Non-commutative Barge-Ghys quasimorphisms

Michael Brandenbursky,[1] Misha Verbitsky,[2]

Abstract. A (non-commutative) Ulam quasimorphism is a map q from a group Γ to a topological group G such that $q(xy)q(y)^{-1}q(x)^{-1}$ belongs to a fixed compact subset of G. Generalizing the construction of Barge and Ghys, we build a family of quasimorphisms on a fundamental group of a closed manifold M of negative sectional curvature, taking values in an arbitrary Lie group. This construction, which generalizes the Barge-Ghys quasimorphisms, associates a quasimorphism to any principal G-bundle with connection on M.

Kapovich and Fujiwara have shown that all quasimorphisms taking values in a discrete group can be constructed from group homomorphisms and quasimorphisms taking values in a commutative group. We construct Barge-Ghys type quasimorphisms taking prescribed values on a given subset in Γ, producing counterexamples to the Kapovich and Fujiwara theorem for quasimorphisms taking values in a Lie group. Our construction also generalizes a result proven by D. Kazhdan in his paper "On ε-representations". Kazhdan has proved that for any $\varepsilon > 0$, there exists an ε-representation of the fundamental group of a Riemann surface of genus 2 which cannot be 1/10-approximated by a representation. We generalize his result by constructing an ε-representation of the fundamental group of a closed manifold of negative sectional curvature taking values in an arbitrary Lie group.

Contents

[1] M. B. acknowledges support of Israel Science Foundation grant 823/23 and Humboldt foundation.

[2] M. V. acknowledges support of HSE University basic research program, FAPERJ E-26/202.912/2018 and CNPq - Process 310952/2021-2.

2020 Mathematics Subject Classification: 53C05, 20F67, 22E25

1 Introduction

1.1 The main results

We start by citing the main results of this paper. We introduce the relevant definitions and motivation for these notions later, and here we just give the statements. The following theorem complements the main result of [FK] by finding a counterexample to their theorem when the quasimorphism takes values in a Lie group.

Theorem 4.31: Let G be a simply connected, connected, non-abelian rational real nilpotent Lie group, and $\Gamma := \pi_1(M)$, where M is a closed manifold of strictly negative sectional curvature, $\dim_{\mathbb{R}} M > 1$. Then there exists a non-constructible HBG-quasimorphism $q_\nabla : \Gamma \longrightarrow G$.

The HBG-quasimorphisms are defined in Definition 3.23 (see also Remark 3.24), and constructibility in Definition 4.27, following [FK].

Another theorem generalizes one of the results of Kazhdan, [Kaz].

Theorem 5.2: Let M be a closed manifold of strictly negative sectional curvature, G a positive-dimensional connected Lie group, and P a trivial principal G-bundle. For any connection ∇ in P, let $q_\nabla : \pi_1(M) \longrightarrow G$ denote the corresponding HBG-quasimorphism (Definition 3.23). Choose a left-invariant metric on G such that the diameter of any closed subgroup is at least $1/3$. Then for each $\varepsilon > 0$, there exists a connection ∇ such that q_∇ is an ε-representation which cannot be $1/3$-approximated by a representation.

We explain the terms "ε-representation" and "δ-approximation" in Subsection 1.6, following [Kaz].

1.2 Ulam quasimorphisms

Throughout this paper, G means a Lie group of algebraic type; usually we assume that G is connected, though this assumption is not always necessary.

Definition 1.1: Let G be a Lie group, and Γ any group. **An Ulam quasimorphism** $q : \Gamma \longrightarrow G$ is a map which satisfies $q(x^{-1}) = q(x)^{-1}$ and $q(x)q(y) \in K \cdot q(xy)$, where $K \subset G$ is a fixed compact, independent from the choice of x, y. Two Ulam quasimorphisms $q_1, q_2 : \Gamma \longrightarrow G$ are called **equivalent** if there exists a compact subset $K \subset G$ such that $q_1(x) \in K \cdot q_2(x)$, for all $x \in \Gamma$.

In this paper, we give a versatile construction of Ulam quasimorphisms associated with connections in vector bundles over a closed manifold M of strictly negative sectional curvature. As a result, we obtain a quasimorphism from $\pi_1(M)$ to a Lie group, called *a Barge-Ghys quasimorphism.* This allows us to find quasimorphisms which are not constructible, in the sense of Fujiwara and Kapovich [FK]. The Fujiwara-Kapovich constructible quasimorphisms are obtained from quasimorphisms taking values in abelian groups. Unlike the constructible quasimorphisms, the Barge-Ghys quasimorphisms we construct cannot be obtained from ones taking values in abelian groups (Theorem 4.31).

Interestingly enough, Fujiwara-Kapovich have proved that *any* Ulam quasimorphism taking values in a discrete group is constructible. It turns out that the Ulam quasimorphisms taking values in Lie groups are of entirely different nature.

Ulam quasimorphisms from a free group to a Lie group G with bi-invariant metric were explored by P. Rolli in [R]. Rolli has constructed non-trivial quasimorphisms from a free group to G. It seems that Rolli's construction also can lead to non-constructible quasimorphisms.

As an application of our approach, we give a construction of ε-representations which cannot be approximated by a representation (Theorem 5.2), generalizing a result of Kazhdan ([Kaz]).

Note that our definition of Ulam quasimorphism may not be the optimal for some purposes. Fujiwara and Kapovich [FK] give several non-equivalent (more relaxed) definitions of a quasimorphism taking values in a non-abelian group G; for abelian G, all these definitions are equivalent. Another, even more relaxed, definition was considered in [HS]. We give a more detailed presentation of these notions in Subsection 4.4.

One of the results we obtain builds on the difference between these notions. Recall that the **geometric quasimorphism** ([FK]) is a map $q: \Gamma \longrightarrow G$ such that there exists a compact subset $K \subset G$ such that $q(xy) \in Kq(x)Kq(y)$ for all $x, y \in \Gamma$.

Theorem 1.2: Let Γ be a fundamental group of a closed manifold of strictly negative curvature, G a non-commutative, simply connected nilpotent Lie group, and Λ a cocompact lattice in G.[1] Then a non-constructible HBG-quasimorphism $q: \Gamma \longrightarrow G$ obtained in Theorem 4.31 can be approximated by a geometric quasimorphism $q_0: \Gamma \longrightarrow \Lambda$, which is also non-constructible.
Proof: Remark 4.33. ■

Notice that, by contrast, any *Ulam* quasimorphism $q_1: \Gamma \longrightarrow \Lambda$ is by [FK] constructible. This is where the difference between the Ulam quasimorphisms and the geometric quasimorphisms becomes apparent.

[1] By Maltsev's theorem ([CG]), existence of a cocompact lattice is equivalent to G being rational.

1.3 Quasimorphisms, bounded cohomology and the commutator length

In the literature, the quasimorphisms are usually considered as maps taking values in $\mathbb{R}$. In this context, a quasimorphism is defined as a map $q : G \longrightarrow \mathbb{R}$ which satisfies $|q(xy) - q(x) - q(y)| < C$, where C is a constant independent from x, y. We shall sometimes call such maps **commutative quasimorphisms**.

In geometric context, this notion originates in the paper of Gromov [G2]. Using the quasimorphisms, R. Brooks proved that the second bounded cohomology $H^2_b(\mathbb{F}_2, \mathbb{R})$ of the free group $\mathbb{F}_2$ is infinite-dimensional ([Br, G2]). Since then, quasimorphisms $q : G \longrightarrow \mathbb{R}$ became prevalent in topology ([C]), symplectic geometry ([EP, Sh]) and dynamics ([BM, GG]).

Barge and Ghys ([BG]) generalized the observation of Brooks to prove that the second bounded cohomology of $\pi_1(S)$ is infinite-dimensional for any Riemann surface S with $g(S) > 1$.

We say that two quasimorphisms $q, q_1 : G \longrightarrow \mathbb{R}$ are **equivalent** if $|q(x) - q_1(x)| < C$, where C is a constant independent from x. The group $\mathcal{Q}$ of equivalence classes of quasimorphisms fits into an exact sequence

$$H^1(G, \mathbb{R}) \longrightarrow \mathcal{Q} \longrightarrow H^2_b(G, \mathbb{R}) \longrightarrow H^2(G, \mathbb{R}), \tag{1.1}$$

where $H^2_b(G, \mathbb{R})$ denotes the bounded cohomology. A quasimorphism $q : G \longrightarrow \mathbb{R}$ is called **homogeneous** if it satisfies $q(x^n) = nq(x)$ for any $x \in G$ and $n \in \mathbb{Z}$. It is possible to see that every quasimorphism $q : G \longrightarrow \mathbb{R}$ is equivalent to a unique homogeneous quasimorphism ([C, Lemma 2.2.1]).

The commutator length of $g \in [G, G]$ is the minimal number m such that g can be represented as a product of m commutators. Recall that **the stable commutator length** of an element $g \in G$ is defined as

$$\mathsf{scl}(g) := \lim_{n\to\infty} \frac{\mathsf{cl}(g^n)}{n},$$

where cl is the commutator length. The homogeneous quasimorphisms are related with the stable commutator length, due to the celebrated theorem of Bavard ([B]). This result is known as *Bavard duality*. In its simplest version, Bavard duality can be stated as follows: for any $g \in [G, G]$, g has non-zero stable commutator length if and only if there exists a homogeneous quasimorphism $q : G \longrightarrow \mathbb{R}$ such that $q(g) \neq 0$.

1.4 Barge-Ghys construction and manifolds of strictly negative curvature

Further on, we are interested in vector bundles over closed manifolds of strictly negative sectional curvature. Sometimes such manifolds are called "hyperbolic", but we don't use this term to avoid confusion with manifolds of constant strictly negative curvature.

We study the Ulam quasimorphisms associated with the holonomy of a connection on vector bundles or on principal G-bundles, where G is a connected Lie group. In the sequel, we use either the language of G-bundles or the language of vector bundles with connection, whatever is more convenient. However, all statements that we make can be easily translated from one language to another, giving equivalent results in two parallel frameworks.

Remark 1.3: We tacitly assume that the base manifold of strictly negative curvature has dimension at least 2.

Barge-Ghys quasimorphisms were introduced in [BG], who used them to prove that the bounded cohomology of a Riemann surface is infinite-dimensional. In the literature these quasimorphisms are variously called de Rham quasimorphisms ([C]) or Barge-Ghys quasimorphisms [EP, PR]; we follow the second convention.

The generalization of Barge-Ghys quasimorphisms to closed manifolds of strictly negative curvature seems to be well known ([Mar]). However, the infinite-dimensionality of the space of Barge-Ghys quasimorphisms, originally proven in [BG] for Riemann surface, is less straightforward. This result was established in [BFMSS].

1.5 Barge-Ghys quasimorphisms associated with a connection

We feel compelled to make a few comments about the terminology used in this paper.

Originally, the Barge-Ghys quasimorphisms were associated with a non-closed 1-form θ on a closed manifold M of strictly negative sectional curvature. Given $\gamma \in \pi_1(M, x)$, we represent γ by a geodesic loop $\underline{\gamma}$ (which is unique, because M has negative curvature), and put $q(\gamma) = \int_{\underline{\gamma}} \theta$.

Taking θ as a connection form in a trivial line bundle, the number $\int_{\underline{\gamma}} \theta$ is interpreted as the holonomy of this connection along $\underline{\gamma}$ (Subsection 3.2).

In Subsection 3.2, we generalize this construction to arbitrary vector bundles and to principal G-bundles with connection. This gives a "non-commutative quasimorphism" (Definition 1.1), which we call "a non-commutative Barge-Ghys quasimorphism associated with a connection".

This quasimorphism does not satisfy $q(x^n) = q(x)^n$, that is, it is not homogeneous (Definition 3.21). In the usual theory of quasimorphisms (which we call throughout this paper "commutative quasimorphisms") for every quasimorphism there exists a unique homogeneous quasimorphism in the same equivalence class. This construction is called "homogenization"; it is obtained by the standard limit construction ([C, Lemma 2.2.1]).

Unfortunately, we were unable to generalize the homogenization construction to the non-commutative case. Instead we use an ad hoc construction, which works in the same generality, and gives a homogeneous quasimorphism (Subsection 3.5). The quasimorphisms obtained this way are called "the homogeneous Barge-Ghys quasimorphisms". They are also associated with a connection in a vector bundle or in a principal G-bundle.

There are disclaimers we need to make at this point. First of all, we could not devise a general definition of "Barge-Ghys quasimorphisms". These two kinds of Barge-Ghys quasimorphisms are the only cases we could came up with.

Second, the Barge-Ghys quasimorphisms associated with a connection are distinct from the homogeneous Barge-Ghys quasimorphisms. These are two distinct classes which rarely intersect, and we do not know if they can be united in a meaningful class of more general quasimorphisms. However, each Barge-Ghys quasimorphism associated with a G-bundle with connection is equivalent to the homogeneous Barge-Ghys quasimorphism associated with the same connection (Claim 3.29).

1.6 Ulam stability and Kazhdan's ε-representations

The earliest mention of quasimorphisms is found in Ulam's 1960 book "A collection of mathematical problems", Chapter 6 ([U]). Ulam defined an ε-automorphism of a topological group as a map $\rho : G \longrightarrow G$ such that $\rho(xy)\rho(x)^{-1}\rho(y)^{-1}$ belongs to an ε-neighborhood of the identity. Ulam asked whether any such map admits an $k\varepsilon$-approximation by an automorphism, for some $k > 0$ which is independent from ε.

More recently, this problem was generalized to representations (see Definition 1.4 below). However, the notion of Ulam stability seems to be present, implicitly, in earlier works of von Neumann ([vN]) and Turing ([Tu]).

A similar question was considered in 1982 by D. Kazhdan ([Kaz]). He defined **an ε-representation** of a group G as a map $\rho: G \longrightarrow U(V)$, where V is a Hilbert space, finite dimensional or infinite dimensional, satisfying

$$\|\rho(xy) - \rho(x)\rho(y)\| < \varepsilon,$$

where $\|\cdot\|$ is the operator norm. The distance between two maps $\rho_1, \rho_2 : G \longrightarrow U(V)$ can be defined as

$$d(\rho_1, \rho_2) := \sup_{x \in G} \|\rho_1(x) - \rho_2(x)\|.$$

Following a suggestion of V. Milman, Kazhdan asked whether for any $\delta > 0$ there exists $\varepsilon > 0$ such that any ε-representation can be δ-approximated by a representation. He proved that this holds true for amenable groups. When $G = \pi_1(S)$, where S is a genus 2 Riemann surface, Kazhdan has constructed an ε-representation $G \longrightarrow U(n)$, which cannot be 1/10-approximated by a representation, for any given $\varepsilon > 0$.

Definition 1.4: The group Γ is called **Ulam stable** ([BOT]) if for any $\delta > 0$ there exists $\varepsilon > 0$ such that any finite-dimensional ε-representation $q: \Gamma \longrightarrow U(V)$ can be δ-approximated by a representation $\rho: \Gamma \longrightarrow U(V)$. It is called **strongly Ulam stable** if the same is true even for infinite-dimensional Hilbert representations.

In [Kaz], D. Kazhdan has proven that all amenable groups are strongly Ulam stable. Using the Barge-Ghys quasimorphisms, we were able to prove this result for the fundamental group of any closed strictly negatively curved manifold. In Theorem 5.2, we generalize [Kaz, Theorem 2] showing that a fundamental group Γ of a complete Riemannian manifold with uniformly bounded strictly negative sectional curvature admits an ε-representation taking values in any given Lie group G, which cannot be 1/3-approximated by a representation[2]. Kazhdan proves this for $\Gamma = \pi_1(S)$, where S is a genus 2 Riemann surface, and $G = U(n)$.

In [BOT], Burger, Ozawa and Thom address the question of strong Ulam stability, obtaining definite results for a large class of groups. Let Γ be a group which contains a subgroup Λ admitting a homogeneous $\mathbb{R}$-valued quasimorphism which is not a homomorphism, that is, such that the map

$$H^2_b(\Lambda, \mathbb{R}) \longrightarrow H^2(\Lambda, \mathbb{R})$$

[2] Our norm conventions are different from Kazhdan's, but the constants 1/3 and 1/10 depend on the choice of normalizations, and are not important in themselves.

has non-trivial kernel. Then Γ has an infinite-dimensional Hilbert ε-representation, for any given $\varepsilon > 0$, which cannot be $\frac{\sqrt{3}}{16}$-approximated by a representation. Also, they observe that any free group admits a finite-dimensional ε-representation taking values in $U(n)$ which cannot be 2-approximated by a representation ([BOT, Proposition 3.3]).

In [GLMR], the Ulam stability was cast in a cohomological setting. Glebsky, Lubotzky, Monod and Rangarajan defined an asymptotic version of bounded cohomology, and proved that vanishing of asymptotic cohomology implies Ulam stability. In the same paper, Ulam stability of $U(1)$-representations is directly related to existence of quasimorphisms.

In this paper, we further generalize Kazhdan's theorem, which is known for ε-representations with values in $U(n)$ ([BOT, Proposition 3.3]) to ε-representations taking values in an arbitrary Lie group (Theorem 5.2).

1.7 Constructible quasimorphisms

The paper [FK] by Fujiwara and Kapovich is the fundamental treatment of Ulam quasimorphisms taking values in a non-commutative group G. Fujiwara and Kapovich considered the case when G is discrete, and their result is mostly negative. Fujiwara and Kapovich defined "constructible quasimorphisms", which are up to equivalence and finite index quasimorphisms which can be constructed in terms of homomorphisms, quasimorphisms with abelian target, and sections of bounded central extensions.

Then they proved that any Ulam quasimorphism taking values in a discrete group is always constructible ([FK, Theorem 1.2]).

In Theorem 4.31 we construct examples of quasimorphisms taking values in a Lie group G, which do not satisfy conclusions of [FK, Theorem 1.2]. There is no contradiction because the group G is not discrete. We prove that for any closed manifold M of strictly negative sectional curvature, there exists an Ulam quasimorphism $\pi_1(M) \longrightarrow G$ which is not constructible.

2 Principal bundles and vector bundles

Almost everything we say in this paper can be formulated for vector bundles or for principal G-bundles, where G is a Lie group. Usually we state only one of two versions, leaving the rest for the reader. In this section, we briefly explain the passage from vector bundles with connection to G-bundles and vice versa.

Definition 2.1: Let G be a Lie group. A **principal G-bundle** is a smooth manifold E equipped with a smooth, free G-action, such that the natural map $E \to E/G$ is a locally trivial fibration.

Example 2.2: Consider the standard action of $U(1)$ on 3-dimensional sphere $S^3 \subset \mathbb{C}^2$, with e^{it} taking (ξ_1, ξ_2) to $(e^{it}\xi_1, e^{it}\xi_2)$. Then $S^3/U(1) = S^2$, and this action defines a principal $U(1)$-bundle on a 2-dimensional sphere. This fibration is known as **Hopf fibration**.

Definition 2.3: Let $\pi: E \longrightarrow M$ be a principal G-bundle and V a space with G-action. Consider the quotient $(E \times V)/G$, where G acts diagonally. Since the action of G is free on fibers of π, the quotient $(E \times V)/G$ is a locally trivial fibration on M with fiber V. It is called **the associated fibration**.

Example 2.4: Let V be a representation of a group G, and $\pi: E \longrightarrow M$ be a principal G-bundle. Then the associated fibration $(E \times V)/G$ is a vector bundle, called **a vector bundle with a G-structure**. We say that **the structure group of a vector bundle B is reduced to G** if B is obtained from a principal G-bundle this way.

Example 2.5: Consider a complex vector bundle B over M equipped with a Hermitian structure, and let $\pi: E \longrightarrow M$ be the space of all orthonormal complex frames in B. Since the group $U(n)$ acts on orthonormal complex frames freely and transitively, E is a principal $U(n)$-bundle. Consider the standard $\mathbb{C}^n$-representation V of $U(n)$. By construction, the vector bundle $(E \times V)/U(n)$ coincides with B, hence this construction gives a reduction of the structure group of B to $U(n)$.

We have described the correspondence between the vector bundles (with appropriate reduction of the structure group) and the principal G-bundles. It turns out that this construction is well compatible with connections; one can define the holonomy and the curvature in both contexts, and these notions are equivalent. This is a part of standard differential geometry course, see e. g. [St, KN].

We will presently give a partial description of this correspondence.

Definition 2.6: Let $\pi: E \longrightarrow M$ be a smooth fibration, with $T_\pi E$ **the bundle of vertical tangent vectors** (vectors tangent to the fibers of π). An **Ehresmann connection** on π is a sub-bundle $T_{\mathsf{hor}} E \subset TE$ such that $TE = T_{\mathsf{hor}} E \oplus T_\pi E$. The **parallel transport** along the path γ :

$[0, a] \longrightarrow Z$ associated with the Ehresmann connection is a diffeomorphism $V_t : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(t))$ smoothly depending on $t \in [0, a]$ and satisfying $\frac{dV_t}{dt} \in T_{\mathsf{hor}} E$.

Definition 2.7: Let B be a vector bundle on M and $\operatorname{Tot} B \xrightarrow{\pi} M$ its total space. An Ehresmann connection on π is called **linear** if it is preserved by the homothety map $\operatorname{Tot} B \longrightarrow \operatorname{Tot} B$ mapping v to λv and by the addition map $\operatorname{Tot}(B \oplus B) \longrightarrow \operatorname{Tot} B$, that is, addition preserves horizontal vectors.

Proposition 2.8: A notion of a linear Ehresmann connection on a vector bundle B coincide with the usual notion of a connection; the corresponding parallel transport maps coincide as well.
Proof: [OV, Proposition 2.21]. ■

Definition 2.9: Let now $\pi : E \longrightarrow M$ be a principal G-bundle. A G-connection on π is a G-invariant Ehresmann connection.

Remark 2.10: Let $\pi : E \longrightarrow M$ be a principal G-bundle with G-connection ∇, and X a smooth manifold with G-action ρ. Then the associated bundle $E_X := (E \times X)/G$ inherits an Ehresmann connection: $TE_X = T_{\mathsf{hor}} E \oplus (TX/\rho(\mathfrak{g}))$, where $\mathfrak{g} = \operatorname{Lie}(G)$ is the Lie algebra of G.

Definition 2.11: Let B be a vector bundle on M with structure group G, V representation of G, and E the corresponding principal G-bundle, $B = (E \times V)/G$. Then for any G-connection on E induces an Ehresmann connection on B as in Remark 2.10. Clearly, this connection is linear, hence induces a connection on B as on a vector bundle (Proposition 2.8). This connection is called **induced by the G-connection ∇^E.**

The holonomy of a G-connection and its induced connection are compatible, in the following sense:

Claim 2.12: Let E the a principal G-bundle on M, V a representation of G, and $B = (E \times V)/G$ the corresponding vector bundle with the structure group G. Let ∇^E a G-connection on E, and ∇ the induced connection on B. Denote the holonomy of ∇^E along a loop γ based on $m \in M$ by $g_\gamma \in G$. Then the holonomy of ∇ along γ is obtained from the action of g_γ on $B\Big|_m = \left(E\Big|_m \times V\right)/G$.
Proof: [KN]. ■

Throughout this paper, we use one of these equivalent languages, and assume tacitly an analogous statement for the other one. It is slightly more convenient to speak of non-commutative Barge-Ghys quasimorphisms in the language of vector bundles: this way, the correspondence with the usual, commutative Barge-Ghys quasimorphism is more apparent. However, it is more natural to state and prove the generalization of Kazhdan's theorem in the language of principal G-bundles. The translation from one of these languages to another is straightforward and is left to the reader as an exercise.

3 Barge-Ghys quasimorphisms on fundamental groups of closed manifolds of strictly negative curvature

3.1 Manifolds with strictly negative sectional curvature

In this preliminary section we list a few arguments of Riemannian geometry, most of them either classical or due to M. Gromov, [G2]. For manifolds of constant negative curvature, all the results we are going to obtain are classical and well known ([Fri, Chapter 8]); however, for arbitrary Riemannian manifolds of strictly negative curvature, a more subtle approach is required.

Theorem 3.1: (Cartan-Hadamard)
Let M be a complete, simply connected Riemannian manifold of non-positive sectional curvature. Then M is contractible.

Proof: We give a sketch of the proof, following [BBI]. In [BBI] this result was stated and proven for CAT(0)-spaces, but the comparison inequalities which are required by the CAT(0)-geometry easily follow from the non-positivity of the sectional curvature.

By Hopf-Rinow theorem, every two points of M can be connected by a geodesic. Let $\gamma_1 : [a,b] \longrightarrow M$ and $\gamma_2 : [c,d] \longrightarrow M$ be segments of geodesics in M, parametrized by the arc length. As shown in [BBI], the distance function $D : [a,b] \times [c,d] \longrightarrow \mathbb{R}^{>0}$ taking x, y to $d(\gamma_1(x), \gamma_2(y))$ is strictly convex, unless the geodesics γ_1, γ_2 are segments of the same geodesic line. This implies, in particular, that any two points are connected by a unique geodesic: indeed, if γ_1 and γ_2 have the same ends, $\gamma_1(a) = \gamma_2(c)$ and $\gamma_1(b) = \gamma_2(d)$ the function D would be equal to 0 in (a, c) and (b, d), hence it is zero on the diagonal, and the images of γ_1 and γ_2 coincide.

Fix a reference point $p \in M$ and consider the function

$$H: M \times [0,1] \longrightarrow M$$

taking $x \in M$ and $t \in [0,1]$ to $\gamma(t \cdot d(p,x))$, where $\gamma: [0, d(p,x)]$ is the geodesic connecting p to x. A similar argument implies that H is continuous; clearly, H is a deformation retraction of M to p, hence M is contractible. ■

We will not use the Cartan-Hadamard theorem, but we use its corollary, which is inherent in its proof.

Corollary 3.2: Let M be a simply connected, complete manifold of non-positive sectional curvature. Fix a reference point $p \in M$ and consider the function $H_p: M \times [0,1] \longrightarrow M$ taking $x \in M$ and $t \in [0,1]$ to $\gamma(t \cdot d(p,x))$, where $\gamma: [0, d(p,x)] \longrightarrow M$ is the geodesic connecting p to x. Then H_p is continuous. ■

We could use the map H_p to define **a straight singular simplex,** following [G2]. Since we need only 2-dimensional simplices, we restrict ourselves to the 2-dimensional case.

Definition 3.3: Let M be a simply connected, complete manifold of non-positive sectional curvature. Let γ be the geodesic segment connecting b to c. We obtain a triangle by connecting a to all points of γ by a unique geodesic. The **geodesic simplex** $\Delta(a,b,c)$, associated with the points $a,b,c \in M$ is the union $\cup_{t\in[0,1]}[H_a(t)(b), H_a(t)(c)]$, where $H_a(t): M \longrightarrow M$ is the homotopy along geodesics passing through a, defined in Corollary 3.2, and $[H_a(t)(b), H_a(t)(c)]$ the geodesic segment connecting $H_a(t)(b)$ and $H_a(t)(c)$.

Remark 3.4: The boundary of the simplex $\Delta(a,b,c)$ is the union of geodesics connecting a to b, b to c and c to a. Indeed, the homothety $H_a(t)$ moves any point $x \in M$ along the geodesic connecting x to a as t goes from 1 to 0, hence the segments of the boundary connecting b to a and c to a are geodesics; the third segment is γ, which is also chosen geodesic.

Remark 3.5: Note that the order of the points $a,b,c \in M$ is important. Indeed, unless M has constant sectional curvature, the simplex $\Delta(a,b,c)$ and (say) $\Delta(b,a,c)$ are different: otherwise, if $\Delta(a,b,c) = \Delta(b,a,c) = \Delta(a,c,b)$, this simplex is a segment of a completely geodesic 2-dimensional submanifod, and a general Riemannian manifold does not have completely geodesic submanifolds ([MW]).

The main technical result about manifolds of strictly negative sectional curvature which we use is the following theorem of Gromov, which is proven in [G2] for any straight singular simplex.

Theorem 3.6: Let M be a simply connected, complete manifold of strictly negative sectional curvature $K(M) < -\varepsilon < 0$, and $\Delta(a,b,c)$ a geodesic simplex defined above. Then the Riemannian area $\mathit{Area}(\Delta(a,b,c))$ satisfies $\mathit{Area}(\Delta(a,b,c)) \leqslant \pi\varepsilon^{-2}$.
Proof: [G2, §I.3]. ■

In the sequel, we will need two statements about uniqueness of geodesics on manifolds of strictly negative sectional curvature. The first is a direct consequence of the Cartan-Hadamard's theorem (Theorem 3.1).

Claim 3.7: Let $\gamma \in \pi_1(M,p)$ be an element of a fundamental group of a closed manifold of strictly negative sectional curvature. Then γ is represented by a unique geodesic loop based at p. ■

Another statement (slightly less trivial) deals with *free geodesic loops*. Recall that a **free geodesic loop** is an immersed submanifold of dimension 1 which is locally geodesic.

Proposition 3.8: Let M be a closed manifold of strictly negative sectional curvature, and $\varphi: \ S^1 \longrightarrow M$ be a smooth map. Then there exists a unique free geodesic loop φ_1 which is free homotopic to φ. Moreover, φ_1 strictly minimizes the length of the loop.
Proof: [Kl, Theorem 3.8.14]. ■

3.2 Non-commutative Barge-Ghys quasimorphisms

Throughout this section, G is a connected Lie group.

Definition 3.9: Let M be a closed manifold with non-positive sectional curvature, and (P,∇) a principal G-bundle with connection. Fix $x \in M$. The **non-commutative Barge-Ghys map** takes $\gamma \in \pi_1(M,x)$ to the holonomy of ∇ along the geodesic path starting and ending at x and homotopic to γ.[1]

[1] Since M has non-positive curvature, this geodesic path is unique in its homotopy class, Proposition 3.8.

Remark 3.10: Throughout this paper, we consider the maps $q: \Gamma \longrightarrow G$, where $\Gamma = \pi_1(M)$ is the fundamental group of a closed manifold of strictly negative curvature. Such groups are Gromov hyperbolic, however, it is not clear yet whether our constructions can be generalized to all Gromov hyperbolic groups. Nevertheless, all results we prove remain valid whenever each geodesic simplex in the universal cover of M can be filled by a disk of bounded area, as in Theorem 3.6. Note that the homological version of the bounded filling result is also true for hyperbolic groups which are not of geometric origin ([M]).

Theorem 3.11: Let M be a closed manifold of strictly negative sectional curvature, and Θ a geodesic n-polygon in M, that is, a contractible loop of n geodesic segments. Consider a principal G-bundle (P, ∇) with connection on M, and let $h(\Theta) \in G$ be the holonomy along the boundary of Θ, considered as a loop starting and ending at $p \in \Theta$. Then $h(\Theta)$ belongs to a compact $K_n \subset G$ which is the same for all n-polygons Θ, but depends on n and (P, ∇) and the bound on the curvature of M.

Proof: By the effective version of the Ambrose-Singer theorem, the holonomy along a path is linearly expressed in terms of the integral of the curvature over a disk filling this path ([RW, Theorem 1], [MO1], [Y]). Therefore, for a left-invariant metric on G, the holonomy is bounded in terms of the integral of the curvature. The area of any geodesic simplex is bounded by Theorem 3.6. The absolute value of the curvature of ∇ is bounded from above because M is compact, and the curvature form on the pullback bundle $(\widetilde{P}, \widetilde{\nabla})$ is obtained as a pullback of the curvature of (P, ∇).

This implies that $h(\Theta)$ belongs to a fixed compact K when $n = 3$ and Θ is a simplex. When $n > 3$, we represent Θ as a boundary of the union of $n-2$ geodesic simplexes $D_1, .., D_{n-2}$ with common vertex p. Then the holonomy $h(\Theta)$ is obtained as a product $h(\Theta) = h(D_1)h(D_2)...h(D_{n-2}) \subset K^{n-2}$. Therefore, $h(\Theta)$ belongs to a fixed compact $K_n := K^{n-2}$, independent from the choice of Θ. ■

Theorem 3.12: Let M be a closed manifold of strictly negative sectional curvature, and (P, ∇) a principal G-bundle with connection. Fix $x \in M$, and let $q: \pi_1(M) \longrightarrow G$ be the non-commutative Barge-Ghys map associated with (P, ∇). Then q is an Ulam quasimorphism (Definition 1.1).

Proof: Denote by $\widetilde{M} \stackrel{\pi}{\longrightarrow} M$ the universal cover of M. Let $a, b \in \pi_1(M)$, and P_a, $P_b \in \operatorname{Diff}(\widetilde{M})$ the corresponding deck transformations. Fix a preimage $\widetilde{x} \in \pi^{-1}(x)$, and denote by $(\widetilde{P}, \widetilde{\nabla})$ the pullback of (P, ∇) to $\widetilde{M}$. By definition, the product $q(ab)q(b)^{-1}q(a)^{-1}$ is represented by the holonomy of $(\widetilde{P}, \widetilde{\nabla})$ along the geodesic simplex connecting the points $\widetilde{x}, P_a(\widetilde{x})$, and $P_b(P_a(\widetilde{x}))$ in $\widetilde{M}$. By Theorem 3.11, this quantity belongs to a compact subset independent from the choice of $x \in M$ and $a, b \in \pi_1(M)$. ■

Remark 3.13: The set of equivalence classes of non-commutative Barge-Ghys quasimorphisms is very big. As follows from Proposition 3.17 below, the usual ("commutative") Barge-Ghys quasimorphism is a special case of the quasimorphism taking values in a Lie group as defined in Definition 3.9. By [BG], [C], the vector space spanned by commutative Barge-Ghys quasimorphisms up to equivalence is infinite-dimensional; in Theorem 4.31 we construct non-commutative Barge-Ghys quasimorphisms which cannot be obtained from the commutative ones.

Remark 3.14: Using the Riemann-Hilbert correspondence ([OV, §2.8]) one can associate a flat principal G-bundle over M with each group homomorphism $\rho : \pi_1(M) \longrightarrow G$. By construction, the holonomy of this flat connection is equal to ρ ([OV, §2.50]). Therefore, the corresponding Barge-Ghys quasimorphism is equal to ρ. In other words, all group homomorphisms to a Lie group can be realized as Barge-Ghys quasimorphisms.

3.3 Commutative and non-commutative Barge-Ghys quasimorphisms

Definition 3.15: Let M be a closed manifold of strictly negative sectional curvature, and $\theta \in \Lambda^1 M$ a 1-form on M. The **(commutative) Barge-Ghys quasimorphism** ([BG]) associated with θ takes $\gamma \in \pi_1(M)$ to the integral of θ over the geodesic path starting and ending at x and homotopic to γ.

Remark 3.16: Note that in all literature on quasimorphisms, one uses the additive notation: $|q(xy) - q(x) - q(y)| < C$. For the non-commutative quasimorphisms, we are forced to use the multiplicative notation,

$$q(xy)q(y)^{-1}q(x)^{-1} \in K.$$

When we speak of commutative Barge-Ghys quasimorphisms in this wider context, this might create a confusion.

When B is an oriented real rank-1 vector bundle on M, the commutative Barge-Ghys quasimorphism is actually equal to the "non-commutative Barge-Ghys quasimorphism" defined in Definition 3.9; we explain this equivalence below.

Topologically, B is always trivial. To trivialize B, we choose a metric on B using the partition of unity, and trivialize B by taking a positive length-1 section u. Fix the connection ∇_0 on B in such a way that $\nabla_0 u = 0$. Then any connection ∇ on B can be written as $\nabla = \nabla_0 + \theta$, where $\theta \in \Lambda^1(M)$ is a 1-form. The holonomy of this connection along a loop γ is given by $\operatorname{Hol}_\gamma(\nabla) = e^{-\int_\gamma \theta}$. Indeed, for any section $u_f := f(t)u$ of B restricted to a geodesic segment γ parametrized by $t \in [a, b]$, the equation

$$\nabla(u_f) = \frac{df}{dt}u + \theta f u = 0$$

is equivalent to $f' = -f\theta$, equivalently, $\frac{d \log f}{dt} = -\theta$.

Proposition 3.17: Let (B, ∇_0) be a trivial real rank 1 vector bundle on a closed manifold M of strictly negative sectional curvature and θ a 1-form on M. Consider the connection $\nabla := \nabla_0 - \theta$. Then the commutative Barge-Ghys quasimorphism associated with θ can be obtained as the logarithm of the "non-commutative" Barge-Ghys quasimorphism associated with (B, ∇).
Proof: Indeed, for any loop in M, one has $\operatorname{Hol}_\gamma(\nabla) = e^{-\int_\gamma \theta}$. ■

Remark 3.18: For any trivialized real rank 1 vector bundle B on M, the connections on B are in bijective correspondence with 1-forms on M. Therefore, Proposition 3.17 defines a bijective correspondence between the set of (non-commutative) Barge-Ghys quasimorphisms associated with B and the set of commutative Barge-Ghys quasimorphisms.

3.4 Translation length in Gromov hyperbolic groups

We proceed with a few observations about Gromov hyperbolic groups, used in the sequel.

Let Γ be a group generated by a finite set $\mathcal{A}$, and $C_{\mathcal{A}}(\Gamma)$ its Cayley graph. **Algebraic translation length** $\tau_{\mathcal{A}}(\gamma)$ of an element $\gamma \in \Gamma$ is defined ([BH]) as

$$\tau_{\mathcal{A}}(\gamma) := \lim_{n\to\infty} \frac{1}{n} d(1_\Gamma, \gamma^n).$$

The limit exists because the function $n \mapsto d(1_\Gamma, \gamma^n)$ is subadditive. It is not hard to see that the map $\gamma \longrightarrow \tau_{\mathcal{A}}(\gamma)$ is conjugation invariant ([BH, Remark Γ.3.14 (1)]). Further on, we shall use the following result, also found in [BH].

Proposition 3.19: Let Γ be a finitely generated Gromov hyperbolic group, and S_R the set of all conjugacy classes of all $\gamma \in \Gamma$ satisfying $\tau_{\mathcal{A}}(\gamma) < R$, where $R > 0$ is a real number. Then S_R is finite.
Proof: [BH, Proposition Γ.3.15]. ■

An element $\gamma \in \Gamma$ is called **primitive** if it cannot be represented as a power $\gamma = \varphi^n$, for any $n > 1$.

Corollary 3.20: Let Γ be a finitely generated Gromov hyperbolic group. Then any non-torsion element of Γ is an integer power of a primitive element.

Proof: By [C, Corollary 3.3.5], $\tau_{\mathcal{A}}(\gamma) > 0$ for all non-torsion γ. Clearly, $\tau_{\mathcal{A}}(\gamma^k) = k\tau_{\mathcal{A}}(\gamma)$. By Proposition 3.19, there exists a number $C \in \mathbb{R}^{>0}$ such that for all non-torsion $u \in \Gamma$ we have $\tau_{\mathcal{A}}(u) > C$. Then γ cannot be represented as n-th power for any $n > C^{-1}\tau_{\mathcal{A}}(\gamma)$. Let m be a maximal number such that γ is an m-th power of an element $\gamma_1 \in \Gamma$. Then γ_1 is primitive.[2] ■

3.5 HBG-quasimorphism associated with a connection

Definition 3.21: A quasimorphism $q : \Gamma \longrightarrow G$ is called **homogeneous** if its restriction to any cyclic subgroup of Γ is a group homomorphism.

Let M be a closed Riemannian manifold of strictly negative sectional curvature, $\Gamma := \pi_1(M)$. By [P, Corollary 6.2.4], Γ is torsion-free. By Corollary 3.20, every non-torsion element of $\pi_1(M)$ is a power of a primitive element, which is unique by [BB, Lemma 2.2].

Given a primitive $\gamma \in \pi_1(M)$, let F_γ be the shortest free geodesic loop representing γ. By Proposition 3.8, F_γ is unique in its free homotopy class. Clearly, the conjugate elements of $\pi_1(M)$ correspond to the same free homotopy class. For each conjugacy class of γ we fix a choice of a point $x \in F_\gamma$. When $\gamma = \gamma_1^d$ and γ_1 is primitive, we denote by F_γ the loop F_{γ_1} iterated d times.

[2] We are grateful to Yves Cornulier and Sam Nead (https://mathoverflow.net/users/1650/sam-nead) who suggested a proof of this statement on Mathoverflow, [MO2].

Let (B, ∇) be a bundle with connection. Fix a point $p \in M$, and $\gamma \in \pi_1(M)$. We are going to define a homogeneous quasimorphism $q : \Gamma \longrightarrow GL(B_p)$, where B_p denotes the fiber of B in p, as follows.

Consider a point $x \in F_\gamma$, and let $\widetilde{F}_\gamma := \nu_{x,\gamma} \circ F_\gamma \circ \nu_{x,\gamma}^{-1}$ be the 3-segment piecewise geodesic path obtained by connecting p to x, going around the loop F_γ starting and ending in x, and going back to p along $\nu_{x,\gamma}$ in the opposite direction. Clearly, this path represents γ in $\pi_1(M, p)$. Denote by $q(\gamma) \in GL(B_p)$ the holonomy along $\widetilde{F}_\gamma$. By construction, q restricted to a cyclic subgroup is always a homomorphism.

Note that q depends on the choice of $x \in F_\gamma$, which has to be fixed for each conjugacy class of $\gamma \in \Gamma$.

Theorem 3.22: Let M be a closed manifold with strictly negative sectional curvature, $p \in M$ a base point, $\Gamma := \pi_1(M)$, and $q : \Gamma \longrightarrow GL(B_p)$ the map defined above. Then $q : \Gamma \longrightarrow GL(B_p)$ is an Ulam quasimorphism. Moreover, q is homogeneous.

Proof: Let $\alpha, \beta, \gamma = (\alpha\beta)^{-1}$ be elements of Γ. Choose any points $a \in F_\alpha$, $b \in F_\beta$, $c \in F_\gamma$. Then $q(\alpha)q(\beta)q(\alpha\beta)^{-1}$ is a holonomy along a contractible geodesic polygon with 9 edges obtained by going along

$$\nu_{a,\alpha}, F_\alpha, \nu_{a,\alpha}^{-1}, \nu_{b,\beta}, F_\beta, \nu_{b,\beta}^{-1}, \nu_{c,\gamma}, F_\gamma, \nu_{c,\gamma}^{-1},$$

see Figure 1. However, the holonomy along any contractible geodesic polygon is bounded by Theorem 3.11. Homogeneity of q is clear, because $q(\gamma^n)$ is the holonomy of ∇ along the loop $\nu_{x,\gamma} \circ F_\gamma^n \circ \nu_{x,\gamma}^{-1}$. ■

Definition 3.23: Let (B, ∇) be a vector bundle with connection, and q the Ulam quasimorphism defined in Theorem 3.22. Then $q : \Gamma \longrightarrow GL(B_p)$ is called **the HBG-quasimorphism associated with (B, ∇).**

Remark 3.24: Instead of vector bundles, Theorem 3.22 can be stated in the setting of principal bundles. In Section 2 we explained how it is done. This allows one to modify Definition 3.23 obtaining a HBG-quasimorphism associated with a principal G-bundle with connection, for arbitrary Lie group G.

Remark 3.25: A homogeneous quasimorphism $q : \Gamma \longrightarrow \mathbb{R}$ is unique in its equivalence class ([C]). Clearly, this is false when G is compact or contains a compact subgroup $K \subset G$: all homomorphisms from $\mathbb{Z}$ to K are equivalent

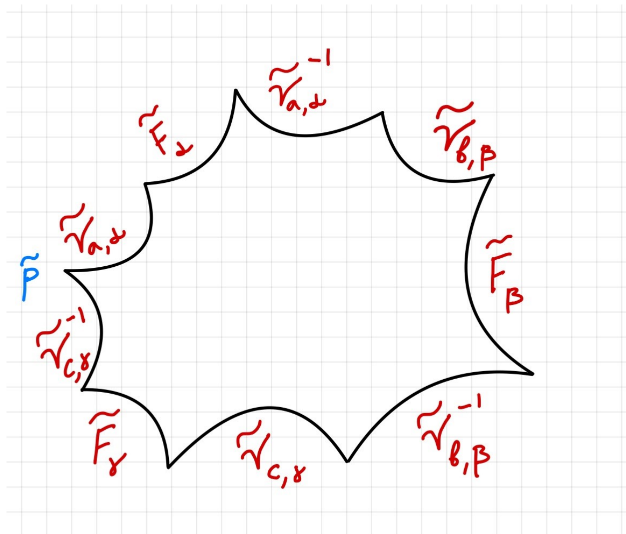

Figure 1: Lift of the geodesic polygon to the universal cover.

and homogeneous. However, the uniqueness is true when G is a simply connected nilpotent Lie group (Claim 3.31). We expect this to be true for any simply connected Lie group which has no non-trivial compact subgroups.

Remark 3.26: For any homogeneous quasimorphism $q: \Gamma \longrightarrow G$ and $r \in G$, the map rqr^{-1} is also a homogeneous quasimorphism. The HBG-quasimorphism $q: \pi_1(M,p) \longrightarrow G$ associated with a G-bundle depends on the choice of the points $x_i \in F_{\gamma_i}$ on the shortest loop representing the primitive element γ_i. However, this dependence is easy to describe: for any pair $x_i, y_i \in F_{\gamma_i}$, consider the geodesic simplex with vertices x_i, y_i, p, with the geodesic from x_i to y_i going in the same direction as in the loop F_{γ_i}. Let r_i denote the loop along this simplex, starting in p, going to x, then to y and to p again. Then $q(\gamma_i)$ is the holonomy along $\nu_{x_i,\gamma_i} F_{\gamma_i} \nu_{x_i,\gamma_i}^{-1}$, or, equivalently, the holonomy along $r_i \nu_{y_i,\gamma_i} F_{\gamma_i} \nu_{y_i,\gamma_i}^{-1} r_i^{-1}$. In other words, replacing the choice of x_i by y_i is equivalent to the conjugation of $q(\gamma_i)$ with the holonomy along a geodesic simplex r_i, which is bounded. These two quasimorphisms are Ulam equivalent because they are equivalent to the original Barge-Ghys quasimorphism (Claim 3.29). When G is a simply connected nilpotent Lie group, these HBG quasimorphisms are actually equal (Claim 3.31).

In the commutative case, any homogeneous quasimorphism is conjugation invariant ([C, §2.2.3]). The following result is a non-commutative Barge-Ghys analogue of this statement.

Claim 3.27: Let $q: \Gamma \longrightarrow G$ be a HBG-quasimorphism, and $\alpha, \beta \in \Gamma$ conjugate elements. Then $q(\alpha)$ is conjugate to $q(\beta)$.[3]

Proof: Let F_γ be the free geodesic loop homotopy equivalent to α and β, and $x \in F_\gamma$ be the chosen point. Then $q(\alpha)$, which is equal to holonomy of ∇ along the loop $\nu_{x,\alpha} \circ F_\gamma \circ \nu_{x,\alpha}^{-1}$, is conjugate to $q(\beta)$, obtained the same way from the loop $\nu_{x,\beta} \circ F_\gamma \circ \nu_{x,\beta}^{-1}$. It is easy to write this conjugation explicitly: $q(\alpha) = Rq(\beta)R^{-1}$, where R is holonomy of ∇ along the loop $\nu_{x,\alpha} \circ \nu_{x,\beta}^{-1}$. ■

When $G \sim \mathbb{R}^n$, $q(\alpha) = q(\beta)$ for α, β conjugate is true by [C, §2.2.3]. However, when $G = S^1$ (or other compact group), it is easy to construct a homogeneous Ulam quasimorphism $q: \Gamma \longrightarrow G$ which does not have this property. For example, let $\Gamma = \mathbb{F}_2$ be a free group, and define an arbitrary map $q: \mathfrak{P} \longrightarrow S^1$, where $\mathfrak{P}$ is the set of all primitive words in $\mathbb{F}_2$. If $q(W) = q(W^{-1})^{-1}$, this map extends to a homogeneous Ulam quasimorphism, and all homogeneous Ulam quasimorphisms $q: \mathbb{F}_2 \longrightarrow S^1$, are obtained this way. Clearly, this map is not necessarily invariant under the conjugation. This implies, in particular, that Claim 3.27 is false for a general Ulam quasimorphism, if G has a compact subgroup.

Question 3.28: Can Claim 3.27 be generalized to an arbitrary homogeneous Ulam quasimorphism $q: \Gamma \longrightarrow G$, when G is a Lie group which has no compact subgroups?

Any commutative quasimorphism is equivalent to its homogenization, which is unique in its equivalence class ([C, §2.2.3]). For Ulam quasimorphisms taking values in a non-commutative group, the homogenization procedure is not yet known. However, the Barge-Ghys quasimorphisms admit a "homogenization", which is unique in its equivalence class by Claim 3.31 below.

Claim 3.29: Let (B, ∇) be a bundle (vector or a principal G-bundle) with connection over a closed manifold of strictly negative sectional curvature, $\Gamma = \pi_1(M, p)$, and $q_\nabla: \Gamma \longrightarrow G$ the corresponding Barge-Ghys quasimorphism. After making the relevant choices, we obtain the HBG-

[3]Clearly, the conjugating element depends on α and β.

quasimorphism $q: \Gamma \longrightarrow G$. Then q is equivalent to q_∇, and, moreover, $q(\gamma)q_\nabla(\gamma)^{-1} \in K_4$, where $K_4 \subset G$ is the compact subset defined in Theorem 3.11, which depends only on M, B and ∇.

Proof: Let $\gamma \in \Gamma$ be any element, which is represented as $\gamma = \gamma_0^n$, where γ_0 is primitive, and let F_{γ_0} be the free geodesic representing the same free homotopy class. Fix a geodesic segment $\nu_{\gamma_0,x}$ connecting $p \in M$ to a point x in F_{γ_0}, and let $\widetilde{\gamma}$ the geodesic segment connecting p to itself and homotopic to γ. Then $q_\nabla(\gamma)$ is a holonomy of ∇ along $\widetilde{\gamma}$, and $q(\gamma)$ is the holonomy of ∇ along the geodesic chain $\nu_{\gamma_0,x} \circ F_{\gamma_0}^n \circ \nu_{\gamma_0,x}^{-1}$. Geometrically, the segment $F_{\gamma_0}^n$ is one geodesic interval, and therefore the polygon $\nu_{\gamma_0,x} \circ F_{\gamma_0}^n \circ \nu_{\gamma_0,x}^{-1} \circ \widetilde{\gamma}^{-1}$ has 4 geodesic sides. By Theorem 3.11, the holonomy along the sides this polygon belongs to K_4. ∎

Remark 3.30: For homogeneous quasimorphisms taking value in $\mathbb{R}$ or $\mathbb{R}^n$ (elsewhere, we call them "commutative"), equivalence implies equality: if $q_1, q_2 : \Gamma \longrightarrow \mathbb{R}^n$ are equivalent homogeneous quasimorphisms, they satisfy $q_1 = q_2$, because

$$q_1(x) - q_2(x) = \lim_{n\to\infty} \frac{q_1(x^n) - q_2(x^n)}{n} = 0.$$

The same is true for all homogeneous Ulam quasimorphisms $q: \Gamma \longrightarrow G$, if G is nilpotent and simply connected:

Claim 3.31: Let $q_1, q_2 : \Gamma \longrightarrow G$ be homogeneous quasimorphisms which are equivalent. Assume that G is a simply connected nilpotent Lie group. Then $q_1 = q_2$.[4]

Proof. Step 1: Let $K \subset G$ be a compact subset such that $q_1(x)q_2(x)^{-1} \in K$ for all $x \in \Gamma$. Since $q_i(x^n) = q_i(x)^n$, this gives $q_1(x)^n q_2(x)^{-n} \in K$. Then Claim 3.31 would follow if we prove that for any distinct $a, b \in G$, the sequence $a^n b^{-n}$ is unbounded, that is, does not belong to a compact set.

Step 2: Let G_i be the lower central series for G. Applying induction and using the long exact sequence

$$\longrightarrow \pi_2(G_i/G_{i-1}) \longrightarrow \pi_1(G_{i-1}) \longrightarrow \pi_1(G_i) \longrightarrow \pi_1(G_i/G_{i-1}) \longrightarrow 0$$

[4]We expect that this statement is true whenever G is a Lie group such that the exponential map $\exp: \mathfrak{g} \longrightarrow G$ is a homeomorphism.

for the Serre's fibration $G_{i-1} \longrightarrow G_i \longrightarrow G_i/G_{i-1}$, we immediately obtain that the groups G_i and G_i/G_{i-1} are simply connected for all i. Since G_i/G_{i-1} is abelian, this group is isomorphic to $\mathbb{R}^n$.

Step 3: We use the induction on $\dim G$. When G is 1-dimensional, this group is commutative, giving $G = \mathbb{R}$; if the sequence $a^n b^{-n} \in \mathbb{R}$ is bounded, we have $a = b$.

Let $Z \subset G$ be the center; it is non-trivial, because G is nilpotent. Since $a^n b^{-n}$ is bounded modulo Z, we may apply the induction hypothesis to the representatives of a, b in G/Z and obtain that $a = b$ modulo Z. Then $az = b$, where $z \in Z$, hence the sequence $a^n b^{-n} = z^n$ is bounded. Since $Z = \mathbb{R}^n$ (Step 2), this may happen only if z is trivial. Then $a = b$. ■

Remark 3.32: Let E be a principal G-bundle with connection with $G = \mathbb{R}^{>0}$, and B the associated line bundle with induced connection. Then the usual (commutative) Barge-Ghys quasimorphism q, taking values in the additive group $\mathbb{R}$, is obtained from our Barge-Ghys quasimorphism q_∇ by taking the logarithm. Let q_0 be the the corresponding HBG-quasimorphism. Then the logarithm $\log q_0$ is homogeneous and equivalent to q by Claim 3.29. Since a homogeneous commutative quasimorphism is unique in its equivalence class ([C, Lemma 2.2.1]), this implies that $\log q_0$ is equal to the homogenization of q.

4 Constructible Ulam quasimorphisms

In the sequel, when we speak of nilpotent Lie groups, we always mean connected, simply connected, algebraic nilpotent Lie groups over $\mathbb{R}$. Sometimes we emphasize "algebraic", but this is not that necessary: every connected simply connected nilpotent Lie group admits a unique algebraic structure by [Ho, Theorem on page 12].

4.1 Zariski closure of discrete subgroups

Definition 4.1: Let Y be a real algebraic variety, and $X \subset Y$ a subset. **The Zariski closure** of X in Y is the intersection of all real algebraic subvarieties $X_i \subset Y$ containing X. A subset $X \subset Y$ is **Zariski dense** if its Zariski closure is Y.

Remark 4.2: Let $\Gamma \subset G$ be a subgroup of a real algebraic group. Then its Zariski closure $\overline{\Gamma}$ is an algebraic subgroup of G. Indeed, the group laws put algebraic constraints on $\overline{\Gamma}$, hence the smallest algebraic subvariety of G containing Γ is closed under the group operations.

The following notion is going to be used in the proof of Theorem 4.31 below. For the readers' convenience, we recall the definition of Hausdorff distance ([G1]).

Definition 4.3: Let Z_1, Z_2 be subsets of a metric space M. Denote by $Z_i(\varepsilon)$ the ε-neighborhood of Z_i, and let

$$d_H(Z_1, Z_2) := \inf\{\varepsilon \in \mathbb{R}^{>0} \cup \infty \ \mid \ Z_1(\varepsilon) \supset Z_2 \text{ and } \ Z_2(\varepsilon) \supset Z_1\}.$$

The number $d_H(Z_1, Z_2)$ is called **the Hausdorff distance** between Z_1 and Z_2. It is not hard to see that d_H defines a metric, taking values in $\mathbb{R}^{>0} \cup \infty$, on the set of all closed subsets of M.

Definition 4.4: Two subsets X, Y of a metric space M are **coarse equivalent** if the Hausdorff distance $d_H(X, Y)$ is finite. When M is a Lie group, and d a left-invariant Riemannian distance, this is equivalent to $K \cdot X \supset Y$ and $K \cdot Y \supset X$ for a compact subset $K \subset G$.

Example 4.5: Any two rank 2 discrete lattices in $\mathbb{R}^2$ with the usual Euclidean metric are coarse equivalent. However, $\mathbb{R}^2$ is not coarse equivalent to a point.

Example 4.6: More generally, $\mathbb{R}^k$ taken with the standard Euclidean metric is not coarse equivalent to any subspace $\mathbb{R}^n \subsetneq \mathbb{R}^k$.

We need to relax the notion of coarse equivalence slightly to accommodate the metrics which are not bi-invariant.

Definition 4.7: Let G be a Lie group. We say that $X \subset G$ is **bi-coarse equivalent** to $Y \subset G$ if there exists a compact subset $K \subset G$ such that $K \cdot X \cdot K \supset Y$ and $K \cdot Y \cdot K \supset X$.

Definition 4.8: Let G be a real algebraic Lie group, and $\Lambda \subset G$ a Zariski dense subgroup. We say that Λ is **bi-coarse Zariski dense in** G if the following property holds. If a subset $\Lambda' \subset G$ is bi-coarse equivalent to Λ, then Λ' is also Zariski dense in G.

We prove that a lattice in a nilpotent Lie group is bi-coarse Zariski dense. Note that simply connected nilpotent Lie groups are algebraic, with the algebraic structure induced by the standard algebraic structure on its Lie algebra. Indeed, for a simply connected nilpotent Lie group, the exponential map from its Lie algebra to the Lie group is polynomial and invertible, and the inverse map, called the logarithm, is also polynomial ([CG, Proposition 1.2.8]).

Definition 4.9: A subset $\Lambda \subset G$ of a Lie group is called **bi-cocompact** if there exists a compact subset $K \subset G$ such that $K \cdot \Lambda \cdot K = G$.

Remark 4.10: Further on, we use the following elementary observation. Clearly, the subgroup $[G, G] \subset G$ is normal, and for any $H \subset G$ the group $H \cdot [G, G]$ generated by H and $[G, G]$ is also normal. Indeed, any subgroup $G_1 \subset G$ which contains $[G, G]$ is normal, because $x^{-1}yxy^{-1} \in [G, G] \subset G_1$ for any $x \in G_1$ and $y \in G$, and therefore yxy^{-1} also belongs to G_1.

Claim 4.11: Let G be a connected, simply connected algebraic nilpotent Lie group, and $H \subset G$ a proper algebraic subgroup. Consider the minimal Lie subgroup $H \cdot [G, G]$ containing H and $[G, G]$. Then $G/(H \cdot [G, G]) \cong \mathbb{R}^n$ for some $n > 0$.

Proof: A quotient of a connected, simply connected nilpotent Lie group by an algebraic subgroup is connected, simply connected, because this algebraic subgroup is connected and simply connected ([CG, Proposition 1.2.8]). Since $G/(H \cdot [G, G])$ is commutative, it is isomorphic to $\mathbb{R}^n$. It remains to show that $n > 0$. Otherwise the projection of $\mathfrak{h}$ to $\frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]}$ is surjective, where $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. However, any set of elements generating $\frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]}$ also generates $\mathfrak{g}$ ([Ko]), hence in this case $G = H$, which is a contradiction because $H \subset G$ is a proper subgroup. ■

Remark 4.12: Let $H \subset G$ be a Zariski closed subgroup in a connected, simply connected nilpotent algebraic Lie group. Then H is also connected and simply connected ([CG, Proposition 1.2.2]).

We need the following trivial sublemma.

Sublemma 4.13: Consider vector spaces $\mathbb{R}^k \subsetneq \mathbb{R}^n$, and let $K \subset \mathbb{R}^n$ be a compact subset. Then $K + \mathbb{R}^k \neq \mathbb{R}^n$. ■

Lemma 4.14: Let $\Lambda \subset G$ be a bi-cocompact subset in a simply connected nilpotent Lie group G. Then the group generated by Λ is Zariski dense.

Proof: Let $H \subset G$ be the Zariski closure of the group generated by Λ. We need to show that $H = G$. Since H is bi-cocompact, $K \cdot H \cdot K = G$ for some compact subset $K \subset G$. Consider the projection map $\pi: \ G \longrightarrow G/[G,G]$. Let $K_1 := \pi(K)$; this set is compact because K is compact and π is continuous. The group $\pi(H)$, which is a commutative nilpotent Lie group, is isomorphic to $\mathbb{R}^n$. Assume, on contrary, that $H \neq G$. By Claim 4.11, the image of H in $G/[G,G]$ is strictly smaller than $\pi(G)$, which is also homeomorphic to $\mathbb{R}^m$, for some $m > n$. Now, $K_1\pi(H)K_1 \neq \pi(G)$ follows from Sublemma 4.13. This gives a contradiction with $K \cdot H \cdot K = G$. ■

Corollary 4.15: Let G be a simply connected nilpotent Lie group, and $\Gamma \subset G$ a lattice, which is cocompact ([W, Theorem 2.2.6]). Then Γ is bi-coarse Zariski dense.

Proof: Since Γ is cocompact, it is bi-cocompact. Any subset $S \subset G$ which is bi-coarse equivalent to a bi-cocompact set is bi-cocompact. Therefore, S is Zariski dense by Lemma 4.14. This implies that Γ is bi-coarse Zariski dense. ■

4.2 Virtually conjugation equivalent cyclic subgroups

In the sequel, we need the following notion.

Definition 4.16: Let $A, B \subset G$ be infinite cyclic subgroups of a group G. We say that A is **virtually conjugation equivalent (VCE equivalent) to** B, denoted $A \sim_{\mathsf{vce}} B$, if there exist $u \in G$ such that the intersection $A \cap B^u$ is infinite, where B^u denotes the subgroup obtained from B by conjugation with u. We also write $x \sim_{\mathsf{vce}} y$ for elements $x, y \in G$ when x, y generate infinite cyclic subgroups which are VCE equivalent.

Remark 4.17: In [DGO], two elements of a group are called **commensurable** if non-zero powers of these elements are conjugate. This is equivalent to VCE equivalence of cyclic subgroups generated by these elements.

Claim 4.18: The relation $\sim_{\mathsf{vce}}$ is, indeed, an equivalence relation.

Proof: Let A, B, C be three cyclic subgroups which satisfy $A \sim_{\mathsf{vce}} B$ and $B \sim_{\mathsf{vce}} C$. Then A is commensurable with B^u and B is commensurable with C^v, that is, the intersections $A \cap B^u$ and $B \cap C^v$ are infinite. Note that any infinite subgroup in a cyclic group has finite index, hence to show that $A \sim_{\mathsf{vce}} C$ it would suffice to show that $A \cap C^{vu}$ is infinite.

By definition, $C^{vu} \cap B^u$ is a finite index subgroup in B^u. On the other hand, B^u contains $A \cap B^u$ as a finite index subgroup. Since an intersection of two infinite subgroups in $\mathbb{Z}$ is always infinite, $A \cap B^u \cap C^{vu}$ is infinite, proving the claim. ■

Remark 4.19: If $\operatorname{rk} H^1(\Gamma, \mathbb{Q}) \geqslant 2$, it is easy to find infinitely many cyclic subgroups which are pairwise VCE non-equivalent. Indeed, if the line in $H^1(\Gamma, \mathbb{Q})$ generated by a cyclic subgroup A is not collinear with the line generated by B, this implies that $A \not\sim_{\mathsf{vce}} B$. To extend this statement to more general hyperbolic groups, we need the following argument.

The notion of hyperbolic embedding was defined in [DGO]. The following result is one of the applications of hyperbolic embeddings

Claim 4.20: Let H be hyperbolically embedded subgroup of Γ, then every (commutative) quasimorphism on H extends to a quasimorphism on Γ.
Proof: The quasimorphisms from H to $\mathbb{R}$ are controlled by kernel of the map $H^2_b(H, \mathbb{R}) \longrightarrow H^2(H, \mathbb{R})$, called **the exact reduced bounded cohomology**, which is clear from (1.1). Every class in second exact reduced bounded cohomology of H can be extended to $H^2_b(\Gamma, \mathbb{R})$, as follows from [HO]. This result was generalized to all cohomology groups in [FPS]. ■

In [O], D. Osin defined acylindrically hyperbolic groups, which includes all hyperbolic groups which are not virtually cyclic. The fundamental group of a closed manifold of strictly negative sectional curvature, which is by convention assumed of dimension > 1, is hyperbolic and not virtually cyclic, hence it is acylindrically hyperbolic. In [O, Theorem 1.2] and [DGO, Theorem 2.24], it was shown that any acylindrically hyperbolic group Γ contains a hyperbolically embedded subgroup $H = \mathbb{F}_2 \times K$, where K is some finite group. This brings the following theorem.

Theorem 4.21: Let Γ be an acylindrically hyperbolic group (this includes fundamental groups of a closed manifold of strictly negative sectional curvature.) Then there exists a free subgroup $H = \mathbb{F}_2 \subset \Gamma$ such that any quasimorphism on H can be extended to a quasimorphism on Γ. ■

Remark 4.22: Brooks quasimorphisms are quasimorphisms on a free group, defined in [Br]; see [C] for more details. We use Theorem 4.21 only Brooks quasimorphisms; however, for Brooks quasimorphisms this result is implicit already in Bestvina-Fujiwara [BF], where the free group is a Schottky subgroup.

The following result immediately follows from the definition of Brooks quasimorphism.

Claim 4.23: Let W_1 and W_2 be two reduced words, considered as elements of $\mathbb{F}_2$. Denote by $q_1 : \mathbb{F}_2 \longrightarrow \mathbb{R}$ the homogenization of the Brooks quasimorphism associated with W_1. Assume that W_1 is not a subword of W_2^i, for all $i \in \mathbb{Z}$. Then $q_1(W_2) = 0$.
Proof: The proof is left as an exercise to the reader. ■

This implies the following proposition which will be used in the sequel.

Proposition 4.24: Let Γ be an acylindrically hyperbolic group.[1] Let k, k_1, l, l_1 be non-negative integers such that neither $k = 0, k_1 = 0$ nor $l = 0, l_1 = 0$, nor $k = k_1, l = l_1$ holds. Then there exists a free subgroup $H = \mathbb{F}_2 \subset \Gamma$ generated by $a, b \in \Gamma$ such that $a^k b^l \not\sim_{\mathsf{vce}} a^{k_1} b^{l_1}$, where the relation $\not\sim_{\mathsf{vce}}$ is taken in Γ.

Proof: By Theorem 4.21, there exists a subgroup $\mathbb{F}_2 \subset \Gamma$ such that any quasimorphism is extended from $\mathbb{F}_2$ to Γ. Let $W_1 = a^{k_1} b^{l_1}$ and $W_2 = a^k b^l$. The negation of the condition "$k = 0, k_1 = 0$ or $l = 0, l_1 = 0$, or $k = k_1, l = l_1$" is equivalent to "W_1 is not a subword of W_2^i, and W_2 is not a subword of W_1^i, for all $i \in \mathbb{Z}$." Let $q_1, q_2 : \mathbb{F}_2 \longrightarrow \mathbb{R}$ be the homogenizations of the corresponding Brooks quasimorphisms. By construction, q_i can be extended to quasimorphisms $q_1, q_2 : \Gamma \longrightarrow \mathbb{R}$. Since homogeneous quasimorphisms are conjugate invariant, and $q_i(W_j) = \delta_{ij}$, the cyclic group $\langle W_1 \rangle$ intersects trivially with any conjugate to $\langle W_2 \rangle$. ■

As an immediate corollary, we obtain the following assertion, which is also implied by [BF].[2]

[1] This includes fundamental groups of a closed manifold of strictly negative sectional curvature.
[2] For non-elementary hyperbolic groups, this result also follows from [EF].

Corollary 4.25: Let Γ be an acylindrical hyperbolic group. Then there exists infinitely many cyclic subgroups U_i which satisfy $U_i \not\sim_{\mathsf{vce}} U_j$ for any $i \neq j$. ■

Remark 4.26: In [DGO, Corollary 6.12], a similar result is shown: the authors construct an arbitrarily large collection of elements in a hyperbolic group which are non-commensurable in the sense of Remark 4.17.

4.3 Non-constructible Ulam quasimorphisms

In [FK], Kapovich and Fujiwara define a constructible quasimorphism taking values in a discrete group. We extend their definition to any Lie group.

Definition 4.27: Let $q: \Gamma \longrightarrow G$ be an Ulam quasimorphism. Consider a finite index subgroup $\Gamma_0 \subset \Gamma$, and let $H \supset q(\Gamma_0)$ be a subgroup of G containing $q(\Gamma_0)$. Assume that H contains a normal subgroup A, abelian and central in H, such that the composition $\Gamma_0 \xrightarrow{q} H \longrightarrow H/A$ is equivalent to a homomorphism $\Gamma_0 \longrightarrow H/A$. Then q is called **a constructible quasimorphism**.

An element $a \in \Gamma$ of a group is called **primitive** if for any $n \in \mathbb{Z}$ and $b \in \Gamma$ such that $b^n = a$, one has $n = \pm 1$. A quasimorphism $q: \Gamma \longrightarrow G$ is called **homogeneous** if its restriction to any cyclic subgroup is a homomorphism (Definition 3.21). We gave a detailed treatment of homogeneous quasimorphisms in Subsection 3.5, where we defined a new type of quasimorphisms, called HBG-quasimorphism, for "homogeneous Barge-Ghys". We use the acronym HBG to avoid the confusion, because the HBG-quasimorphisms are not, in fact, "Barge-Ghys quasimorphisms", in the sense of Definition 3.9. One should think of a HBG-quasimorphism as of a "homogenization" of a Barge-Ghys quasimorphism. However, the notion of homogenization, which is well known for the usual ($\mathbb{R}$-valued) quasimorphisms, is not defined (yet) for the quasimorphisms taking values in a Lie group.

In Subsection 5.2, we prove the following theorem.

Theorem 4.28: Let M be a closed manifold of strictly negative curvature, G a connected non-abelian Lie group, and $x_1, ..., x_n \in \Gamma := \pi_1(M)$ a collection of primitive elements satisfying $x_i \not\sim_{\mathsf{vce}} x_j$ for $i \neq j$.[3] Take any collection of

[3] Since dim $M > 1$, there exist infinitely many such subgroups, see Corollary 4.25.

elements $g_i \in G$, with $i = 1, ..., n$. Then

(i) There exists a connection ∇ on a trivial principal bundle P such that the corresponding HBG-quasimorphism $q_\nabla : \Gamma \longrightarrow G$ takes x_i to g_i, $i = 1, 2, ..., n$.

(ii) Moreover, for any countable family of elements z_i such that $z_i \not\sim_{\mathsf{vce}} x_j$ for all z_i, x_j, the connection ∇ can be chosen such that $q_\nabla(z_i)$ is not central in G.

Proof: See Subsection 5.2 below. ■

We use this result to prove Theorem 4.31 below.

Definition 4.29: A real nilpotent Lie group is called **rational** if its Lie algebra $\mathfrak{g}$ contains a rational Lie subalgebra $\mathfrak{g}_{\mathbb{Q}}$ such that $\mathfrak{g} = \mathfrak{g}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$. By Maltsev's theorem, this property is equivalent to existence of cocompact lattices ([CG]).

Example 4.30: Consider the **Heisenberg group** $\mathscr{H}$, that is, the group of upper triangular matrices 3×3. This is a 3-dimensional nilpotent Lie group. The subgroup $\mathscr{H}_{\mathbb{Z}}$ of integer upper triangular matrices is cocompact in $\mathscr{H}$, which is an exercise left to the reader. This implies, in particular, that $\mathscr{H}$ is rational.

Theorem 4.31: Let G be a simply connected, connected, non-abelian rational real nilpotent Lie group, and $\Gamma := \pi_1(M)$, where M is a closed manifold of strictly negative sectional curvature, $\dim_{\mathbb{R}} M > 1$. Then there exists a non-constructible HBG-quasimorphism $q_\nabla : \Gamma \longrightarrow G$.

Proof. Step 1: By Maltsev's theorem, G contains a lattice Λ ([CG, Theorem 5.8.1]), which is a posteriori cocompact ([CG, Corollary 5.4.6]). This lattice is finitely generated ([CG, Corollary 5.1.7]). Denote its generators by g_i. By Corollary 4.25, Γ contains infinitely many x_i which satisfy $x_i \not\sim_{\mathsf{vce}} x_j$ for all $i \neq j$. Using Theorem 4.28, we find $x_1, ..., x_n \in \Gamma$ and a HBG-quasimorphism $q_\nabla : \Gamma \longrightarrow G$ taking each x_i to g_i. We choose, in addition, elements $a, b \in \Gamma$ such that $a^l b^m \not\sim_{\mathsf{vce}} a^{l_1} b^{m_1}$ unless $l = l_1, m = m_1$ or $l = l_1 = 0$ or $m = m_1 = 0$ (Proposition 4.24). Since the choice of x_i is arbitrary, we may assume also that $x_i \not\sim_{\mathsf{vce}} a$ and $x_i \not\sim_{\mathsf{vce}} b$ for all i.

Denote the identity element in G by 1_G. We apply Theorem 4.28 to x_i as above, z_i equal to $a^l b^m$ such that $l, m \neq 0$. Then we obtain a HBG-quasimorphism such that $q_\nabla(x_i) = g_i$, $q_\nabla(a) = 1_G$, $q_\nabla(b) = 1_G$, and $q_\nabla(a^i b^j)$ is not central for infinitely many $i, j > 0$.

We are going to show that q_∇ is not constructible.

Step 2: By contradiction, suppose that q_∇ is constructible, that is, there exists a finite index subgroup $\Gamma_0 \subset \Gamma$ and $q_\nabla\Big|_{\Gamma_0}$ satisfies the assumptions of Definition 4.27.

Since Γ_0 is of finite index in Γ, it contains powers $x_i^{n_i}$ of each x_i, for some $n_i \in \mathbb{Z}^{>0}$. Using the central series for the lattice $\Lambda \subset G$, we can easily see that the elements $q_\nabla(x_i^{n_i}) = g_i^{n_i}$ generate a finite index sublattice $\Lambda_0 \subset \Lambda$, which is bi-coarse equivalent to G because it is cocompact.

Since Λ_0 is bi-coarse Zariski dense (Corollary 4.15), any map $\Gamma_0 \longrightarrow G$ which is equivalent to q_∇ has a Zariski dense image.

Since q_∇ is constructible, there exists a homomorphism $h: \Gamma_0 \longrightarrow H/A$, where $H \subset G$ is a subgroup and A its central subgroup, which can be lifted to a homomorphism $h_1: \Gamma_0 \longrightarrow H$ which is Ulam equivalent to $q_\nabla\Big|_{\Gamma_0}$. Then $\operatorname{im} h$ is Zariski dense in G, as indicated above. Therefore the group H is also Zariski dense in G, and A belongs to the center $Z \subset G$.

Clearly, the composition of $h: \Gamma_0 \longrightarrow G/A$ and $\pi: G/A \longrightarrow G/Z$ is a homomorphism. It remains to show that the composition of $q_\nabla\Big|_{\Gamma_0}$ and $\pi: G \longrightarrow G/Z$ is not equivalent to any homomorphism, which will give a contradiction.

Step 3: A sequence $\{x_i \in G\}$ is called **bounded** if it belongs to a compact subset of G. For any compact subset $K \subset G$ in a topological group G, and any sequences $\{x_i\}$, $\{y_i \in K x_i K\}$, the sequence $\{x_i\}$ is bounded if and only if $\{y_i\}$ is bounded. Indeed, $y_i \in K x_i K \Leftrightarrow x_i \in K^{-1} y_i K^{-1}$, and for any bounded $\{x_i \in K_1\}$, the sequence $y_i \in K x_i K \subset K K_1 K$ is clearly bounded.

Step 4: A simply connected nilpotent Lie group G does not contain a bounded subgroup. Indeed, the closure of a bounded group is compact, because a closed bounded set is compact. By Cartan's theorem, its closure is a Lie subgroup $K \subset G$. Let $G \supset G_1 \supset G_2 \supset ...$ be the lower central series for G, with $G_{i+1} = [G_i, G]$, and i the smallest number such that $K \subset G_i$. Then the projection of K to $\frac{G_i}{[G_i, G_i]}$ is a non-trivial compact subgroup. However, $\frac{G_i}{[G_i, G_i]} = \mathbb{R}^n$, and $\mathbb{R}^n$ does not have non-trivial compact subgroups.

Step 5: Let $q_1 : \Gamma \longrightarrow G/Z$ denote the composition of $q_\nabla : \Gamma \longrightarrow G$ and the projection to G/Z. Arguing by contradiction, suppose that $h : \Gamma_0 \longrightarrow G/Z$ is a homomorphism which is equivalent to q_1, where $\Gamma_0 \subset \Gamma$ is a finite index subgroup. Then there exists a compact $K \subset G$ such that

$$h(z) \in Kq_1(z)K \tag{4.1}$$

for all $z \in \Gamma_0$. Since Γ_0 has finite index in Γ, there exist $i, j > 0$ such that $a^i, b^j \in \Gamma_0$. Since $q_1(a^{ki}) = q_1(b^{kj}) = 1_G$ for all $k \in \mathbb{Z}$, this gives $h(a^{ik}) = h(a^i)^k \in K \cdot K$ and $h(b^{jk}) = h(b^j)^k \in K \cdot K$. However, for any element x in a simply connected nilpotent Lie group, a sequence $\{x^n, n \in \mathbb{Z}\}$ can be bounded only if $x = 1_G$, hence $h(a^i) = 1_G$ and $h(b^j) = 1_G$. Since h is a homomorphism, this implies that $h(a^i b^j) = 1_G$. By construction, the HBG-homomorphisms are "homogeneous", that is, satisfy $q_\nabla(x^n) = (q_\nabla(x))^n$. This implies that q_1 is also homogeneous. Since $q_1(a^i b^j) \neq 1_G$, the sequence $q_1((a^i b^j)^n), n \in \mathbb{Z}$ is not bounded (Step 4), which implies that the sequence $h((a^i b^j)^n) \in Kq_1((a^i b^j)^n)K$ is also unbounded (Step 3), giving a contradiction. ■

Remark 4.32: In Theorem 4.31, we construct a HBG-quasimorphism which is not equivalent to a constructible one. However, the proof we use brings a more powerful result, which was the chief aim of an earlier version of this paper. Let $q, q' : \Gamma \longrightarrow G$ be Ulam quasimorphisms. We say that q, q' are **algebraically equivalent** if there exists a compact subset $K \subset G$ such that for all $x \in \Gamma$, one has $q(x) \in Kq'(x)K$. In Theorem 4.31, we construct a HBG-quasimorphism q_∇ which is not algebraically equivalent to a constructible quasimorphism.

4.4 Different notions of quasimorphisms: geometric, algebraic and Ulam

The notion of Ulam quasimorphism is not the only notion of a quasimorphism considered in the literature. In [HS], Hartnick and Schweizer define an alternative notion of "a quasimorphism" $\varphi : G \longrightarrow H$ as a map such that for any quasimorphism $q : H \longrightarrow \mathbb{R}$ its composition with φ is a quasimorphism $q \circ \varphi : G \longrightarrow \mathbb{R}$. When H is a hyperbolic group, it has many quasimorphisms to $\mathbb{R}$ ([EF]), and this condition is quite restrictive. However, when H is a lattice of high rank, such as $SL(n, \mathbb{Z})$, $n > 2$, ([BM1, BM2]), all quasimorphisms to $\mathbb{R}$ are equivalent to homomorphisms. However, the lattices of finite rank satisfy property T ([BHV]), hence they have finite abelinization. Therefore, all quasimorphisms on H are bounded; for such H the Hartnick-Schweizer condition is quite weak.

In [FK], Fujiwara and Kapovich considered other versions of the notion of quasimorphism, taking values in a non-commutative group H; when H is commutative, all these notions are equivalent to the notion of Ulam quasimorphism. They define the **algebraic quasimorphisms** as maps $q: G \longrightarrow H$ such that there exists a compact set K and $q(xy) \in Kq(x)Kq(y)K$ for all $x, y \in G$, and **geometric quasimorphisms** as ones such that $q(xy) \in Kq(x)Kq(y)$.

For comparison, Ulam quasimorphisms are maps $q: G \longrightarrow H$ which satisfy $q(xy) \in Kq(x)q(y)$; this notion is stronger than the notion of a geometric quasimorphism.

In [He, Definition 2.2], N. Heuer introduced a version of this definition. Given a map $q: G \longrightarrow H$, he defines **the defect** of q as a subgroup generated by $q(xy)q(y)^{-1}q(y)^{-1}$. In [He, Proposition 2.3], Heuer proves that q is an Ulam quasimorphism if and only if the defect is finite.

Since $q(x^{-1}) = q(x)^{-1}$, the condition $q(xy) \in Kq(x)q(y)$ is equivalent to $q(xy) \in q(x)q(y)K$. Indeed, $q((xy)^{-1}) \in Kq(y^{-1})q(x^{-1})$ implies $q(xy) \in (Kq(y^{-1})q(x^{-1}))^{-1} = q(x)q(y)K^{-1}$. The same argument works for the geometric quasimorphisms if we also assume that $q(x^{-1}) = q(x)^{-1}$.

Earlier, we defined an equivalence of quasimorphisms: $q \sim q'$ if there exists a compact $K \subset H$ such that $q(x) \in Kq'(x)$ for all $x \in G$. Interestingly enough, a map which is equivalent to an Ulam quasimorphism is no longer an Ulam quasimorphism (see Remark 4.33 below). However, a map $q': G \longrightarrow H$ which is equivalent to a geometric quasimorphism $q: G \longrightarrow H$ is again a geometric quasimorphism:

$$\begin{aligned} & q(xy) \in Kq(x)Kq(y) \Rightarrow \\ & q'(xy) \in K_1 q(xy) \subset K_1 Kq(x)Kq(y) \subset K_1 K K_1^{-1} q'(x) K K_1^{-1} q'(y). \end{aligned}$$

Here the "equivalence" is understood as $q'(x) \in K_1 q(x)$, and the "geometric quasimorphism" condition as $q(xy) \in Kq(x)Kq(y)$.

The reason why the "geometric quasimorphism" condition is sometimes more appropriate stems from the following observation. Let $q: G \longrightarrow H$ be an Ulam quasimorphism, and $d: H \times H \longrightarrow \mathbb{R}^{\geqslant 0}$ a left-invariant metric such that all closed balls in H are compact. Then the condition

$$d(q(xy), q(x)q(y)) \leqslant C$$

is equivalent to being an Ulam quasimorphism. However, a map $q' :\ G \longrightarrow H$ such that $d(q(x), q'(x)) \leqslant C_1$ is no longer an Ulam quasimorphism, but only a geometric one. This leads to the following observation.

Remark 4.33: Let $q :\ \Gamma \longrightarrow G$ be a non-constructible quasimorphism (Theorem 4.31) taking values in a nilpotent Lie group G containing a cocompact lattice Λ. A notion of constructible quasimorphisms is defined only for Ulam quasimorphisms, but we can generalize it to geometric quasimorphisms as follows: a geometric quasimorphism is constructible if it is equivalent, in the sense of Definition 1.1, to a constructible Ulam quasimorphism. Pick a right-invariant metric d on G, and let R be the diameter of the fundamental domain of Λ acting on G. For each $q(x) \in G$, let us choose a closest element $q'(x) \in \Lambda$. Then $d(q(x), q'(x)) < R$, hence the map $q' :\ \Gamma \longrightarrow \Lambda$ is equivalent to q, and q' is a non-constructible geometric quasimorphism. This gives an example of a geometric quasimorphism which is non-constructible, even when the target group is discrete. By [FK], any Ulam quasimorphism $q'' :\ \Gamma \longrightarrow \Lambda$ into a discrete group is constructible, hence q' is not equivalent to any Ulam quasimorphism.

5 HBG-quasimorphisms with prescribed values

5.1 Connections with prescribed holonomy

We prove the following preliminary lemma, which will be used later in this section.

Lemma 5.1: Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra, and P a trivial G-bundle on an interval $[0, 1]$. Fix an element $g \in G$. Denote by ∇_0 the trivial connection on P. Then there exists a $\mathfrak{g}$-valued 1-form A with compact support, such that the holonomy $\operatorname{Hol}(\nabla)$ of the connection $\nabla := \nabla_0 + A$ is equal to g.

Proof: Write A as $a(t)dt$, where $a \in \mathfrak{g}$ and dt is the standard 1-form on $[0, 1]$. Then $\operatorname{Hol}(\nabla) = \int_0^1 a(t)dt$. Since G is connected, we can connect the unit 1_G to g by a path $\gamma :\ [0, 1] \longrightarrow G$. Reparametrizing γ, we may assume that γ is constant in a small neighborhood of 0 and of 1. By Newton-Leibniz formula, $\int_0^1 (\gamma(t)^{-1})^* \dot\gamma dt = g$. Setting $a(t) := (\gamma(t)^{-1})^* \dot\gamma$, we obtain a connection form which satisfies $\operatorname{Hol}(\nabla) = \int_0^1 (\gamma(t)^{-1})^* \dot\gamma dt = g$. Since γ is constant in a neighborhood of 0 and of 1, the form $a(t)dt$ has compact support. ■

5.2 Constructing the HBG-quasimorphisms

In this section, we prove Theorem 4.28. We repeat its statement for convenience.

Theorem 4.28: Let M be a closed manifold of strictly negative curvature, G a non-abelian connected Lie group, and $x_1, ..., x_n \in \Gamma := \pi_1(M)$ a collection of primitive elements satisfying $x_i \not\sim_{\mathsf{vce}} x_j$ for all $i \neq j$. Take any collection of elements $g_i \in G$, with $i = 1, ..., n$. Then

(i) There exists a connection ∇ on a trivial principal bundle P such that the corresponding HBG-quasimorphism $q_\nabla : \Gamma \longrightarrow G$ takes x_i to g_i, $i = 1, 2, ..., n$.

(ii) Moreover, for any countable family of elements $z_j \in \Gamma$ satisfying $z_j \not\sim_{\mathsf{vce}} x_i$ for all i, j, the connection ∇ can be chosen in such a way that $q_\nabla(z_j)$ is not central in G for all z_j.

Proof. Step 1: For each primitive element $x \in \pi_1(M)$, denote by F_x the minimal free geodesic loop representing x (Proposition 3.8). This geodesic loop is unique and determines x up to conjugation. Fix a point p on M, and a point p_i on each F_{x_i}. Let γ_{x_i} be the piecewise smooth loop connecting p to p_i by a geodesic segment $\nu_{p_i,F_{x_i}}$, going around F_{x_i}, and back from p_i to p by the same geodesic segment $\nu_{p_i,F_{x_i}}$ reversed. For each connection ∇ on P, the corresponding HBG-quasimorphism takes x_i to the holonomy of ∇ along γ_{x_i}.

We are going to choose a connection ∇ on P such that the holonomy of ∇ along γ_{x_i} is equal to g_i.

Step 2: Since $x_i \not\sim_{\mathsf{vce}} x_j$, none of the loops $F_{x_1}, ..., F_{x_n}$ is contained in another of these loops. This is actually the only reason why we care about the VCE equivalence. Fix an open set B_{x_i} containing a segment of F_{x_i} and not intersecting the rest of the loops. We can choose B_{x_i} in such a way that it does not intersect the geodesic segment connecting p to p_i (Step 1). Denote by ∇_0 the trivial connection on P. Using Lemma 5.1, we can construct a connection 1-form on each open set B_{x_i} in such a way that the holonomy of the corresponding connection along $\gamma_i \cap B_{x_i}$ is equal to any given element of G.

This gives a new connection on P, equal to the old one outside of B_{x_i}, with prescribed holonomy $\mathfrak{H}$ on F_{x_i}. The holonomy of this connection along $\nu_{p_i,F_{x_i}} \circ F_{x_i} \circ \nu^{-1}_{p_i,F_{x_i}}$ is $\mathfrak{S}\mathfrak{H}\mathfrak{S}^{-1}$, where $\mathfrak{S} : P\Big|_p \longrightarrow P\Big|_{p_i}$ is the holonomy of this connection along $\nu_{p_i,F_{x_i}}$. Since the map $\mathfrak{H} \mapsto \mathfrak{S}\mathfrak{H}\mathfrak{S}^{-1}$ is bijective, by an appropriate choice of the connection on B_{x_i}, we can obtain any element of G as holonomy of the connection along $\nu_{p_i,F_{x_i}} \circ F_{x_i} \circ \nu^{-1}_{p_i,F_{x_i}}$.

Using a partition of unity, we can glue this 1-form to the connection form in ∇_0, obtaining another connection which is equal to ∇_0 outside of B_{x_i}, and has prescribed holonomy on $\gamma_i \cap B_{x_i}$. This allows us to modify ∇_0 on each open set B_{x_i} in such a way that the holonomy of ∇ along $\gamma_i \cap B_{x_i}$ is equal to g_i.

We built a connection ∇ which satisfies Theorem 4.28 (i). To finish the proof of Theorem 4.28, it remains to show that ∇ can be chosen in such a way that (ii) is satisfied.

Step 3: Suppose that ∇ is chosen in such a way that it satisfies Theorem 4.28 (i). We modify the connection ∇ on a sequence of small open sets $D_1, ..., D_k$ with each D_l intersecting γ_{z_l} and not intersecting $\gamma_{x_1}, \gamma_{x_2}, ..., \gamma_{x_n}$. The result of these successive modifications is a connection denoted ∇_k.

We choose ∇_k in such a way that $q_{\nabla_k}(z_i)$ is not central for $i = 1, ..., k$. The passage from ∇_{k-1} to ∇_k is expressed by $\nabla_k = \nabla_{k-1} + \theta_k$, where θ_k is a $\mathfrak{g}$-valued 1-form with support in D_k.

We chose the 1-form θ_k very small, in such a way that the series $\nabla + \sum \theta_i$ converges to a connection $\widetilde{\nabla}$. By construction, the holonomy of $\widetilde{\nabla}$ along γ_{z_k} is equal to the holonomy of ∇_k, hence $q_{\widetilde{\nabla}}(z_k)$ is not central whenever $q_{\nabla_k}(z_k) = q_{\widetilde{\nabla}}(z_k)$ is not central.

The $\mathfrak{g}$-valued 1-form θ_k is chosen, on each step, using the same argument as in the proof of Lemma 5.1. Using this lemma, the 1-form θ_k is determined by the value of $q_{\nabla_k}(z_k)$, which can be chosen arbitrarily small, provided that $q_{\nabla_k}(z_k)$ is not central. Then the series $\nabla + \sum_{i=1}^{\infty} \theta_i$ converges to a connection $\widetilde{\nabla}$ which satisfies Theorem 4.28 (ii). Since the support of the form $\sum_{i=1}^{\infty} \theta_i$ does not intersect the paths $\gamma_{x_1}, ..., \gamma_{x_n}$, we have $q_{\nabla}(x_i) = q_{\widetilde{\nabla}}(x_i) = g_i$, hence $\widetilde{\nabla}$ satisfies both Theorem 4.28 (i) and Theorem 4.28 (ii). ■

5.3 ε-representations

As an application of the construction given in Subsection 5.2, we generalize Kazhdan's theorem [Kaz, Theorem 2] to the fundamental group of an arbitrary closed manifold of strictly negative sectional curvature. We define the notion of an ε-representation and a δ-approximated ε-representation as follows (compare with [Kaz] and Subsection 1.6).

Let G be a topological group. Recall that **Chabauty topology** on the set of closed subgroups $C(G)$ is defined by the base of neighborhoods, $W_U(\Gamma)$

given for each open subset $U \subset G$ and a subgroup $\Gamma \subset G$

$$W_U(\Gamma) = \{\Gamma' \in C(G) \ \mid \ U \cdot \Gamma' \supset \Gamma \ \text{ and } \ U \cdot \Gamma \supset \Gamma'\}.$$

If the topology of G is induced by a left-invariant metric, the Chabauty topology is induced by the Hausdorff metric on the space of subsets of G (Definition 4.3).

By [F, p. 474], or [dH, Proposition 2 (vi)], the space $C(G)$ is compact.

Recall that a group G **has no small subgroups** ([Tao, Exercise 1.5.6, Corollary 1.5.8]) if there exists a neighborhood of identity which does not contain a non-trivial subgroup; this property is clearly satisfied by any Lie group. When G has no small subgroups, the space of $C_0(G)$ closed non-trivial subgroups is also compact.

Clearly, the diameter of a subset of a metric space M is a continuous function in the topology on 2^M induced by the Hausdorff metric. Since $C_0(G)$ is compact, the diameter of non-trivial subgroups in G is bounded from below for any metric on G.

Fix a left-invariant metric on G such that any non-trivial subgroup has diameter at least $1/3$. We motivate the choice of the constant $1/3$ as follows. For compact group G, we can always choose a bi-invariant metric; its geodesics are translations of one-parametric subgroups. We normalize the metric such that the diameter of any compact one-parametric subgroup is bounded from below by $1/2$.

For a bi-invariant Riemannian metric on a Lie group, the geodesics are obtained by translation of one-parametric group: the Lie-theoretic exponential map coincides with the Riemannian. Every compact Lie group admits a bi-invariant Riemannian metric.

Since every two points are connected by a geodesic, every element of a compact group is an exponent of an element of the Lie algebra, and every finite cyclic subgroup belongs to a circle subgroup. If we normalize the metric such that the diameter of each circle subgroup is bounded from below by $1/2$, then the diameter of each finite subgroup is at least $1/3$, which is realized by $\mathbb{Z}/3\mathbb{Z}$. In particular, of we choose the bi-invariant metric on the group $SU(2) = S^3$ such that $\mathsf{diam}(SU(2)) = 1/2$ (this is equivalent to each meridian circle being of length 1), the bound $1/3$ is realized.

An ε-representation of a group Γ is a map $q : \ \Gamma \longrightarrow G$ such that $d(q(x)q(y), q(xy)) < \varepsilon$ for any $x, y \in \Gamma$. An ε-representation can be **δ-approximated by a representation** if there exists a representation $\rho : \Gamma \longrightarrow G$ such that $d(\rho(x), q(x)) < \delta$ for all $x \in \Gamma$.

Theorem 5.2: Let M be a closed manifold of strictly negative sectional curvature, G a positive-dimensional connected Lie group, and P a trivial principal G-bundle. For any connection ∇ in P, let $q_\nabla : \pi_1(M) \longrightarrow G$ denote the corresponding HBG-quasimorphism (Subsection 3.5). Choose a left-invariant metric on G such that the diameter of any closed subgroup is at least $1/3$. Then for each $\varepsilon > 0$, there exists a connection ∇ such that q_∇ is an ε-representation which cannot be $1/3$-approximated by a representation.

Proof. Step 1: Let $a_1, ..., a_n$ be the generators of $\pi_1(M)$. Using Corollary 4.25, we find $b \in \pi_1(M)$ such that $b \not\sim_{\mathsf{vce}} a_i$. Choose an open set U_b which intersects F_b and does not intersect $F_{a_1}, F_{a_2}, ..., \nu(p, p_i)$, $i = 1, ..., n$, where $\nu(p, p_i)$ are geodesics connecting the marked point p with F_{a_i}. Let p_b be the chosen fixed point on F_b, and ∇_0 be the trivial connection on a trivial G-bundle P. Choose a connection $\nabla_0 + \theta$ which is trivial outside of U_b. Consider the path $\gamma = \nu(p, p_b) F_b \nu(p, p_b)^{-1}$ as a map from $[0, 1]$ to M. Then the holonomy of $\nabla + \theta$ along γ is equal to $\int_0^1 \gamma^*(\theta)$. For any non-trivial $g = e^u \in G$ in the image of the exponential map $\mathrm{Lie}(G) \to G$, we can choose a 1-form θ on M such that $\gamma^*(\theta) = f(t)udt$, for some function $f : [0, 1] \longrightarrow \mathbb{R}$ with compact support. Choosing f such that $g = \int_0^1 f(t)udt$, we obtain a 1-form θ with values in the Lie algebra of G such that the holonomy of $\nabla_0 + \theta$ along γ is equal to g. Moreover, replacing θ with $\frac{1}{m}\theta$, we obtain a connection with holonomy $e^{\frac{1}{m}u} = g^{1/m}$.

For m sufficiently large, this would give an ε-representation q_∇ such that $q_\nabla(a_i) = 1_G$, and $q_\nabla(b)^m = g$.

Step 2: It remains to show that the ε-representation q_∇ cannot be $1/3$-approximated by a representation ρ. By contradiction, assume that q_∇ is $1/3$-approximated by a representation $\rho : \pi_1(M) \longrightarrow G$. Since

$$d(\rho(a_i^n), q_\nabla(a_i)^n) < 1/3,$$

the closure of a subgroup of G generated by $\rho(a_i)$ has diameter less than $1/3$. Since the diameter of non-trivial subgroups of G is $\geqslant 1/3$, this implies that $\rho(a_i) = 1_G$, and ρ is trivial. Therefore, $\rho(b) = 1_G$. However, for all $n \in \mathbb{Z}$, we have

$$d(\rho(b)^n, q_\nabla(b)^n) < 1/3,$$

because q_∇ is $1/3$-approximated by ρ. Then the diameter of the subgroup of G generated by $g = q_\nabla(b)$ is less than $1/3$, which again implies that $g = 1_G$, leading to contradiction. ■

Acknowledgments: We are very grateful to anonymous referees, who provided an excellent commentary to our work and gave many insightful suggestions and ideas. We are also grateful to IMPA and Ben Gurion University of the Negev who provided us with an excellent research atmosphere and working conditions.

References

[BG] J. Barge, É. Ghys, *Surfaces et cohomologie bornée,* Invent. Math. 92 (1988), no. 3, 509-526. (Cited on pages 5, 6, and 16.)

[BB] L. Bartholdi and O. Bogopolski, *On abstract commensurators of groups*, J. Group Theory, 13(6):903-922, 2010. (Cited on page 18.)

[BFMSS] L. Battista, S. Francaviglia, M. Moraschini, F. Sarti, and A. Savini, *Bounded cohomology classes of exact forms*, Proc. Amer. Math. Soc., To appear. arXiv:2211.16125, 2022. (Cited on page 6.)

[B] C. Bavard, *Longueur stable des commutateurs*, Enseign. Math. (2) 37 (1991), no. 1-2, 109-150. (Cited on page 5.)

[BHV] B. Bekka, P. de la Harpe, A. Valette, *Kazhdan's Property (T)*, Volume 11 of New Mathematical Monographs, Cambridge University Press, 2008. (Cited on page 32.)

[BF] M. Bestvina and K. Fujiwara *Bounded cohomology of subgroups of mapping class groups*, Geom. Topol., 6, pp 69-89, 2002. (Cited on page 28.)

[BM] M. Brandenbursky, M. Marcinkowski, *Entropy and quasimorphisms,* Source: Journal of Modern Dynamics (2019), Vol. 15, p 143-163. (Cited on page 5.)

[BH] M. R. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der mathematischen Wissenschaften, 319. Springer-Verlag, Berlin, 1999. (Cited on pages 17 and 18.)

[Br] R. Brooks, *Some remarks on bounded cohomology*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), pp. 53-63, Ann. of Math. Stud., 97, Princeton Univ. Press, Princeton, N.J., 1981. (Cited on pages 5 and 28.)

[BBI] D. Burago, Yu. Burago, S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, 33. American Mathematical Society, Providence, RI, 2001. (Cited on page 12.)

[BM1] M. Burger and N. Monod, *Bounded cohomology of lattices in higher rank Lie groups*, J. Eur. Math. Soc. (JEMS) 1 (2) (1999), 199-235. (Cited on page 32.)

[BM2] Burger, M., and Monod, N., *Continuous bounded cohomology and applications to rigidity theory*, Geom. Funct. Anal. 12 (2) (2002), 219-280. (Cited on page 32.)

[BOT] M. Burger, N. Ozawa, A. Thom, *On Ulam stability,* Israel J. Math. 193 (2013), no. 1, 109-129. (Cited on pages 8 and 9.)

[C] D. Calegari, *scl,* MSJ Memoirs, 20. Mathematical Society of Japan, Tokyo, 2009. (Cited on pages 5, 6, 7, 16, 18, 19, 21, 23, and 28.)

[CG] L. Corwin, F. P. Greenleaf, *Representations of Nilpotent Lie Groups and Their Applications. Part I, Basic Theory and Examples,* Cambridge Univ. Press, Cambridge, UK, 1990 (Cited on pages 4, 25, and 30.)

[DGO] F. Dahmani, V. Guirardel, and D. Osin. *Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces.* Mem. Amer. Math. Soc., 245(1156):v+152, 2017. (Cited on pages 26, 27, and 29.)

[dH] P. de la Harpe, *Spaces of closed subgroups of locally compact groups*, 2008, arXiv:0807.2030. (Cited on page 37.)

[EF] D. B. A. Epstein, K. Fujiwara, *The second bounded cohomology of word hyperbolic groups,* Topology 36, (1997), 1275-1289. (Cited on pages 28 and 32.)

[EP] M. Entov, L. Polterovich, *Calabi quasimorphism and quantum homology*, Int. Math. Res. Not. 2003, no. 30, 1635-1676. (Cited on pages 5 and 6.)

[F] J. M. G. Fell, *A Hausdorff topology for the closed subsets of a locally compact non Hausdorff space,* Proc. Amer. Math. Soc. 13 (1962), 472-476. (Cited on page 37.)

[FPS] R. Frigerio, M. B. Pozzetti, and A. Sisto., *Extending higher-dimensional quasi-cocycles.* J.Topol., 8(4):1123-1155, 2015. (Cited on page 27.)

[Fri] R. Frigerio, *Bounded Cohomology of Discrete Groups*, Mathematical Surveys and Monographs 227, AMS, 2017. (Cited on page 12.)

[FK] K. Fujiwara, M. Kapovich, *On quasihomomorphisms with noncommutative targets,* Geom. Funct. Anal. 26 (2016), no. 2, 478-519. (Cited on pages 2, 3, 4, 9, 29, 33, and 34.)

[GG] J.-M. Gambaudo, É. Ghys, *Commutators and diffeomorphisms of surfaces.* Ergodic Theory Dynam. Systems 24 (2004), no. 5, 1591-1617. (Cited on page 5.)

[GLMR] L. Glebsky, A. Lubotzky, N. Monod, B. Rangarajan, *Asymptotic Cohomology and Uniform Stability for Lattices in Semisimple Groups*, arXiv:2301.00476. (Cited on page 9.)

[G1] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Based on the 1981 French original. With appendices by M. Katz, P. Pansu and S. Semmes. Translated from the French by Sean Michael Bates. Progress in Mathematics, 152. Birkhäuser Boston, Inc., Boston, MA, 1999. xx+585 pp. (Cited on page 24.)

[G2] M. Gromov, *Volume and bounded cohomology,* Inst. Hautes Études Sci. Publ. Math. No. 56 (1982), 5-99 (1983). (Cited on pages 5, 12, 13, and 14.)

[HO] M. Hull, D. Osin. *Induced quasicocycles on groups with hyperbolically embedded subgroups*, Algebr. Geom. Topol., 13 (5), pp. 2635-2665, 2013. (Cited on page 27.)

[HS] T. Hartnick, P. Schweitzer, *On quasioutomorphism groups of free groups and their transitivity properties*, Journal of Algebra, Volume 450 (2016), pp. 242-281, arXiv:1403.2786. (Cited on pages 4 and 32.)

[He] N. Heuer, *Low-dimensional bounded cohomology and extensions of groups,* Math. Scand., 126(1):5-31, 2020. (Cited on page 33.)

[Ho] G. Hochschild, *Note on Algebraic Lie Algebras*, Proceedings of the American Mathematical Society, Vol. 29, No. 1 (Jun., 1971), pp. 10-16 (Cited on page 23.)

[Kaz] D. Kazhdan, *On ε-representations*, Israel Journal of Mathematics, vol. **43**, pages 315-323 (1982), (Cited on pages 3, 4, 7, 8, and 36.)

[Kl] W. P. A. Klingenberg, *Riemannian Geometry*, second edition (1995), de Gruyter. (Cited on page 14.)

[KN] S. Kobayashi and K. Nomizu, *Foundations of differential geometry,* Wiley-Interscience, New York (1969). (Cited on pages 10 and 11.)

[Ko] H. Koch *Generator and relation ranks for finite-dimensional nilpotent lie algebras,* Algebra and Logic volume 16, pages 246-253 (1977) (Cited on page 25.)

[Mar] D. Marasco, *Cup product in bounded cohomology of negatively curved manifolds*, Proc. Amer. Math. Soc., 151(6):2707-2715, 2023. (Cited on page 6.)

[MO1] Users of MathOverflow,
https://mathoverflow.net/questions/423909/holonomy-bounded-in-terms-of-area-and-the-curvature
(Cited on page 15.)

[MO2] Y. Cornuiler, S. Nead, Mathoverflow question
https://mathoverflow.net/questions/423909/holonomy-bounded-in-terms-of-area-and-the-curvature
(Cited on page 18.)

[M] I. Mineyev, *Straightening and bounded cohomology of hyperbolic groups*, Geom. Funct. Anal., 11(4):807-839, 2001. (Cited on page 15.)

[MW] T. Murphy, F. Wilhelm, *Random manifolds have no totally geodesic submanifolds* Michigan Math. J. 68 (2019), no. 2, 323-335. (Cited on page 13.)

[vN] J. von Neumann, *Beweis des Ergodensatzes und des H-Theorems in der neuen Mechanik*, Z. Physik, 57:30-70, 1929. (Cited on page 7.)

[RW] H. Reckziegel, E. Wilhelmus, *How the Curvature Generates the Holonomy of a Connection in an Arbitrary Fibre Bundle*, Results in Mathematics volume 49, pages 339-359 (2006). (Cited on page 15.)

[OV] L. Ornea, M. Verbitsky, *Principles of Locally Conformally Kähler Geometry*, arXiv:2208.07188. (Cited on pages 11 and 16.)

[O] D. Osin, *Acylindrically hyperbolic groups.* Trans. Amer. Math. Soc., 368(2):851-888, 2016. (Cited on page 27.)

[P] P. Petersen, *Riemannian Geometry*, Third Edition, Graduate Texts in Mathematics, Springer Verlag. (Cited on page 18.)

[PR] L. Polterovich, D. Rosen, *Function theory on symplectic manifolds*, CRM Monograph Series, 34, AMS, Providence, RI, 2014. (Cited on page 6.)

[R] P. Rolli, *Quasi-morphisms on free groups*, arXiv preprint arXiv:0911.4234, 2009. (Cited on page 4.)

[Sh] E. Shelukhin, *The Action homomorphism, quasimorphisms and moment maps on the space of compatible almost complex structures*, Commentarii Mathematici Helvetici (2014), Volume 89, Issue 1, 2014, 69–123. (Cited on page 5.)

[St] S. Sternberg *Lectures on differential geometry*, (1964) New York: Chelsea (1093). (Cited on page 10.)

[Tao] T. Tao, *Hilbert's fifth problem and related topics*, Graduate Studies in Mathematics, AMS, 2014. (Cited on page 37.)

[Tu] A. M. Turing, *Finite approximations to Lie groups*, Ann. of Math. (2), 39(1):105-111, 1938. (Cited on page 7.)

[U] S. M. Ulam, *A collection of mathematical problems,* Interscience Tracts in Pure and Applied Mathematics, no. 8 Interscience Publishers, New York-London 1960. (Cited on page 7.)

[W] Zh. Wang, *Chapter 2: Lattices and nilmanifolds,* lecture notes to "MATH 597: Dynamical Systems on Nilmanifolds", Penn State University, Fall 2018, http://www.personal.psu.edu/zxw14/MATH597/Chapter2.pdf (Cited on page 26.)

[Y] D. Yang, *Holonomy equals curvature*, https://cims.nyu.edu/~yangd/papers/holonomy.pdf (Cited on page 15.)

Michael Brandenbursky
Ben Gurion University of the Negev
Beer Sheva, Israel
brandens@bgu.ac.il

Misha Verbitsky
Instituto Nacional de Matemática Pura e Aplicada (IMPA)
Estrada Dona Castorina, 110
Jardim Botânico, CEP 22460-320
Rio de Janeiro, RJ - Brasil
also:
Laboratory of Algebraic Geometry,
Faculty of Mathematics, National Research University HSE,
6 Usacheva Str. Moscow, Russia
`verbit@impa.br`