(\mathbf{P}_n(\Sigma)) \to \mathbf{Q}(\mathrm{Ham}(\Sigma)),$$

from the space of homogeneous quasimorphisms on the pure braid group to the space of homogeneous quasimorphisms on the group of Hamiltonian diffeomorphisms of the surface.

There are two main problems in proving the unboundedness of the autonomous norm. The first, which is a general one, is to show that the above construction yields nontrivial quasimorphisms. The second is to show that among these nontrivial quasimorphisms there are ones which are bounded on the set of autonomous diffeomorphisms. These are the main objectives of the present paper as well as the earlier ones [2, 3, 4].

The solution of the first problem has two parts. The first one, which is essentially the same for all surfaces, is the claim that the kernel of the composition $\mathbf{Q}(\mathbf{B}_n(\Sigma)) \to \mathbf{Q}(\mathbf{P}_n(\Sigma)) \to \mathbf{Q}(\mathrm{Ham}(\Sigma))$ consists of homomorphisms. The idea of the proof is due to Ishida who did it in the case of the disc and the sphere [9] and his argument was generalized to all surfaces in [3]. The second part is to construct nontrivial quasimorphisms on the full braid group. Here, the solution depends on the genus.

The problem of identifying quasimorphisms on braid groups which yield quasimorphisms vanishing on autonomous diffeomorphisms is the main problem in all the cases and, again, solutions depend on the genus. In what follows we provide a proof of Theorem 1.1 by reducing the argument to several results which are then proved in the rest of the paper.

**Proof of Theorem 1.1.** The structure of the proof is presented in the following composition of linear maps.

$$\mathbf{Q}(\mathbf{F}_2;\mathbf{Z}/2\times\mathbf{Z}/2)\longrightarrow\mathbf{Q}(\mathbf{F}_2)\xrightarrow{\pi^*}\mathbf{Q}(\mathbf{P}_2(\mathbf{T}^2))\xrightarrow{\mathcal{G}}\mathbf{Q}(\mathrm{Ham}(\mathbf{T}^2))$$

Here, $\mathbf{Q}(G)$ denotes a space of homogeneous quasimorphisms on a group $G$ and $\mathbf{Q}(\mathbf{F}_2;\mathbf{Z}/2\times\mathbf{Z}/2)\subset\mathbf{Q}(\mathbf{F}_2)$ is the subspace of quasimorphisms invariant under the action generated by inverting generators.

- The construction of Gambaudo and Ghys (Section 2.4), provides a linear map $\mathcal{G}\colon\mathbf{Q}(\mathbf{P}_n(\mathbf{T}^2))\to\mathbf{Q}(\mathrm{Ham}(\mathbf{T}^2))$ from the space of homogeneous quasimorphisms of the pure braid group to the space of homogeneous quasimorphisms of the group $\mathrm{Ham}(\mathbf{T}^2)$ of Hamiltonian diffeomorphisms of the torus. This map has a nontrivial kernel in general and the goal is to construct a suitable quasimorphism $\psi$ on the pure braid group such that its image $\mathcal{G}(\psi)$ is a nontrivial quasimorphism bounded on the set of autonomous diffeomorphisms.
- In our proof, we specify the braid group to two strings. There is an isomorphism $\mathbf{P}_2(\mathbf{T}^2)\cong\mathbf{F}_2\times\mathbf{Z}^2$ (Lemma 2.6). We construct a suitable quasimorphism on the pure braid group by constructing a quasimorphism $\psi\colon\mathbf{F}_2\to\mathbf{R}$ on the free group and composing it with the projection $\pi\colon\mathbf{P}_2(\mathbf{T}^2)\to\mathbf{F}_2$. The free group here is the quotient of the braid group by its center and hence the projection is canonical (i.e. every automorphism of $\mathbf{P}_2(\mathbf{T}^2)$ descends to an automorphism of the quotient $\mathbf{F}_2$). Thus the refined goal is to construct a quasimorphism $\psi\colon\mathbf{F}_2\to\mathbf{R}$ such that the image $\mathcal{G}(\psi\circ\pi)$ is nontrivial and bounded on the set of autonomous elements.
- Let $\mathbf{F}_2=\langle a,b\rangle$ and let $\sigma_a,\sigma_b\in\mathrm{Aut}(\mathbf{F}_2)$ be automorphisms defined by $\sigma_a(a)=a^{-1}$, $\sigma_a(b)=b$, $\sigma_b(a)=a$ and $\sigma_b(b)=b^{-1}$. They generate an action of $\mathbf{Z}/2\times\mathbf{Z}/2$ on $\mathbf{F}_2$. Let $\mathbf{Q}(\mathbf{F}_2;\mathbf{Z}/2\times\mathbf{Z}/2)$ denote the space of homogeneous quasimorphisms which are invariant under this action. The composition
$$\mathbf{Q}(\mathbf{F}_2;\mathbf{Z}/2\times\mathbf{Z}/2)\stackrel{\pi^*}{\to}\mathbf{Q}(\mathbf{P}_2(\mathbf{T}^2))\stackrel{\mathcal{G}}{\to}\mathbf{Q}(\mathrm{Ham}(\mathbf{T}^2))$$
is injective (Proposition 3.1). Moreover, the space $\mathbf{Q}(\mathbf{F}_2;\mathbf{Z}/2\times\mathbf{Z}/2)$ is infinite dimensional (Proposition 4.3). We obtain this way an infinite dimensional space of quasimorphisms on the group $\mathrm{Ham}(\mathbf{T}^2)$ and the next step is to prove that it contains quasimorphisms bounded on autonomous diffeomorphisms.
- We prove in Lemma 5.1 that if a quasimorphism $\psi\in\mathbf{Q}(\mathbf{F}_2)$ vanishes on primitive elements and on the commutator $[a,b]$ of the generators

then the quasimorphism $\mathcal{G}(\pi^*\psi)$ vanishes on autonomous elements. This reduces our task to showing that the space $\mathbf{Q}(\mathbf{F}_2; \mathbf{Z}/2 \times \mathbf{Z}/2)$ contains quasimorphisms vanishing on primitive elements and the commutator of the generators. Observe that the second condition is automatic. Indeed, if $\psi \in \mathbf{Q}(\mathbf{F}_2; \mathbf{Z}/2 \times \mathbf{Z}/2)$ then we have the following computation in which we use invariance under $\sigma_a$ and homogeneity (which implies conjugation invariance).

$$\begin{aligned}\psi[a,b] &= \psi\left(aba^{-1}b^{-1}\right)\\ &= \psi\left(\sigma_a\left(a^{-1}bab^{-1}\right)\right)\\ &= \psi\left(a^{-1}bab^{-1}\right)\\ &= \psi\left(a^{-1}\left(bab^{-1}a^{-1}\right)a\right)\\ &= \psi[b,a] = -\psi[a,b].\end{aligned}$$

It follows that $\psi[a,b] = 0$.

- Let $\sigma = \sigma_a \circ \sigma_b \in \mathrm{Aut}(\mathbf{F}_2)$ be the automorphism acting on a word by inverting all its letters. If a quasimorphism $\psi \in \mathbf{Q}(\mathbf{F}_2)$ is invariant under the action of $\mathbf{Z}/2 \times \mathbf{Z}/2$ then it is invariant under the action generated by $\sigma$. This, in turn, implies that $\psi$ vanishes on palindromes (see the proof of Corollary 5.5). It has been observed by Bardakov, Shpilrain and Tolstykh [1] that a primitive element of the free group $\mathbf{F}_2$ is a product of two palindromes. Hence every quasimorphism from $\mathbf{Q}(\mathbf{F}_2; \mathbf{Z}/2 \times \mathbf{Z}/2)$ vanishes on primitive elements. This finishes the proof. □

*Remark* 1.2. In the proof Proposition 3.1 which claims the injectivity of the homomorphism $\mathbf{Q}(\mathbf{F}_2; \mathbf{Z}/2 \times \mathbf{Z}/2) \to \mathbf{Q}(\mathrm{Ham}(\mathbf{T}^2))$, we provide explicit examples of Hamiltonian diffeomorphisms on which quasimorphisms of the form $\mathcal{G}(\pi^*\psi)$ evaluate nontrivially (Example 3.3). Such examples are quite standard and have been considered, for example, by Khanevsky [10] and Polterovich-Shelukhin [14].

*Remark* 1.3. Another side result is concerned with the Calabi property and continuity of the quasimorphisms we construct in the paper, see Section 6.6 for a discussion of the Calabi property, and a new example of a Calabi-type quasimorphism. More precisely, if $\psi \in \mathbf{Q}(\mathbf{F}_2)$ is a nontrivial quasimorphism vanishing on palindromes then $\mathcal{G}(\pi^*\psi)$ is nontrivial (see Corollary 6.4 and the discussion that follows it). Moreover, as proven in [15] and [7, Proposition 4.1],

- if $\psi[a,b] \neq 0$ then $\mathcal{G}(\pi^*\psi)$ has the Calabi property;

- if $\psi[a,b]=0$ then $\mathcal{G}(\pi^*\psi)$ is nontrivial and continuous in $C^0$-topology.

It follows that the quasimorphisms constructed in the proof of Proposition 4.3 are $C^0$-continuous and nontrivial. Since by Lemma 5.1 such quasimorphisms vanish on autonomous diffeomorphisms, we can strengthen Theorem 1.1 to the following statement:

**Theorem 1.4.** *The group* $\mathrm{Ham}(\mathbf{T}^2)$ *equipped with the word norm associated with the* $C^0$*-closure of the set of autonomous diffeomorphisms has infinite diameter.*

## 2. Preliminaries

In this section we provide necessary definitions, review in detail the construction of Gambaudo-Ghys and state some known results which we need for the proof.

**Definition 2.1.** Let $G$ be a group. A function $\|\cdot\|\colon G\to[0,\infty)$ is called a *conjugation invariant norm* on $G$ if it satisfies the following conditions:

(1) $\|f\|=0$ if and only if $f=1$,

(2) $\|f^{-1}\|=\|f\|$,

(3) $\|fg\|\leq\|f\|+\|g\|$,

(4) $\|gfg^{-1}\|=\|f\|$.

**Definition 2.2.** A function $\psi\colon G\to\mathbf{R}$ is called a *quasimorphism* if there exist $D_\psi\geq 0$ such that the inequality

$$|\psi(f)-\psi(fg)+\psi(g)|\leq D_\psi$$

holds for all $f,g\in G$. A quasimorphism $\psi$ is called homogeneous if

$$\psi\left(f^n\right)=n\psi(f),$$

for all $f\in G$ and $n\in\mathbf{Z}$. The space of all homogeneous quasimorphisms on a group $G$ is denoted by $\mathbf{Q}(G)$. Let $\mathtt{S}\subset G$. We denote by $\mathbf{Q}(G;\mathtt{S})$ the space of homogeneous quasimorphism which vanish on $\mathtt{S}$.

If $\psi\colon G\to\mathbf{R}$ is a quasimorphism then its homogenization $\overline{\psi}\colon G\to\mathbf{R}$, the unique homogeneous quasimorphism that differs from $\psi$ by a bounded function, satisfies

$$\overline{\psi}(g)=\lim_{n\to\infty}\frac{\psi(g^n)}{n}.$$

Moreover, the homogenization behaves well with respect to group actions.

**Lemma 2.3.** *Let $\psi\colon G \to \mathbf{R}$ be a quasimorphism and let $\alpha\colon H \to \mathrm{Aut}(G)$ be an action of a group $H$ on the group $G$. If $\psi$ is invariant under the action $\alpha$ then so is its homogenization.*

*Proof.* The invariance of $\psi$ under the action $\alpha$ means that $\psi(\alpha(h)(g)) = \psi(g)$ for every $h \in H$ and every $g \in G$. The statement is a consequence of the following straightforward computation.

$$\begin{aligned}
\overline{\psi}(\alpha(h)(g)) &= \lim_{n\to\infty} \frac{\psi\left((\alpha(h)(g))^n\right)}{n}\\
&= \lim_{n\to\infty} \frac{\psi\left(\alpha(h)(g^n)\right)}{n}\\
&= \lim_{n\to\infty} \frac{\psi\left(g^n\right)}{n}\\
&= \overline{\psi}(g).
\end{aligned}$$

□

2.4. **The Gambaudo-Ghys construction.** Let $\Sigma$ be an oriented surface and let $\mathbf{X}_n(\Sigma)$ denote the space of configurations of ordered $n$-tuples of points in $\Sigma$. Its quotient by the $n$-th symmetric group is the space of unordered configurations and it is denoted by $\mathbf{C}_n(\Sigma)$. The fundamental groups

$$\begin{aligned}
\mathbf{P}_n(\Sigma) &:= \pi_1(\mathbf{X}_n(\Sigma))\\
\mathbf{B}_n(\Sigma) &:= \pi_1(\mathbf{C}_n(\Sigma))
\end{aligned}$$

are called the *pure braid group* and the (full) *braid group* of the surface $\Sigma$ respectively.

Let $(z_1, \ldots, z_n) \in \mathbf{X}_n(\Sigma)$ be an $n$-tuple of distinct points which is the basepoint in the configuration space. We fix an auxiliary Riemannian metric on $\Sigma$ and for every point $x \in \Sigma$ we fix a geodesic $\gamma_{i,x}$ of minimal length from $z_i$ to $x$. We denote by $\overline{\gamma}_{i,x}$ the reversed geodesic.

Let $h_t \in \mathrm{Ham}(\Sigma)$ be a Hamiltonian isotopy from the identity to a diffeomorphism $h = h_1 \in \mathrm{Ham}(\Sigma)$ and let $(x_1, \ldots, x_n) \in \mathbf{X}_n(\Sigma)$ be a point in the configuration space. Let $\gamma(h, x_1, \ldots, x_n) \in \mathbf{P}_n(\Sigma)$ be the braid represented by the loop $[0,3] \to \mathbf{X}_n(\Sigma)$ defined by

$$s \mapsto \begin{cases} (\gamma_{1,x_1}(s), \ldots, \gamma_{n,x_n}(s)) & \text{for } 0 \le s \le 1 \\ (h_{s-1}(x_1), \ldots, h_{s-1}(x_n)) & \text{for } 1 \le s \le 2 \\ (\overline{\gamma}_{1,h(x_1)}(s-2), \ldots, \overline{\gamma}_{n,h(x_n)}(s-2)) & \text{for } 2 \le s \le 3. \end{cases}$$

This braid is only well defined on a set of points $(x_1, \ldots, x_n)$ of full measure.

Let $\psi \colon \mathbf{P}_n(\Sigma) \to \mathbf{R}$ be a homogeneous quasimorphism and let

$$\mathcal{G} \colon \mathbf{Q}(\mathbf{P}_n(\Sigma)) \to \mathbf{Q}(\mathrm{Ham}(\Sigma))$$

be defined by

$$\mathcal{G}(\psi)(h) := \lim_{p\to\infty} \frac{1}{p} \int_{\mathbf{X}_n(\Sigma)} \psi(\gamma(h^p, x_1, \ldots, x_n))\, dx_1 \wedge \cdots \wedge dx_n.$$

The fact that the value $\mathcal{G}(\psi)$ is a homogeneous quasimorphism, when $\psi$ is a signature quasimorphism, was first proved by Gambaudo and Ghys [8] for the case of the disc and the sphere and later extended to all $\psi$ and all surfaces by Brandenbursky [3]. The map $\mathcal{G}$ is linear and, in general, has a nontrivial kernel. In Section 3 we prove that $\mathcal{G}$ is injective on a certain subspace of $\mathbf{Q}(\mathbf{P}_2(\mathbf{T}^2))$.

2.5. **Braid groups on two strings.** We use the following presentations of the braid groups the free group:

$$\begin{aligned} \mathbf{B}_2(\mathbf{T}^2) &= \langle a_1, a_2, b_1, b_2, \sigma \,|\, \texttt{Relations} \rangle \\ \mathbf{P}_2(\mathbf{T}^2) &= \langle a_1, a_2, b_1, b_2, \sigma^2 \,|\, \texttt{Relations} \rangle \\ \mathbf{F}_2 &= \langle a, b \rangle. \end{aligned}$$

We omit the relations because they are quite complicated and we don't need them in our discussions. They can be found in [17, Theorem 1.3 and 1.4]. The generators are presented in Figure 2.1, which should be understood as follows. For example, the generator $a_1$ is a braid in

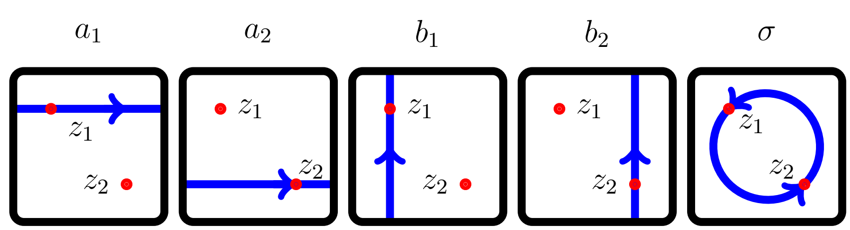


FIGURE 2.1. Generators of the braid group $\mathbf{B}_2(\mathbf{T}^2)$.

which the first basepoint traces the horizontal loop going once around the torus and the second basepoint remains still.

**Lemma 2.6.** *The map* $\Phi\colon \mathbf{X}_2(\mathbf{T}^2) \to \mathbf{T}^2 \setminus \{0\} \times \mathbf{T}^2$ *defined by*

$$\Phi(x, y) := (x - y, y)$$

*is a diffeomorphism. It induces an isomorphism*

$$\Phi_*\colon \mathbf{P}_2(\mathbf{T}) \to \mathbf{F}_2 \times \mathbf{Z}^2,$$

*which on the generators is given by*

$$\begin{array}{lll} a_1 \mapsto (a, (0,0)) & a_2 \mapsto (a^{-1}, (1,0)) & \sigma^2 \quad \mapsto ([a,b], (0,0)) \\ b_1 \mapsto (b, (0,0)) & b_2 \mapsto (b^{-1}, (0,1)). & \end{array}$$

*Proof.* The fact that $\Phi$ is a diffeomorphism is straightforward. Let $\pi\colon \mathbf{P}_2(\mathbf{T}^2) \to \mathbf{F}_2$ denote the projection onto the free factor. The following figures describe the value of $\pi$ on a generator.

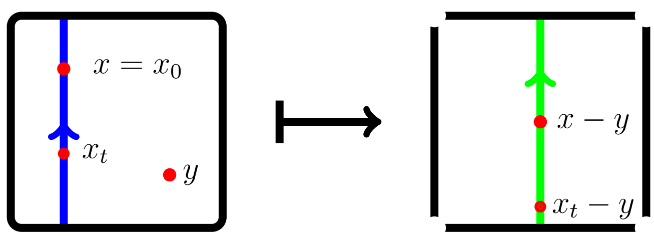


FIGURE 2.2. The image $\pi(b_1) = b \in \mathbf{F}_2$.

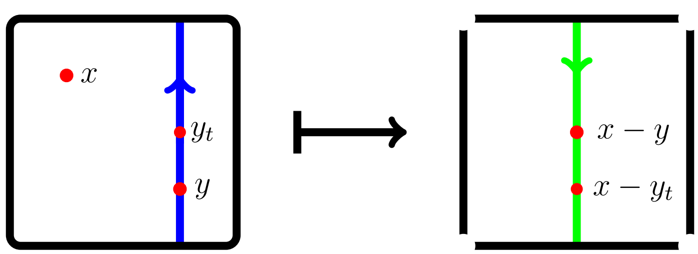


FIGURE 2.3. The image $\pi(b_2) = b^{-1} \in \mathbf{F}_2$.

The left hand side of the figure represents the image of the generator $b_1$ (blue) with respect to the projection onto the torus (the black square). The generator $b_1$ moves the point $x$ along the meridian of the torus and keeps the point $y$ fixed. The generator $b_2$ keeps the point $x$ fixed and moves the point $y$ along the meridian of the torus. The right hand sides of the figures present the free part of $\Phi_*(b_i)$ as loops on the punctured torus. The abelian parts are straightforward to see. The values on the generators $a_i$ are computed analogously. □

It follows from the above proposition that the quotient of the pure braid group $\mathbf{P}_2(\mathbf{T}^2)$ by its center is isomorphic to the free group $\mathbf{F}_2$. Let $\pi\colon \mathbf{P}_2(\mathbf{T}^2) \to \mathbf{F}_2$ denote the projection. It induces the linear map

$$\pi^*\colon \mathbf{Q}(\mathbf{F}_2) \to \mathbf{Q}(\mathbf{P}_2(\mathbf{T}^2)).$$

In the second part of the paper we will need quasimorphisms on the full braid group. In what follows we identify those quasimorphisms $\psi$ on the free groups such that $\pi^*\psi$ extends to the full braid group.

**Definition 2.7.** A word $w$ in $\mathbf{F}_2 = \langle a, b\rangle$ is called a *palindrome* if $w$ is equal to itself read from right to left. Let $\mathtt{PAL} \subset \mathbf{F}_2$ denote the set of all palindromes.

**Proposition 2.8.** *A quasimorphism $\psi \in \mathbf{Q}(\mathbf{F}_2)$ vanishes on palindromes if and only if the quasimorphism $\pi^*\psi$ extends to $\mathbf{B}_2(\mathbf{T}^2)$. In particular, we get a linear map*

$$\mathbf{Q}(\mathbf{F}_2, \mathtt{PAL}) \to \mathbf{Q}(\mathbf{B}_2(\mathbf{T}^2)).$$

*Proof.* The pure braid group is a normal subgroup of finite index in the full braid group. According to [11, Lemma 4.2], a homogeneous quasimorphism $\psi\colon H \to \mathbf{R}$ on a finite index normal subgroup $H \triangleleft G$ extends to the group $G$ if and only if for every $h \in H$ and every $g \in G$ we have that $\psi(ghg^{-1}) = \psi(h)$.

It follows that a quasimorphism on the pure braid group extends if and only if it is invariant under the automorphism defined by the conjugation by $\sigma$. Since $\mathbf{F}_2$ is the quotient of the pure braid group by the center, the conjugation by $\sigma$ descends to an automorphism of the free group. By abuse of notation we denote it by $\sigma \in \mathrm{Aut}(\mathbf{F}_2)$. Observe that $\sigma$ is defined by specifying its values on generators as $\sigma(a) = a^{-1}$ and $\sigma(b) = b^{-1}$.

We conclude that if $\psi \in \mathbf{Q}(\mathbf{F}_2)$ then the quasimorphism $\pi^*\psi$ extends to the full braid group if and only if $\psi$ is invariant under $\sigma$. That is, $\psi(\sigma(g)) = \psi(g)$ for every $g \in \mathbf{F}_2$.

Observe that $\sigma(g) = g^{-1}$ if and only if $g$ is a palindrome. In particular, every element of the form $\sigma(g)g^{-1}$ is a palindrome.

If $\psi$ vanishes on palindromes then $\psi(\sigma(g)g^{-1}) = 0$ for every $g$. The following computation shows that $\psi$ is invariant under $\sigma$. Let $w \in \mathbf{F}_2$

be any element.

$$\begin{aligned}|\psi(\sigma(w))-\psi(w)| &= \frac{1}{n}|\psi(\sigma(w^n))-\psi(w^n)|\\ &\leq \frac{1}{n}\left(|\psi(\sigma(w^n)w^{-n})|+D_\psi\right)=\frac{D_\psi}{n}.\end{aligned}$$

Conversely, if $\psi$ is invariant with respect to $\sigma$ then if $g\in\mathbf{F}_2$ is a palindrome we get that

$$\psi(g^{-1})=\psi(\sigma(g))=\psi(g)$$

and by homogeneity we obtain that $\psi(g)=0$. □

## 3. The injectivity theorem

Let $\sigma_a,\sigma_b\in\mathrm{Aut}(\mathbf{F}_2)$ be automorphisms defined by

$$\begin{aligned}\sigma_a(a)&=a^{-1}, &\sigma_a(b)&=b\\ \sigma_b(a)&=a, &\sigma_b(b)&=b^{-1}.\end{aligned}$$

They generate an action of $\mathbf{Z}/2\times\mathbf{Z}/2$ on the free group $\mathbf{F}_2$.

**Proposition 3.1.** *Let $\mathbf{Q}(\mathbf{F}_2;\mathbf{Z}/2\times\mathbf{Z}/2)\subset\mathbf{Q}(\mathbf{F}_2)$ be the space of homogeneous quasimorphisms which are invariant under the above action. The composition*

$$\mathbf{Q}(\mathbf{F}_2,\mathbf{Z}/2\times\mathbf{Z}/2)\xrightarrow{\pi^*}\mathbf{Q}(\mathbf{P}_2(\mathbf{T}^2))\xrightarrow{\mathcal{G}}\mathbf{Q}(\mathrm{Ham}(\mathbf{T}^2))$$

*is injective.*

**Lemma 3.2.** *Let $h\in\mathrm{Ham}(\mathbf{T}^2)$ be a Hamiltonian diffeomorphism and let $x,y\in\mathbf{T}^2$ be two points. If $h(x)=x$ and $h(y)=y$ then*

$$\gamma(h^p,x,y)=\gamma(h,x,y)^p$$

*Proof.* Immediate from the definition of $\gamma(-,-,-)$. □

*Proof.* Let $\psi\in\mathbf{Q}(\mathbf{F}_2,\mathbf{Z}/2\times\mathbf{Z}/2)$. We shall prove that $\mathcal{G}(\pi^*\psi)\neq 0$ in $\mathbf{Q}(\mathrm{Ham}(\mathbf{T}^2))$ by constructing explicit examples of Hamiltonian diffeomorphisms on which $\mathcal{G}(\pi^*\psi)$ evaluates nontrivially.

Let $s\in(0,\frac{1}{4})$ and let $0<\epsilon<10^{-3}s$. Let $F_s\colon[0,1]\to\mathbf{R}$ be a smooth function with the following properties:

(1) $F_s(x)=0$ for $x\in\left[0,\frac{1}{4}-s-\epsilon\right]\cup\left[\frac{1}{4}+s+\epsilon,1\right]$,

(2) $F_s'(x)=1$ for $x\in\left[\frac{1}{4}-s,\frac{1}{4}-\epsilon\right]$,

(3) $F_s\left(\frac{1}{4}-x\right)=F_s\left(\frac{1}{4}+x\right)$ for $x\in\left[0,\frac{1}{4}\right]$,

see Figure 3.1.

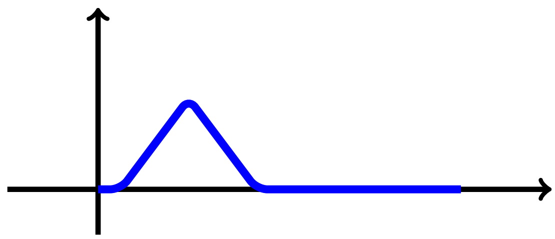

FIGURE 3.1. The function $F_s$.

Let $H, V\colon \mathbf{T}^2 \to \mathbf{R}$ be defined by $H(x,y)=F_s(1-y)$ and $V(x,y)=F_s(x)$ respectively. Let $h_t, v_t \in \mathrm{Ham}(\mathbf{T}^2)$ be the corresponding Hamiltonian flows and let $t_0>0$ be a real number chosen so that the restriction of $v_{t_0}$ to the annulus $\left[\frac{1}{4}-s, \frac{1}{4}-\epsilon\right]\times\mathbf{S}^1$ is the identity. Let $h := h_{t_0}$ and $v := v_{t_0}$. The support of $v$ is marked green and the support of $h$ is marked blue in Figure 3.2 below. The isotopy $\{v_t\}$ is supported between the green lines and the support of the isotopy $\{h_t\}$ is between the blue lines.

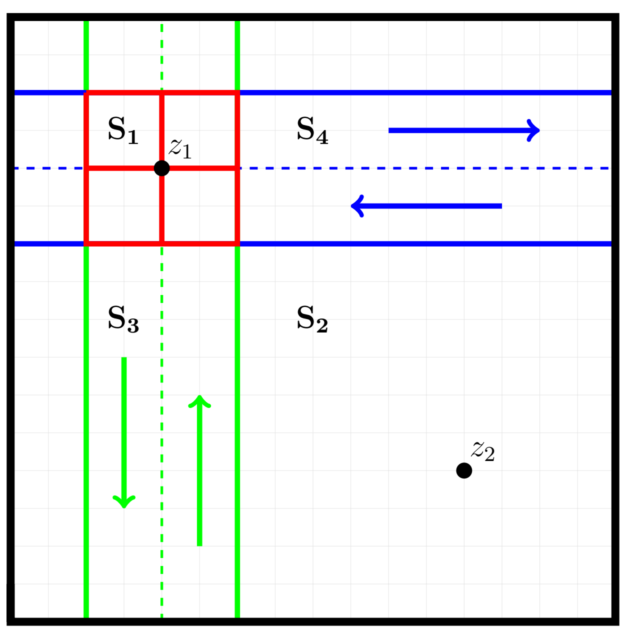


FIGURE 3.2. Diffeomorphisms $h$ and $v$.

Define the following pairwise disjoint open subsets of the torus:

- $S_1 := \left(\frac{1}{4} - s, \frac{1}{4} + s\right) \times \left(\frac{3}{4} - s, \frac{3}{4} + s\right)$ – the red square,
- $S_2 := \mathbf{T}^2 \setminus \left(\left[\frac{1}{4} - s, \frac{1}{4} + s\right] \times [0,1] \cup [0,1] \times \left[\frac{3}{4} - s, \frac{3}{4} + s\right]\right)$ – the complement of the union of the blue and green annuli,
- $S_3 := \left(\frac{1}{4} - s, \frac{1}{4} + s\right) \times [0,1] \setminus \overline{S_1}$ – the green annulus minus the closure of the red square,
- $S_4 := [0,1] \times \left(\frac{1}{4} - s, \frac{1}{4} + s\right) \setminus \overline{S_1}$ – the blue annulus minus the closure of the red square.

Let $\psi\colon \mathbf{F}_2 \to \mathbf{R}$ be a nontrivial homogeneous quasimorphism invariant under the action of $\mathbf{Z}/2 \times \mathbf{Z}/2$ and let $w(a,b) \in \mathbf{F}_2$ be an element such that $\psi(w(a,b)) > 0$. Let $g = w(v,h) \in \mathrm{Ham}(\mathbf{T}^2)$. Now we investigate the value of the integral

$$\int_{\mathbf{X}_2(\mathbf{T}^2)} \psi(\pi(\gamma(g,x,y)))\, dx \wedge dy$$

by decomposing it into a sum of integral over subsets of the configuration space. First observe that the subset

$$\bigcup_{i \neq j} S_i \times S_j \cup \bigcup_i \mathbf{X}_2(S_i) \subset \mathbf{X}_2(\mathbf{T}^2)$$

is open and dense so we have

$$\begin{aligned}\int_{\mathbf{X}_2(\mathbf{T}^2)} \psi(\pi(\gamma(g,x,y)))\, dx \wedge dy &= \sum_{i \neq j} \int_{S_i \times S_j} \psi(\pi(\gamma(g,x,y)))\, dx \wedge dy \\ &\quad + \sum_i \int_{\mathbf{X}_2(S_i)} \psi(\pi(\gamma(g,x,y)))\, dx \wedge dy\end{aligned}$$

It will be useful to know the volumes of the sets $S_i \times S_j$. They are as follows:

- $\mathrm{vol}(S_1 \times S_1) = 16s^4$,
- $\mathrm{vol}(S_1 \times S_2) = 4s^2(1-2s)^2 = 4s^2 - 16s^3 + 16s^4$,
- $\mathrm{vol}(S_1 \times S_3) = \mathrm{vol}(S_1 \times S_4) = 4s^2(1-2s)2s = 8s^3 - 16s^4$,
- $\mathrm{vol}(S_2 \times S_2) = (1-2s)^2 = 1 - 4s + 4s^2$,
- $\mathrm{vol}(S_2 \times S_3) = \mathrm{vol}(S_2 \times S_4) = (1-2s)^2(1-2s)2s = 2s - 12s^2 + 24s^3 - 16s^4$,
- $\mathrm{vol}(S_3 \times S_3) = \mathrm{vol}(S_4 \times S_4) = \mathrm{vol}(S_3 \times S_4) = 4s^2(1-2s)^2 = 4s^2 - 16s^3 + 16s^4$.

The volumes are polynomials of $s$ and what will be important below is their degrees. Let us consider the element $\pi(\gamma(w(v,h),x,y)) \in \mathbf{F}_2$ for various configurations:

- $(x,y) \in S_1 \times S_2$; depending on the position of $x$ in the red square we obtain:
  - (top left) $\pi(\gamma(w(v,h),x,y)) = w(a,b)$,
  - (top right) $\pi(\gamma(w(v,h),x,y)) = w(a^{-1},b)$,
  - (bottom left) $\pi(\gamma(w(v,h),x,y)) = w(a,b^{-1})$,
  - (bottom right) $\pi(\gamma(w(v,h),x,y)) = w(a^{-1},b^{-1})$,

  Since the quasimorphism $\psi$ is invariant under inverting generators we get that $\psi(\pi(\gamma(g,x,y))) = \psi(w(a,b)) \neq 0$. Thus
  $$\begin{aligned}\int_{S_1\times S_2} \psi(\pi(\gamma(g,x,y)))\,dx\wedge dy &= \mathrm{vol}(S_1\times S_2)\psi(w(a,b))\\ &= (4s^2-16s^3+16s^4)\psi(w(a,b)).\end{aligned}$$
- $(x,y) \in S_1\times(S_3\cup S_4)$; for fixed $g$ the braid $\gamma(g,x,y)$ can attain finitely many values in this case and we let $C_1 := \max|\psi(\pi(\gamma(g,x,y)))|$.
  $$\begin{aligned}\left|\int_{S_1\times S_3} \psi(\pi(\gamma(g,x,y)))\,dx\wedge dy\right| &\leq \mathrm{vol}(S_1\times S_2)C_1\\ &= (6s^3-16s^4)C_1.\end{aligned}$$
- $(x,y) \in S_2\times(S_3\cup S_4)$; we get that $\pi(\gamma(g,x,y))$ is either a power of $a$ or a power of $b$ so $\psi(\pi(\gamma(g,x,y))) = 0$.
- $(x,y) \in S_3\times S_4$; here the situation is similar to the first case and the value of $\pi(\gamma(g,x,y))$ depends on the positions $x$ and $y$ in the strips and we obtain that $\psi(\pi(\gamma(g,x,y))) = \psi(w(a,b)) \neq 0$. We get
  $$\begin{aligned}\int_{S_3\times S_4} \psi(\pi(\gamma(g,x,y)))\,dx\wedge dy &= \mathrm{vol}(S_3\times S_4)\psi(w(a,b))\\ &= (4s^2-16s^3+16s^4)\psi(w(a,b)).\end{aligned}$$
- $(x,y) \in \mathbf{X}_2(S_1)$; for fixed $g$ the braid $\gamma(g,x,y)$ can attain finitely many values and let $C_2 := \max|\psi(\pi(\gamma(g,x,y)))|$. We have that
  $$\begin{aligned}\left|\int_{\mathbf{X}_2(S_1)} \psi(\pi(\gamma(g,x,y)))\,dx\wedge dy\right| &\leq \mathrm{vol}(S_1\times S_1)C_1\\ &= 16s^4C_2.\end{aligned}$$

- $(x,y) \in \mathbf{X}_2(S_3) \cup \mathbf{X}_2(S_4)$; in this case the braid $\gamma(g,x,y)$ is equal to either $a_1^m a_2^n$ or $b_1^m b_2^n$ and hence $\pi(\gamma(g,x,y))$ is equal to a power of a generator and $\psi(\pi(\gamma(g,x,y))) = 0$.

Chose $s \in \left(0, \frac{1}{4}\right)$ small enough so that

$$(6s^3 - 16s^4)C_1 + 16s^4 C_2 < 2(4s^2 - 16s^3 + 16s^4)\psi(w(a,b)).$$

For such an $s$ we obtain that

$$\int_{\mathbf{X}_2(\mathbf{T}^2)} \psi(\pi(\gamma(g,x,y)))\, dx \wedge dy \neq 0.$$

Since $g(x) = x$ and $g(y) = y$ for $(x,y)$ outside the subset of arbitrarily small measure (depending on the number $\epsilon$), we have that $\gamma(g^p,x,y) = \gamma(g,x,y)^p$, for $(x,y)$ in the set of measure which is arbitrarily close to full, according to Lemma 3.2. This implies that $\psi(\pi(\gamma(g^p,x,y))) = p\psi(\pi(\gamma(g,x,y)))$ and finally we get that

$$\lim_{p\to\infty} \int_{\mathbf{X}_2(\mathbf{T}^2)} \frac{1}{p} \psi(\pi(\gamma(g^p,x,y)))\, dx \wedge dy \neq 0.$$

which finishes the proof. □

**Example 3.3** (Eggbeater)**.** Let $g = h^{2m} v^{2m-1} \cdots h^2 v \in \mathrm{Ham}(\mathbf{T}^2)$, where $h, v \in \mathrm{Ham}(\mathbf{T}^2)$ are Hamiltonian diffeomorphisms defined in the above proof. It follows from Proposition 4.3 below and the above proof that the quasimorphism $\mathcal{G}(\psi_{c_m} \circ \pi)$ is unbounded on the cyclic subgroup of $\mathrm{Ham}(\mathbf{T}^2)$ generated by $g$. Here $\psi_{c_m}\colon \mathbf{F}_2 \to \mathbf{R}$ is the quasimorphism associated with the function $c_m\colon \mathbf{Z}^m \to \mathbf{Z}$ given by

$$c_m(i_1, \ldots, i_m) = \mathrm{sgn}(|i_1| - |i_m|),$$

see Lemma 4.1. Since the quasimorphism $\mathcal{G}(\psi_{c_m} \circ \pi)$ vanishes on autonomous elements, we get that the cyclic subgroup generated by $g$ is unbounded with respect to the autonomous norm.

♢

## 4. Quasimorphisms with vanishing properties

In this section we prove that the space $\mathbf{Q}(\mathbf{F}_2; \mathbf{Z}/2 \times \mathbf{Z}/2)$ of quasimorphisms on the free group invariant under the action of the Klein group is infinite dimensional. It is done by constructing explicit examples. Our construction is inspired by the example from the proof of Theorem 1.1 in [1].

Let $w \in \mathbf{F}_n$ be a reduced word. A *syllable* in $w$ is a maximal power of a generator occurring in $w$. The exponent of a syllable $s$ is denoted by $e(s)$. For example, the commutator $[a,b] \in \mathbf{F}_2$ has four syllables all with exponents equal to 1. Let $w = s_1 s_2 \dots s_k$ be a reduced word, where $s_i$ are syllables. It defines a $k$-tuple of integers $(e(s_1), e(s_2), \dots, e(s_k))$.

Let $c\colon \mathbf{Z}^m \to \mathbf{Z}$ be a bounded function which satisfies the identity:

$$c(i_1, \dots, i_m) = -c(-i_m, \dots, -i_1). \tag{4.1}$$

Let $\psi_c\colon \mathbf{F}_n \to \mathbf{R}$ be defined as follows. Let $w = s_1 \dots s_k \in \mathbf{F}_n$. If $k < m$ then $\psi_c(w) = 0$. If $k \geq m$ then

$$\psi_c(s_1 s_2 \dots s_k) := \sum_{i=1}^{k-m+1} c(e(s_i), \dots, e(s_{i+m-1})).$$

**Lemma 4.1.** *Let $c\colon \mathbf{Z}^m \to \mathbf{Z} \cap [-B, B]$ be a bounded function satisfying the identity (4.1). Then the function $\psi_c$ is a quasimorphism with defect bounded by $3(m+1)B$.*

*Proof.* Let $s = s_1 \cdots s_k$ and $t = t_1 \cdots t_l$ be reduced words such that $st$ is also reduced. We have the following expression for the value of $\psi_c(st)$:

$$\psi_c(st) = \begin{cases} \psi_c(s) + \psi_c(t) \\ + \sum_{i=1}^{m-1} c(e(s_{k-m+1+i}), \dots, e(s_k), e(t_1), \dots, e(t_i)) \\ \text{ if last letter of } s \text{ is different from the first letter of } t, \text{ or} \\ \psi_c(s) + \psi_c(t) \\ + \sum_{i=1}^{m-1} c(e(s_{k-m+1+i}), \dots, e(s_k t_1), \dots, e(t_i)) \\ -c(e(s_{k-m+1}), \dots, e(s_k)) - c(e(t_1), \dots, e(t_m)), \\ \text{ otherwise.} \end{cases}$$

It follows that in this special case we have the following estimate:

$$|\psi_c(s) - \psi_c(st) + \psi_c(t)| \leq (m+1)B.$$

Let us now consider the general case. Let $s = s_1 \cdots s_k$, $t = t_1 \cdots t_l, u = u_1 \cdots u_p \in \mathbf{F}_n$ be reduced words, where $s_i, t_i$ and $u_i$ are syllables. Suppose that $su$, $u^{-1}t$ and $st$ are reduced. Using the previous inequality we obtain the following estimate of the defect:

$$\begin{aligned} &\left|\psi_c(su) - \psi_c(st) + \psi_c(u^{-1}t)\right| \leq \\ &\leq \left|\psi_c(s) + \psi_c(u) - \psi_c(s) - \psi_c(t) + \psi_c(u^{-1}) + \psi_c(t)\right| + 3(m+1)B \\ &= \left|\psi_c(u) + \psi_c(u^{-1})\right| + 3(m+1)B \\ &\leq 3(m+1)B. \end{aligned}$$

The fact that $\psi_c(u^{-1}) = -\psi_c(u)$ follows from the identity (4.1). This proves that $\psi_c$ is a quasimorphism with defect $3(m+1)B$. □

**Example 4.2.** Let $c\colon \mathbf{Z}^2 \to \mathbf{Z}$ be defined by $c(m,n) = \mathrm{sgn}(|m| - |n|)$. The associated quasimorphism $\psi_c\colon \mathbf{F}_2 \to \mathbf{Z}$ is clearly invariant under the action of $\mathbf{Z}/2\times\mathbf{Z}/2$. To see that it is unbounded consider the cyclic subgroup generated by $a^4b^3a^2b$. We have that

$$\psi_c\left((a^4b^3a^2b)^n)\right) = 2n+1.$$

which implies that the homogenization of this quasimorphism is non-trivial. It follows from Lemma 2.3 that the homogenization is also invariant.

**Proposition 4.3.** *The subspace $\mathbf{Q}(\mathbf{F}_2;\mathbf{Z}/2\times\mathbf{Z}/2) \subset \mathbf{Q}(\mathbf{F}_2)$ of homogeneous quasimorphism invariant under the action of $\mathbf{Z}/2\times\mathbf{Z}/2$ is infinite dimensional.*

*Proof.* Let $c_m\colon \mathbf{Z}^m \to \mathbf{Z}$ for $m \geq 2$ be a function defined by

$$c_m(i_1,\ldots,i_m) := \mathrm{sgn}(|i_1| - |i_m|).$$

Consider the sequence $\psi_{c_m}$ of quasimorphisms defined in the beginning of this section. Since the function $c_m$ depends only on the absolute values, the quasimorphisms $\psi_{c_m}$ are invariant under inverting generators.

Let $w_m = a^{2m}b^{2m-1}\cdots a^2b$. We get that

$$\psi_{c_m}(w^n) = 2n+m-1.$$

Let $k \in \mathbf{N}$ be a positive integer. Consider the square $k\times k$-matrix with entries $a_{ij} = \psi_{c_{3i}}(w_{3j})$ and observe that it is upper triangular with positive entries on the diagonal and hence it has a positive determinant. It implies that the functions $\psi_{c_{3i}}$ for $i = 1,\ldots,k$ are linearly independent for any $k \in \mathbf{N}$. This shows that there exists an infinite dimensional subspace of quasimorphisms invariant under inverting generators. It follows from Lemma 2.3 that $\dim \mathbf{Q}(\mathbf{F}_2;\mathbf{Z}/2\times\mathbf{Z}/2) = \infty$. □

## 5. Vanishing on autonomous diffeomorphisms

**Lemma 5.1.** *Let $\psi\colon \mathbf{F}_2 = \langle a,b\rangle \to \mathbf{R}$ be a homogeneous quasimorphism. If it vanishes on primitive elements and on the commutator $[a,b]$ then the quasimorphism*

$$\mathcal{G}(\psi\circ\pi)\colon\ \mathrm{Ham}(\mathbf{T}^2) \to \mathbf{R}$$

*vanishes on the set of autonomous diffeomorphisms.*

*Proof.* Let $H\colon \mathbf{T}^2 \to \mathbf{R}$ be a smooth function and let $h_t \in \mathrm{Ham}(\mathbf{T}^2)$ be the autonomous flow induced by $H$. Let

$$\mathcal{O}(h_t, x) := \{h_t(x) \in \mathbf{T}^2 \,|\, t \in \mathbf{R}\,\}$$

denote the orbit of the point $x$ with respect to the flow $h_t$. Such an orbit is either periodic (including constant) or it is an interval between one (homoclinic) or two (heteroclinic) fixed points.

Let $z_1, z_2 \in \mathbf{T}^2$ be basepoints and let $x, y \in \mathbf{T}^2$. In what follows we analyze the braid $\gamma(h^p, x, y)$ for $p \in \mathbf{N}$. We break it down into cases depending on the form of the orbits $\mathcal{O}(h_t, x)$ and $\mathcal{O}(h_t, y)$. We consider the following cases:

(1) $\mathcal{O}(h_t, x) = \{x\}$.

  (a) If $\mathcal{O}(h_t, y) = \{y\}$ then $\gamma(h^p, x, y)$ is trivial.

  (b) If $\mathcal{O}(h_t, y)$ is a contractible periodic orbit bounding a disc containing the point $x$ then $\gamma(h^p, x, y)$ is an integer power of $\sigma^2$.

  (c) If $\mathcal{O}(h_t, y)$ is a contractible periodic orbit bounding a disc not containing the point $x$ then $\gamma(h^p, x, y)$ is either trivial or equal to $\sigma^2$.

  (d) If $\mathcal{O}(h_t, y)$ is a homotopically nontrivial periodic orbit then its image is a simple closed curve. There exists a symplectic diffeomorphism of the torus $f \in \mathrm{Symp}(\mathbf{T}^2)$ preserving the basepoints $z_1$ and $z_2$ such that the image $f(\mathcal{O}(h_t, y))$ of the orbit represents the standard generator $(1, 0) \in \mathbf{Z}^2 = \pi_1(\mathbf{T}^2)$ disjoint from $f(x)$. Thus the braid $\gamma(fh^pf^{-1}, f(x), f(y))) = (F, A)$, where both $F \in \mathbf{F}_2$ and $A \in \mathbf{Z}^2$ are powers of primitive elements. Observe that $\gamma(fh^pf^{-1}, f(x), f(y))) = f_*(\gamma(h^p, x, y))$, where $f_*\colon \mathbf{P}_2(\mathbf{T}^2) \to \mathbf{P}_2(\mathbf{T}^2)$ is the automorphism induced by $f$. Since $f$ induces an automorphism of the quotient $\mathbf{F}_2$ we get that $\pi(\gamma(h^p, x, y))$ is a power of a primitive element.

  (e) If the orbit $\mathcal{O}(h_t, y)$ is nonperiodic then there exists $p_0 \in \mathbf{N}$ such that

$$\#\{\gamma(h^p, x, y) \in \mathbf{P}_2(\mathbf{T}^2) \,|\, p \geq p_0\,\} \leq 2.$$

  Indeed, let $y_+ := \lim_{t\to\infty} h_t(y)$ be the limit point and let $\epsilon > 0$. There exists $p_0$ such that $|h^p(y) - y_+| < \epsilon$ for every $p \geq p_0$. Depending on a relative position of the points $z_1, z_2, x$ and $h^p(y)$ the braids $\gamma(h^p, x, y)$ and $\gamma(h^q, x, y)$ for $p, q \geq p_0$ may differ by at most one crossing arising when the endpoints $h^p(y)$ or $h^q(y)$ and $x$ are joined to the basepoints.

(2) The orbit $\mathcal{O}(h_t, x)$ is nonperiodic. Let $x_+ := \lim_{t\to\infty} h_t(x)$.

(a) If $\mathcal{O}(h_t, y)$ is either constant or nonperiodic then

$$\#\{\gamma(h^p, x, y) \in \mathbf{P}_2(\mathbf{T}^2) \mid p \geq p_0\} \leq 2$$

as in the previous case. The only difference is that for given $\epsilon$ one has to choose $p_0$ such that both $|h^p(x) - x_+| < \epsilon$ and $|h^p(y) - y_+| < \epsilon$ for all $p \geq p_0$.

(b) If $\mathcal{O}(h_t, y)$ is a contractible periodic orbit such that $\mathcal{O}(h_t, x)$ is contained in the disc bounded by $\mathcal{O}(h_t, y)$ then $\gamma(h^p, x, y)$ is a power of $\sigma^2$.

(c) If $\mathcal{O}(h_t, y)$ is a contractible periodic orbit such that $\mathcal{O}(h_t, x)$ is not contained in the disc bounded by $\mathcal{O}(h_t, y)$ then $\gamma(h^p, x, y)$ is either trivial or equal to $\sigma^2$.

(d) If $\mathcal{O}(h_t, y)$ is a homotopically nontrivial periodic orbit then, as in the case (1)(d) above, we get that $\gamma(h^p, x, y)$ is a power of a primitive element.

(3) The orbit $\mathcal{O}(h_t, x)$ is contractible periodic.

(a) The case when $\mathcal{O}(h_t, y)$ is either constant or nonperiodic has been dealt with above.

(b) If the orbits $\mathcal{O}(h_t, x)$ and $\mathcal{O}(h_t, y)$ are concentric then $\gamma(h^p, x, y)$ is a power of $\sigma^2$.

(c) If the orbits $\mathcal{O}(h_t, x)$ and $\mathcal{O}(h_t, y)$ are contractible and not concentric then $\gamma(h^p, x, y)$ is either trivial or equal to $\sigma^2$.

(d) If $\mathcal{O}(h_t, y)$ is periodic and homotopically nontrivial then the braid $\gamma(h^p, x, y)$ is a power of a primitive element and the argument is the same as in the analogous cases above.

(4) If $\mathcal{O}(h_t, x)$ is periodic and homotopically nontrivial then the only case which has not been done above is when the orbit $\mathcal{O}(h_t, y)$ is periodic and not contractible. In this case the images of both orbits are disjoint simple closed curves and thus there exists a symplectic diffeomorphism $f \in \mathrm{Symp}(\mathbf{T}^2)$ preserving basepoints $z_1$ and $z_2$ such that both $f(\mathcal{O}(h_t, x))$ and $f(\mathcal{O}(h_t, y))$ are disjoint simple closed curves representing the generator $(1, 0) \in H_1(\mathbf{T}^2; \mathbf{Z})$ (recall that the intersection form on $H_1(\mathbf{T}^2; \mathbf{Z})$ is non-degenerate and

anti-symmetric). In this case we have

$$\gamma(fh^pf^{-1}, f(x), f(y)) = \begin{cases} b_i^m\, b_j^n \\ b_j^m\, b_i^n\, \sigma^2, \end{cases}$$

for some $m, n \in \mathbf{Z}$, where $i, j \in \{1, 2\}$ are distinct. To see this recall that the conjugation by $\sigma$ swaps the generators $b_1$ and $b_2$. It may also be useful to use the following computation

$$\sigma b_i^m b_j^n \sigma = \sigma b_i^m b_j^n \sigma^{-1}\sigma^2 = b_j^m b_i^n \sigma^2.$$

Thus the image of the above braid in the free group is equal to either $b^{m-n}$ or $b^{m-n}[a, b]$. Thus the image of the braid $\gamma(h^p, x, y)$ in the free group is a product of a power of a primitive element and a commutator of two primitive elements.

According to a theorem of Nielsen [12], the commutator of two primitive elements is conjugate to $[a, b]^{\pm 1}$.

As a conclusion we obtain that the projection $\pi(\gamma(h^p, x, y))$ is equal to either one of the following:

- an integer power of the commutator $[a, b]$,
- an integer power of a primitive element,
- a product of a power of a primitive element and a conjugate of the commutator $[a, b]$ or its inverse,

or there exists $p_0 \in \mathbf{N}$ such that

$$\#\{\gamma(h^p, x, y) \in \mathbf{P}_2(\mathbf{T}^2) \,|\, p \geq p_0\} \leq 2.$$

Let $\psi\colon \mathbf{F}_2 \to \mathbf{R}$ be a homogeneous quasimorphism vanishing on primitive elements and on the commutator $[a, b]$. If $\gamma(h^p, x, y)$ attains finitely many values for $p \geq p_0$ then

$$\lim_{p\to\infty} \frac{1}{p}\psi(\gamma(h^p, x, y)) = 0$$

and hence $\mathcal{G}(\psi)(h) = 0$. If $\pi(\gamma(h^p, x, y))$ is a power of either a primitive element or a conjugate of the commutator $[a, b]$ then $\psi(\pi(\gamma(h^p, x, y))) = 0$ by the hypothesis and we also get that $\mathcal{G}(\psi)(h) = 0$. Finally, if $\pi(\gamma(h^p, x, y))$ is a product of a power of a primitive element and a power of a conjugate of the commutator $[a, b]$ then

$$|\mathcal{G}(\psi)(h)| \leq D_\psi.$$

Since for every autonomous diffeomorphism $h$ and $n \in \mathbf{N}$, $h^n$ is also autonomous we get

$$|\mathcal{G}(\psi)(h)| = \left|\frac{\mathcal{G}(\psi)(h^n)}{n}\right| \leq \frac{D_\psi}{n}.$$

This concludes the proof of the vanishing property of $\mathcal{G}(\psi)$ on the set of autonomous diffeomorphisms. $\square$

5.2. **Palindromes and primitive elements.** The following observation and its proof are due to Bardakov, Shpilrain and Tolstykh [1].

**Lemma 5.3.** *Every primitive element $w \in \mathbf{F}_2$ of the free group of rank 2 is a product of up to two palindromes.*

*Proof.* Let $\sigma\colon \mathbf{F}_2 \to \mathbf{F}_2$ be an automorphism defined on generators by $\sigma(a) = a^{-1}$ and $\sigma(b) = b^{-1}$. Consider the extension

$$\mathbf{F}_2 \xrightarrow{\iota} \mathrm{Aut}(\mathbf{F}_2) \xrightarrow{\pi} \mathrm{Out}(\mathbf{F}_2) = \mathrm{GL}(2, \mathbf{Z}),$$

where the quotient is identified with the automorphism group of the abelianisation $\mathbf{Z}^2$ of the free group $\mathbf{F}_2$.

Let $\theta \in \mathrm{Aut}(\mathbf{F}_2)$ be an automorphism. Since the image $\pi(\sigma) \in \mathrm{Out}(\mathbf{F}_2)$ is equal to $-\mathrm{Id}$, the image $\pi[\sigma, \theta]$ of the commutator is trivial. This implies that $[\sigma, \theta] = I_{p(\theta)}$ is an inner automorphism for some $p(\theta) \in \mathbf{F}_2$. For example, if $\tau \in \mathrm{Aut}(\mathbf{F}_2)$ is defined by $\tau(a) = ab$ and $\tau(b) = b$ then $p(\tau) = a$.

The following computation proves that $\sigma(p(\theta)) = p(\theta)^{-1}$ which means that $p(\theta)$ is a palindrome (notice that $\sigma$ is an involution).

$$\begin{aligned} I_{\sigma(p(\theta))} &= \sigma\, I_{p(\theta)}\, \sigma^{-1} = \sigma[\sigma, \theta]\sigma^{-1} \\ &= [\theta, \sigma] = [\sigma, \theta]^{-1} = I_{p(\theta)^{-1}}. \end{aligned}$$

The second observation is that

$$p(\theta\xi) = p(\theta)\theta(p(\xi))$$

for any $\theta, \xi \in \mathrm{Aut}(\mathbf{F}_2)$. Indeed,

$$\begin{aligned} I_{p(\theta\xi)} &= [\sigma, \theta\xi] \\ &= \sigma\theta\sigma\theta^{-1} \cdot \theta\sigma\xi\sigma\xi^{-1}\theta^{-1} \\ &= [\sigma, \theta]\theta[\sigma, \xi]\theta^{-1} \\ &= I_{p(\theta)}\theta I_{p(\xi)}\theta^{-1} \\ &= I_{p(\theta)} I_{\theta(p(\xi))} = I_{p(\theta)\theta(p(\xi))}. \end{aligned}$$

Evaluating this identity on the automorphism $\tau$ defined above we get that

$$\begin{aligned} p(\theta\tau) &= p(\theta)\theta(p(\tau)) \\ p(\theta\tau) &= p(\theta)\theta(a) \\ p(\theta)^{-1}p(\theta\tau) &= \theta(a), \end{aligned}$$

which shows that any primitive element $\theta(a)$ is a product of two palindromes.

□

**Corollary 5.4.** *If $\psi\colon \mathbf{F}_2 \to \mathbf{R}$ is a homogeneous quasimorphism which vanishes on palindromes then it vanishes on primitive elements.*

*Proof.* Let $w \in \mathbf{F}_2$ be a primitive element. It follows from Lemma 5.3 that $w = uv$, where $u, v \in \mathbf{F}_2$ are palindromes. If $n \in \mathbf{N}$ is a positive integer then

$$w^n = (uv)^n = \left((uv)^{n-1}u\right)v,$$

which shows that $w^n$ is a product of two palindromes. It implies that $|\psi(w^n)| \le D_\psi$. Since $\psi$ is homogeneous we get that it vanishes on primitive elements. □

**Corollary 5.5.** *If $\psi \in \mathbf{Q}(\mathbf{F}_2; \mathbf{Z}/2 \times \mathbf{Z}/2)$ then the quasimorphism $\mathcal{G}(\pi^*\psi) \in \mathbf{Q}(\mathrm{Ham}(\mathbf{T}^2)$ vanishes on autonomous diffeomorphisms.*

*Proof.* Let $\sigma = \sigma_a\sigma_b \in \mathrm{Aut}(\mathbf{F}_2)$. It acts on words by inverting all letters. In particular, an element $w \in \mathbf{F}_2$ is a palindrome if and only if $\sigma(w) = w^{-1}$. If $\psi \in \mathbf{Q}(\mathbf{F}_2; \mathbf{Z}/2 \times \mathbf{Z}/2)$ then $\psi$ is invariant under the action of $\sigma$:

$$\psi(\sigma(w)) = \psi(w).$$

If $w \in \mathbf{F}_2$ is a palindrome then $\psi(w^{-1}) = \psi(\sigma(w)) = \psi(w)$ and it follows from the homogeneity of $\psi$ that it vanishes on palindromes. It is then a consequence of Corollary 5.4 that $\psi$ vanishes on primitive elements.

As explained in the computation on page 5, $\psi$ vanishes on the commutator $[a, b]$ of the generators of $\mathbf{F}_2$ and hence, according to Lemma 5.1, the quasimorphism $\mathcal{G}(\pi^*\psi)$ vanishes on autonomous diffeomorphisms. □

## 6. Further results

6.1. **A lift to symplectic diffeomorphisms.** The braid $\gamma(h,x,y)$ is not well defined for $h \in \mathrm{Symp}_0(\mathbf{T}^2)$. That is, it depends on the isotopy from the identity to $h$. For example, the isotopy defined $h_t(u,v) = (u+t,v)$ is a loop based at the identity and $\gamma(h_t,x,y) = a_1a_2$.

**Lemma 6.2.** *Let $\ell\colon [0,1] \to \mathrm{Symp}_0(\mathbf{T}^2)$ be a loop based at the identity. Then for every $x,y \in \mathbf{T}^2$ the braid $\gamma(\ell,x,y)$ is central.*

*Proof.* Since the inclusion $\mathbf{T}^2 \to \mathrm{Symp}_0(\mathbf{T}^2)$ (where the torus acts on itself by translations) is a homotopy equivalence, the loop $\ell$ is isotopic to a concatenation of loops $h_a\colon (u,v) \mapsto (u+t,v)$ and $h_b\colon (u,v) \mapsto (u,v+s)$. This implies that

$$\gamma(\ell,x,y) = a_1^m a_2^m b_1^n b_2^n \in \mathbf{P}_2(\mathbf{T}^2),$$

for some $m,n \in \mathbf{Z}$. Observe that the center of $\mathbf{P}_2(\mathbf{T}^2)$ is isomorphic to $\mathbf{Z}^2$ and is generated by $a_1a_2$ and $b_1b_2$ (see Lemma 2.6. $\square$

It follows from the lemma that if $\psi \in \mathbf{Q}(\mathbf{P}_2(\mathbf{T}^2))$ vanishes on the center then $\psi(\gamma(h,x,y))$ is well defined for $h \in \mathrm{Symp}_0(\mathbf{T}^2)$. We thus have a lift of the Gambaudo-Ghys homomorphism

$$\widehat{\mathcal{G}}\colon \mathbf{Q}(\mathbf{P}_2(\mathbf{T}^2), \mathtt{CENTER}) \to \mathbf{Q}(\mathrm{Symp}_0(\mathbf{T}^2)).$$

**Lemma 6.3.** *The composition*

$$\mathbf{Q}(\mathbf{B}_2(\mathbf{T}^2), \mathtt{CENTER}) \xrightarrow{\iota^*} \mathbf{Q}(\mathbf{P}_2(\mathbf{T}^2), \mathtt{CENTER}) \xrightarrow{\widehat{\mathcal{G}}} \mathbf{Q}(\mathrm{Symp}_0(\mathbf{T}^2))$$

*is injective.*

The proof of this lemma is essentially the same as the proof of Theorem 2 in [2]. If $\psi \in \mathbf{Q}(\mathbf{F}_2)$ then the composition $\psi \circ \pi$ vanishes on the center and hence every quasimorphism on $\mathbf{F}_2$ yields a quasimorphism on the group of symplectic diffeomorphisms. In order to ensure that it is nontrivial we require that $\psi \circ \pi$ extends to the full braid group and this holds if the quasimorphism $\psi$ vanishes on palindromes (see Proposition 2.8).

**Corollary 6.4.** *The composition*

$$\mathbf{Q}(\mathbf{F}_2, \mathtt{PAL}) \xrightarrow{\pi^*} \mathbf{Q}(\mathbf{P}_2(\mathbf{T}^2)) \xrightarrow{\widehat{\mathcal{G}}} \mathbf{Q}(\mathrm{Symp}_0(\mathbf{T}^2))$$

*is injective.* $\square$

Since $\mathrm{Ham}(\mathbf{T}^2)$ is equal to the commutator subgroup of $\mathrm{Symp}_0(\mathbf{T}^2)$ the kernel of the homomorphism $\mathbf{Q}(\mathrm{Symp}_0(\mathbf{T}^2)) \to \mathbf{Q}(\mathrm{Ham}(\mathbf{T}^2))$ induced by the inclusion consists of homomorphisms. Since $\mathbf{Q}(\mathbf{F}_2, \mathtt{PAL})$ contains no homomorphism we get that the composition

$$\mathbf{Q}(\mathbf{F}_2, \mathtt{PAL}) \to \mathbf{Q}(\mathrm{Ham}(\mathbf{T}^2))$$

is injective.

*Remark* 6.5. This is a slightly stronger statement than Proposition 3.1 and it could serve as an alternative part of the proof of the main Theorem 1.1. We chose a more direct approach in order to have a complete proof which makes the paper selfcontained and also because the proof of Proposition 3.1 provides explicit examples of diffeomorphisms. On the other hand, the above arguments allow us to provide examples of Calabi quasimorphisms which are presented next.

6.6. **The Calabi property and continuity.** Let $(M, \omega)$ be a symplectic manifold and let $B \subset M$ is a displaceable symplectic ball. A homogeneous quasimorphism $\Psi\colon \mathrm{Ham}(M, \omega) \to \mathbf{R}$ is called *Calabi* (or has the Calabi property) if its restriction to a subgroup $\mathrm{Ham}(B)$ coincides with the Calabi homomorphism. The definition is due to Entov and Polterovich [6] who constructed first examples of Calabi quasimorphisms using quantum homology. Their examples include the sphere $\mathbf{S}^2$ and they asked whether there were Calabi quasimorphisms for other surfaces. Pierre Py gave a positive answer to this question in [16, 15].

Here we provide a Calabi quasimorphism by producing a slightly modified example.

**Example 6.7** (The snake quasimorphism)**.** Let $w \in \mathbf{F}_2$ be an element. It defines a path on the plane starting at the origin, consisting of horizontal and vertical segments of integer length with turning points on the integer lattice. See Figure 6.1 for an example.

Let $\xi\colon \mathbf{F}_2 \to \mathbf{Z}$ be defined by $\xi(w) := L(w) - R(w)$, where $L(w)$ and $R(w)$ denote the number of the left and right turns of the path defined by $w$. Thus the value at an element drawn in Figure 6.1 is $\xi(w) = 5 - 4 = 1$.

If $w \in \mathbf{F}_2$ is a palindrome then the induced path is symmetric with respect to the half turn about its mid point and hence the initial turns become the opposite terminal turns and hence they cancel. Thus $\xi$ vanishes on palindromes and hence its homogenization vanishes both on palindromes and primitive elements. Since $\xi\left([a,b]^n\right) = 4n - 1$ we

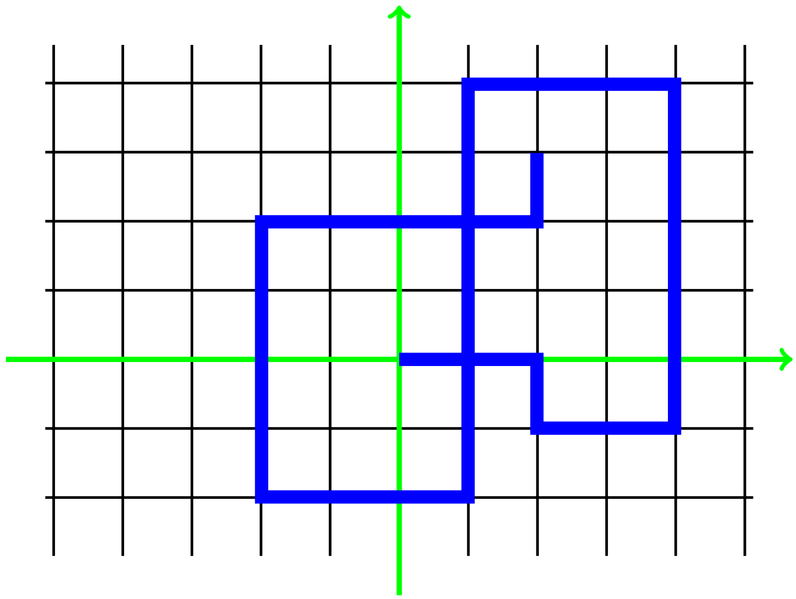

FIGURE 6.1. The element $w = a^2b^{-1}a^2b^5a^{-3}b^{-6}a^{-3}b^4a^4b$

see that $\xi$ is unbounded. Hence its homogenization $\widehat{\xi}$ is nontrivial and $\widehat{\xi}([a,b]) = 4$. Thus the quasimorphism $\mathcal{G}(\pi^*\xi)$ is nontrivial and has the Calabi property. $\diamondsuit$

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## References


[1] Valery Bardakov, Vladimir Shpilrain, and Vladimir Tolstykh. On the palindromic and primitive widths of a free group. *J. Algebra*, 285(2):574–585, 2005.

[2] Michael Brandenbursky. Bi-invariant metrics and quasi-morphisms on groups of Hamiltonian diffeomorphisms of surfaces. *Internat. J. Math.*, 26(9):1550066, 29 pages, 2015.

[3] Michael Brandenbursky and Jarek Kędra. On the autonomous metric on the group of area-preserving diffeomorphisms of the 2-disc. *Algebr. Geom. Topol.*, 13(2):795–816, 2013.

[4] Michael Brandenbursky and Egor Shelukhin. On the $L^p$-geometry of autonomous Hamiltonian diffeomorphisms of surfaces. To appear in Math. Res. Lett.

[5] Dmitri Burago, Sergei Ivanov, and Leonid Polterovich. Conjugation-invariant norms on groups of geometric origin. In *Groups of diffeomorphisms*, volume 52 of *Adv. Stud. Pure Math.*, pages 221–250. Math. Soc. Japan, Tokyo, 2008.

[6] Michael Entov and Leonid Polterovich. Calabi quasimorphism and quantum homology. *Int. Math. Res. Not.*, (30):1635–1676, 2003.

[7] Michael Entov, Leonid Polterovich, and Pierre Py. On continuity of quasimorphisms for symplectic maps. In *Perspectives in analysis, geometry, and topology*, volume 296 of *Progr. Math.*, pages 169–197. Birkhäuser/Springer, New York, 2012. With an appendix by Michael Khanevsky.

[8] Jean-Marc Gambaudo and Étienne Ghys. Commutators and diffeomorphisms of surfaces. *Ergodic Theory Dynam. Systems*, 24(5):1591–1617, 2004.

[9] Tomohiko Ishida. Quasi-morphisms on the group of area-preserving diffeomorphisms of the 2-disk via braid groups. *Proc. Amer. Math. Soc. Ser. B*, 1:43–51, 2014.

[10] Michael Khanevsky. Hamiltonian commutators with large Hofer norm. Sep 2014. Available at http://arxiv.org/abs/1409.7420v2.

[11] A. V. Malyutin. Pseudocharacters of braid groups and the simplicity of links. *Algebra i Analiz*, 21(2):113–135, 2009.

[12] J. Nielsen. Die Isomorphismen der allgemeinen, unendlichen Gruppe mit zwei Erzeugenden. *Math. Ann.*, 78(1):385–397, 1917.

[13] Leonid Polterovich. Floer homology, dynamics and groups. In *Morse theoretic methods in nonlinear analysis and in symplectic topology*, volume 217 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 417–438. Springer, Dordrecht, 2006.

[14] Leonid Polterovich and Egor Shelukhin. Autonomous Hamiltonian flows, Hofer's geometry and persistence modules. *Selecta Math. (N.S.)*, 22(1):227–296, 2016.

[15] Pierre Py. Quasi-morphismes de Calabi et graphe de Reeb sur le tore. *C. R. Math. Acad. Sci. Paris*, 343(5):323–328, 2006.

[16] Pierre Py. Quasi-morphismes et invariant de Calabi. *Ann. Sci. École Norm. Sup. (4)*, 39(1):177–195, 2006.

[17] G. P. Scott. Braid groups and the group of homeomorphisms of a surface. *Proc. Cambridge Philos. Soc.*, 68:605–617, 1970.

[18] Takashi Tsuboi. On the uniform simplicity of diffeomorphism groups. In *Differential geometry*, pages 43–55. World Sci. Publ., Hackensack, NJ, 2009.

[19] Takashi Tsuboi. On the uniform perfectness of the groups of diffeomorphisms of even-dimensional manifolds. *Comment. Math. Helv.*, 87(1):141–185, 2012.

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