LIPSCHITZ GEOMETRY OF SURFACE GERMS IN $\mathbb{R}^4$: METRIC KNOTS

LEV BIRBRAIR*, MICHAEL BRANDENBURSKY**, AND ANDREI GABRIELOV†

Abstract. A link at the origin of an isolated singularity of a two-dimensional semialgebraic surface in $\mathbb{R}^4$ is a topological knot (or link) in S^3. We study the connection between the ambient Lipschitz geometry of semialgebraic surface germs in $\mathbb{R}^4$ and the knot theory. Namely, for any knot K, we construct a surface X_K in $\mathbb{R}^4$ such that: the link at the origin of X_K is a trivial knot; the germs X_K are outer bi-Lipschitz equivalent for all K; two germs X_K and $X_{K'}$ are ambient bi-Lipschitz equivalent only if the knots K and K' are isotopic. We show that the Jones polynomial can be used to recognize ambient bi-Lipschitz non-equivalent surface germs in $\mathbb{R}^4$, even when they are topologically trivial and outer bi-Lipschitz equivalent.

1. Introduction

We study the difference between the outer and ambient bi-Lipschitz equivalence of semialgebraic surface germs at the origin in $\mathbb{R}^4$. Two surface germs are outer bi-Lipschitz equivalent if they are bi-Lipschitz equivalent as abstract metric spaces with the outer metric $d(x,y) = \|x-y\|$. Ambient bi-Lipschitz equivalence means that there exists a germ of a bi-Lipschitz, orientation preserving, homeomorphism of the ambient space mapping one of them to the other one. Note that in Singularity Theory the homeomorphism is not required to be orientation preserving. We add this condition to be consistent with the isotopy equivalence relation in Knot Theory. Also, to avoid confusion between the Singularity Theory and Knot Theory notions of the link, we always write "the link at the origin" speaking of the link of a surface germ.

If a surface germ in $\mathbb{R}^4$ with a connected link at the origin has an isolated singularity then its link is a knot in S^3. The results of [2] show that the ambient equivalence is different from the outer equivalence even when there are no topological obstructions. This phenomenon is called "metric knots." We consider the following question: How different are these equivalence relations? In particular, we show that the question makes sense even when "there is no topology," for the germs with unknotted links at the origin. We prove Universality Theorem (Theorem 3.1 below) which implies that the ambient bi-Lipschitz classification in this case "contains all of the knot theory." Namely, for any knot K, one can construct a germ of a surface X_K in $\mathbb{R}^4$ such that:
1. The link at the origin of X_K is a trivial knot;
2. The germs X_K are outer bi-Lipschitz equivalent for all K;
3. Two germs X_K and $X_{K'}$ are ambient bi-Lipschitz equivalent only if the knots K and K' are isotopic.

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The second theorem (Theorem 3.2 below) states that, for each germ X_K in Universality Theorem, there are infinitely many ambient bi-Lipschitz non-equivalent realizations.

We define (β_1, β_2)-bridges and the saddle and crossing moves. A (β_1, β_2)-bridge and the saddle move are closely related to the broken bridge construction in [2]. A surface germ containing a single (β_1, β_2)-bridge and metrically conical outside it is called a *one-bridge surface*. The saddle and crossing moves relate the metric problem of ambient Lipschitz equivalence of two one-bridge surfaces in $\mathbb{R}^4$ with the topological problem of isotopy of the knots in S^3 corresponding to the links at the origin of the surfaces obtained from these one-bridge surfaces by the saddle and crossing moves. That is why topological knot invariants, such as the Jones polynomial, yield metric knot invariants, which can be used to recognize ambient Lipschitz non-equivalence of surface germs.

Actually, one-bridge surfaces are the most simple examples of not normally embedded surfaces. But even in this case one can have rather non-trivial Ambient Lipschitz geometry. Another version of Universality Theorem (Theorem 3.6 below) states that, for any two knots K and L, one can construct a one-bridge surface X_{KL} such that:
1. The link at the origin of X_{KL} is isotopic to L;
2. For any knots K and L, all surface germs X_{KL} are outer bi-Lipschitz equivalent;
3. Surface germs X_{K_1L} and X_{K_2L} are ambient bi-Lipschitz equivalent only if the knots K_1 and K_2 are isotopic.

In section 4 we consider the Jones polynomial of the links at the origin of surfaces, obtained from one-bridge surfaces by the crossing moves. Since the topology of those surfaces is an Ambient Lipschitz invariant, the corresponding Jones polynomials also become Ambient Lipschitz invariant of one-bridge surfaces. In particular, when those links at the origin are torus links, we compute the corresponding Jones polynomials completely.

If we do not suppose the surface to be a one-bridge surface, we obtain a stronger version of the Universality Theorem (Theorem 3.7 below). It states that, for any two knots K and L, one can construct a surface $X^{\alpha\beta}_{KL}$ such that:
1. The link at the origin of $X^{\alpha\beta}_{KL}$ is isotopic to L;
2. For any fixed α and β, all surface germs $X^{\alpha\beta}_{KL}$ are outer bi-Lipschitz equivalent;
3. For a fixed knot K, the tangent link of $X^{\alpha\beta}_{KL}$ is isotopic to K.

2. Definitions and Notations

We consider germs at the origin of semialgebraic (or definable in a polynomially bounded o-minimal structure) surfaces (two-dimensional sets) in $\mathbb{R}^4$. A surface X can be considered as a metric space, equipped with either the outer metric $d(x, y) = \|x - y\|$ or the inner metric $d_{inner}(x, y)$ defined as the minimal length of a path in X connecting x and y.

The *link at the origin* of X is the set $L_X = \{X \cap S^3_{0,\varepsilon}\}$ for small positive ε. We write "the link at the origin" speaking of this notion of the link from Singularity Theory, reserving the word "link" to the notion of the link in Knot Theory. If X has an isolated singularity at the origin then each connected component of L_X is a knot in S^3.

The *tangent link* of X at the origin is the link of the tangent cone of X at the origin.

Definition 2.1. Two surface germs $(X, 0)$ and $(Y, 0)$ are called *outer bi-Lipschitz equivalent* if there exists a bi-Lipschitz with respect to the outer metric homeomorphism $H : (X, 0) \to (Y, 0)$. The germs are called *ambient bi-Lipschitz equivalent* if there exists an orientation preserving bi-Lipschitz homemorphism $\widetilde{H} : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$, such that $\widetilde{H}(X) = Y$.

An *arc* in the set X is a germ at the origin of a mapping $\gamma : [0, \epsilon) \to X$ such that $\gamma(0) = 0$. Unless otherwise specified, arcs are parameterized by the distance to the origin, i.e., $\|\gamma(t)\| = t$.

Definition 2.2. Let $f \not\equiv 0$ be (a germ at the origin of) a function defined on an arc γ. The *order* α of f on γ (notation $\alpha = ord_\gamma f$) is the value $\alpha \in \mathbb{Q}$ such that $f(\gamma(t)) = ct^\alpha + o(t^\alpha)$ as $t \to 0$, where $c \neq 0$. If $f \equiv 0$ on γ, we set $ord_\gamma f = \infty$.

For any two arcs γ_1 and γ_2 one can define two orders of contact : inner and outer.

Definition 2.3. *The outer order of contact* $tord_{outer}(\gamma_1, \gamma_2)$ is defined as $ord_{\gamma_1}(||\gamma_1(t) - \gamma_2(t)||)$. The *inner order of contact* $tord_{inner}(\gamma_1, \gamma_2)$ is defined as $ord_{\gamma_1}(d_p(\gamma_1(t), \gamma_2(t))$, where d_p is a definable pancake metric (see [3]) equivalent to the inner metric. These two orders of contact are rational numbers (or elements of the field of exponents of an o-minimal structure), $1 \leq tord_{inner}(\gamma_1, \gamma_2) \leq tord_{outer}(\gamma_1, \gamma_2)$.

Let $\beta > 1$ be a rational number. Consider the space $\mathbb{R}^3$ with coordinates (x, y, t). For a fixed $t \geq 0$, we define the subsets $W_t^\pm$ of the square with the side $2t$ in the (x, y)-plane, bounded by the line segments $I_t^\pm = \{|x| \leq t,\ y = \pm t\}$ and the line segments connecting the endpoints of $I_t^\pm$ with the points $(0, \pm t^\beta)$, respectively (shaded areas in Figure 1a). Let $W_t = W_t^+ \cup W_t^-$, and let $W = \bigcup_{t \geq 0} W_t \subset \mathbb{R}^3$. Note that the tangent cone of W is the cone over the set $\{(x, y) : |x| \leq |y| \leq 1\}$.

Definition 2.4. Given a rational number $\beta > 1$, the part of the boundary of the set W belonging to the interior of the cone over the square is called a *β-bridge*.

Let $1 < \beta_1 \leq \beta_2$ be two rational numbers. In the xy-plane $\{t = \text{const}\}$ consider the points (see Figure 2)

$$p_1(t) = (-t, t),\ p_2(t) = (-t^{\beta_1}, t^{\beta_2}),\ p_3(t) = (t^{\beta_1}, t^{\beta_2}),\ p_4(t) = (t, t),$$
$$p_1'(t) = (-t, -t),\ p_2'(t) = (-t^{\beta_1}, -t^{\beta_2}),\ p_3'(t) = (t^{\beta_1}, -t^{\beta_2}),\ p_4'(t) = (t, -t).$$

Let us connect $p_1(t)$ with $p_2(t)$, $p_2(t)$ with $p_3(t)$, $p_3(t)$ with $p_4(t)$ by line segments, and define U_t^+ as the quadrilateral bounded by these segments. Similarly, connect $p_1'(t)$ with $p_2'(t)$, $p_2'(t)$ with $p_3'(t)$, $p_3'(t)$ with $p_4'(t)$ by line segments, and define U_t^- as the quadrilateral bounded by these segments. Let V_t be the union of these segments and let $V = \bigcup_{t \geq 0} V_t \subset \mathbb{R}^3$. Let $U_t = U_t^+ \cup U_t^-$

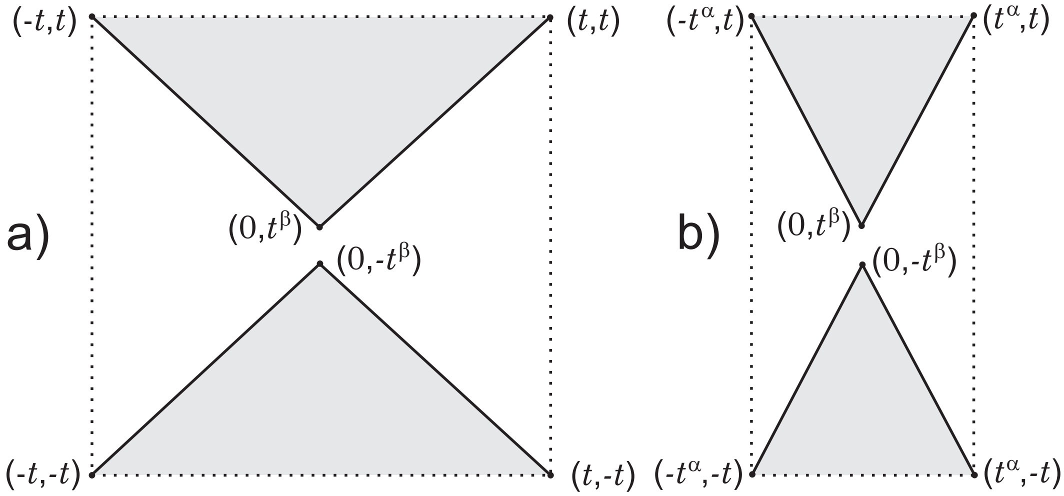


FIGURE 1. a) The set W_t. b) The set $W_t^{\alpha\beta}$.

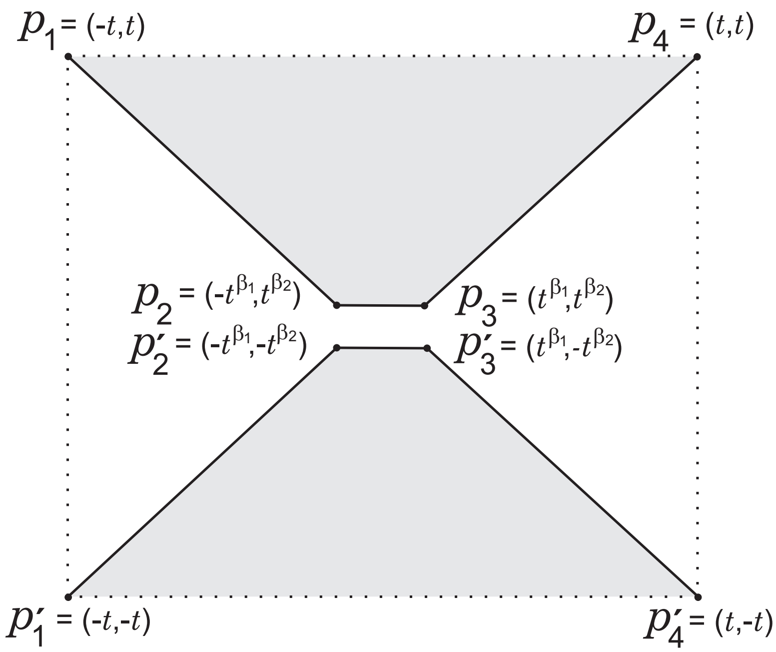


FIGURE 2. The set U_t.

(shaded area in Figure 2), and let $U = \bigcup_{t\geq 0} U_t \subset \mathbb{R}^3$. Note that U and W have the same tangent cones at the origin.

Definition 2.5. Given two rational numbers $1 < \beta_1 \leq \beta_2$, the part of the boundary of the set U belonging to the interior of the cone over the square is called a (β_1, β_2)-*bridge*. Note that, for $\beta_1 = \beta_2 = \beta$, the (β_1, β_2)-bridge is outer bi-Lipschitz equivalent to the β-bridge. The germ of a surface containing a subset outer bi-Lipschitz equivalent to a (β_1, β_2)-bridge and metrically conical outside that part, is called a *one-bridge surface*.

Let $\alpha > 1$, $\beta > 1$ be rational numbers. Consider the space $\mathbb{R}^3$ with coordinates (x, y, t). For a fixed $t \geq 0$, we define the subsets $W_t^{\alpha\beta\pm}$ of the rectangle with the side $2t^\alpha$ in the (x, y)-plane, bounded by the line segments $I_t^{\alpha\pm} = \{|x| \le t^\alpha,\ y = \pm t\}$ and the line segments connecting the endpoints of $I_t{}^{\alpha\pm}$ with the points $(0, \pm t^\beta)$, respectively (shaded areas in Figure 1b).

Let $W_t^{\alpha\beta} = W_t^{\alpha\beta+} \cup W_t^{\alpha\beta-}$, and let $W^{\alpha\beta} = \bigcup_{t\ge 0} W_t^{\alpha\beta} \subset \mathbb{R}^3$. Note that the tangent cone of $W^{\alpha\beta}$ is the cone over the set: $\{(x,y) : x = 0,\ |y| \le 1\}$. Let $\widetilde{W^\alpha}$ be the set defined as the union over $t \ge 0$ of rectangles $\{(x,y) : -t^\alpha \le x \le t^\alpha,\ -t \le y \le t\}$.

Definition 2.6. Given two rational numbers $\alpha > 1$ and $\beta > 1$, the part of the boundary of the set $W^{\alpha\beta}$ belonging to the interior of $\widetilde{W^\alpha}$ is called an (α,β)-*wedge*.

3. Metric Knots

Theorem 3.1 (Universality Theorem)**.** *Let $K \subset S^3$ be a knot. Then one can associate with K a semialgebraic one-bridge surface germ $(X_K, 0)$ in $\mathbb{R}^4$ so that the following holds:*

1) *The link at the origin of each germ X_K is a trivial knot;*
2) *All germs X_K are outer bi-Lipschitz equivalent;*
3) *Two germs X_{K_1} and X_{K_2} are ambient bi-Lipschitz equivalent only if the knots K_1 and K_2 are isotopic.*

Proof. Let $F_K \subset S^3$ be a smooth semialgebraic embedded surface diffeomorphic to $S^1 \times [0,1]$, such that the boundary of F_K contains two components $\widetilde{K}$ and $\widetilde{K}'$ isotopic to the same knot K. Let $\widetilde{Y}_K$ be the cone over F_K and let $\widetilde{X}_K$ be the cone over the boundary of F_K.

Let ξ be an interior point of F_K. Let S_K be a slice of F_K bi-Lipschitz homeomorphic to a square $D = [-1,1] \times [-1,1]$ and containing ξ, and let $\varphi : S_K \to D$ be a bi-Lipschitz homeomorphism such that $\varphi(\xi) = 0$. Suppose, moreover, that the parts of the boundary of F_K belonging to S_K are mapped by φ_K to the sides $\{x = \pm 1\}$ of D, and the parts of the boundary of S_K belonging to the interior of F_K are mapped to the top and bottom sides $\{y = \pm 1\}$ of D.

Let $M_K \subset \mathbb{R}^4$ be the cone over S_K, and let $\widetilde{W}$ be the cone over D. The set W, constructed in the previous section, can be considered as a subset of $\widetilde{W}$. The map φ_K is naturally extended to a map $\Phi_K : M_K \to \widetilde{W}$ by $\Phi_K(t\chi, t) = (t\varphi_K(\chi), t)$ for $\chi \in S$. Note that Φ_K is a bi-Lipschitz map.

Let $V_K = \Phi_K^{-1}(W)$ and $Y_K = \left(\widetilde{Y}_K \setminus M_K\right) \cup V_K$. Define X_K to be the boundary of Y_K. By the construction X_K is a one-bridge surface. Let us show that X_K satisfies the conditions of the theorem.

1) Link of X_K at zero is a trivial knot, because it bounds part of F_K homeomorphic to a disk.

2) Let K_1 and K_2 be two different knots. To show that X_{K_1} and X_{K_2} are bi-Lipschitz equivalent with respect to the outer metrics, we construct a map $\Psi : Y_{K_1} \to Y_{K_2}$ as follows.

We define separately the restrictions $\Psi|_{M_{K_1}}$ and $\Psi|_{Y_{K_1} \setminus M_{K_1}}$. Set $\Psi|_{M_{K_1}} = \Phi_{K_2}^{-1} \circ \Phi_{K_1}$. Since Φ_K is a bi-Lipschitz map for any K, the composition is bi-Lipschitz. Since $Y_K \setminus M_K$ is bi-Lipschitz equivalent to the cone over a square, the corresponding homeomorphism $\widetilde{\Phi}_K : Y_K \setminus M_K \to \widetilde{W}$ is bi-Lipschitz. Set $\Psi|_{Y_{K_1} \setminus M_{K_1}} = \widetilde{\Phi}_{K_2}^{-1} \circ \Phi_{K_1}$. Note that Φ_{K_1}, $\widetilde{\Phi}_{K_1}$, Φ_{K_2} and $\widetilde{\Phi}_{K_2}$ can be chosen so

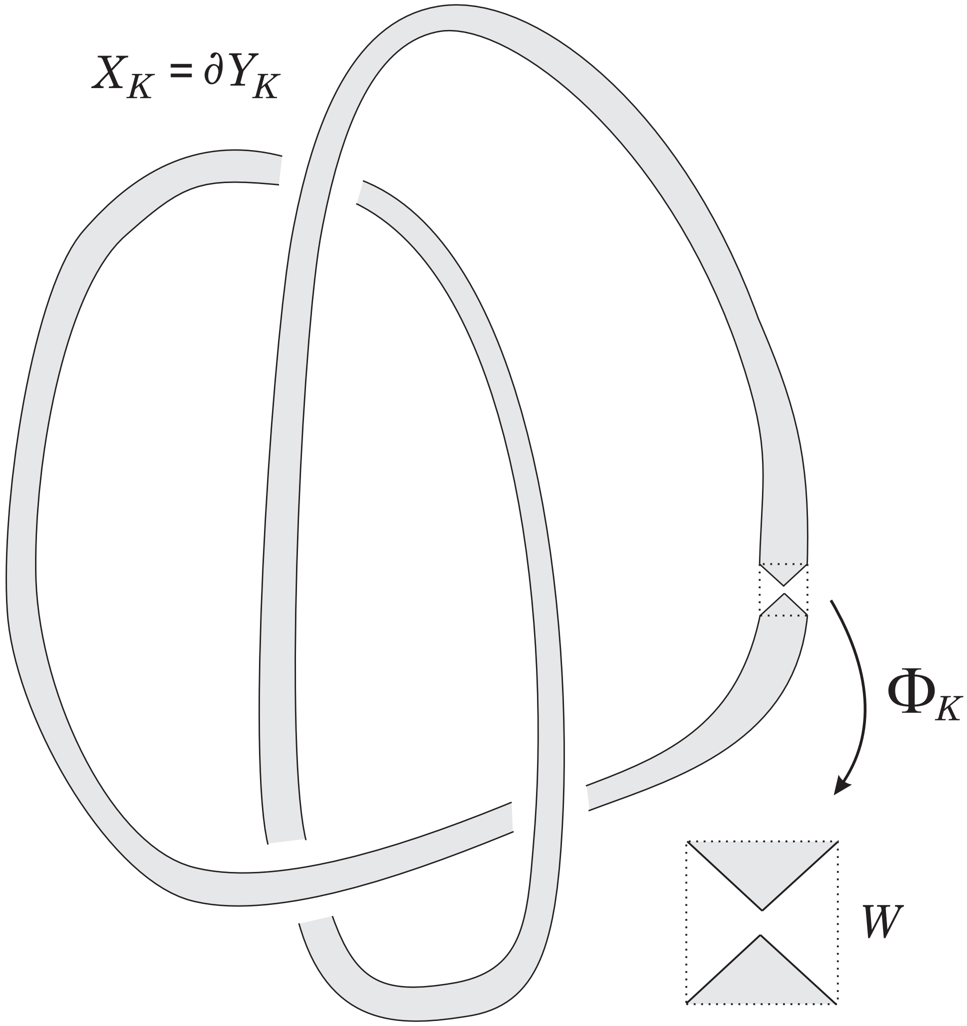


FIGURE 3. The links of the sets $X_K = \partial Y_K$ and W in the proof of Theorem 3.1.

that Ψ is continuous. Since $\Psi(X_{K_1}) = X_{K_2}$ by construction, X_{K_1} and X_{K_2} are outer bi-Lipschitz equivalent.

3) Note that, for any knot K, the link of the tangent cone C_0X_K of the set X_K is the union of two knots isotopic to K, pinched at one point. Thus if K_1 and K_2 are not isotopic, then the tangent cones $C_0X_{K_1}$ and $C_0X_{K_2}$ are not ambient topologically equivalent. This contradicts Sampaio's Theorem (see [6]) which states that the tangent cones of ambient Lipschitz equivalent semialgebraic sets are also ambient Lipschitz equivalent. In our case, the links of the tangent cones are not even ambient topologically equivalent.

Theorem 3.2. *For any knot $K \subset S^3$ and all integers $i \geq 0$, there exist surface germs $(X'_{K,i}, 0)$ in $\mathbb{R}^4$ such that:*

1) *The tangent cones at the origin of all $X'_{K,i}$ are topologically equivalent to the cone over two isotopic copies of K pinched at one point.*

2) *All $X'_{K,i}$ are outer bi-Lipschitz equivalent.*

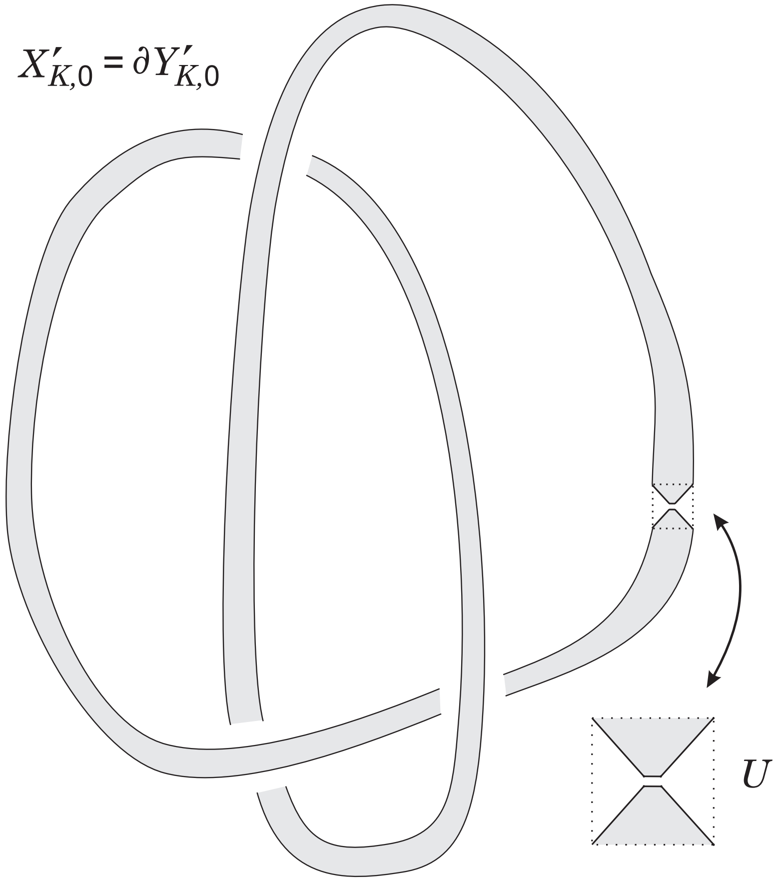


FIGURE 4. The links of the surface $X'_{K,0} = \partial Y'_{K,0}$ and U in the proof of Theorem 3.2.

3) *$X'_{K,i}$ and $X'_{K,j}$ are ambient bi-Lipschitz equivalent only when $i = j$.*

Let $Y'_{K,0}$ be the set obtained by replacing W with U in the construction made in the proof of Theorem 3.1, and let $X'_{K,0}$ be its boundary (see Figure 4). Then the surface germ $(X'_{K,0}, 0)$ satisfies conditions of Theorem 3.1.

Let $F'_{K,i}$ be the set obtained by removing the slice S_k from F_K, making i complete twists and gluing back U (see Figure 5). Let $Y'_{K,i}$ be the set obtained from $F'_{K,i}$ the same way $Y'_{K,0}$ was obtained from F_K, and let $X'_{K,i}$ be its boundary. The same arguments as in the proof of Theorem 3.1 show that the link of $X'_{K,i}$ is a trivial knot and the tangent cone of $X'_{K,i}$ is a cone over the union of two knots isotopic to K, pinched at one point.

We are going to prove that $X'_{K,i}$ and $X'_{K,j}$ are not ambient bi-Lipschitz equivalent if $i \neq j$. For this purpose we define two moves.

Definition 3.3. *Saddle move.* Replace line segments $[p_2(t), p_3(t)]$ and $[p'_2(t), p'_3(t)]$ in each section V_t of U with the segments $[p_2(t), p'_2(t)]$ and $[p_3(t), p'_3(t)]$. Let $S(X'_{K,i})$ be the surface obtained after this move (see Figure 6a). This operation is called *saddle move.*

Definition 3.4. *Crossing move.* Consider the space $\mathbb{R}^3$ with coordinates x, y, t as a subspace $\{z = 0\}$ of the space $\mathbb{R}^4$ with coordinates x, y, t, z. Replace line segments $[p_2(t), p_3(t)]$ and

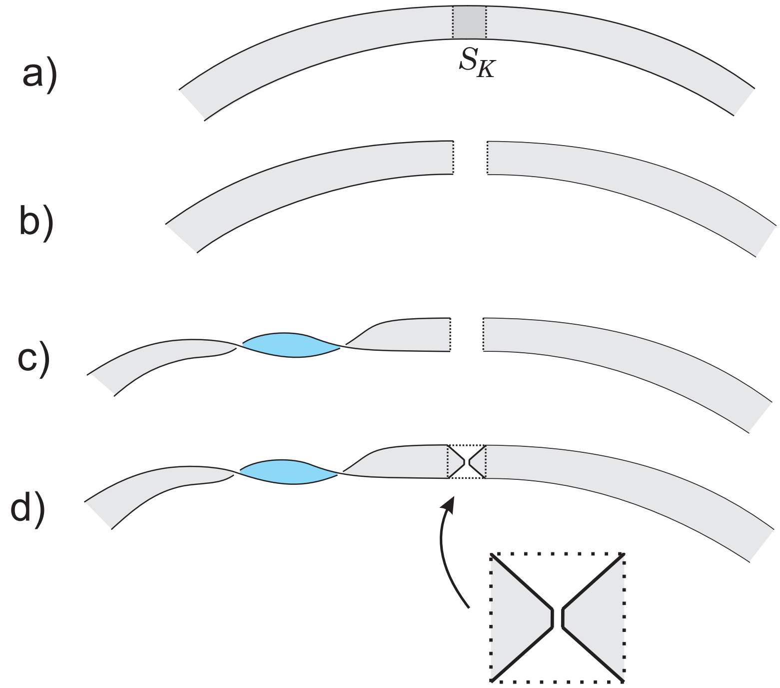


FIGURE 5. Cut and twist in the proof of Theorem 3.2.

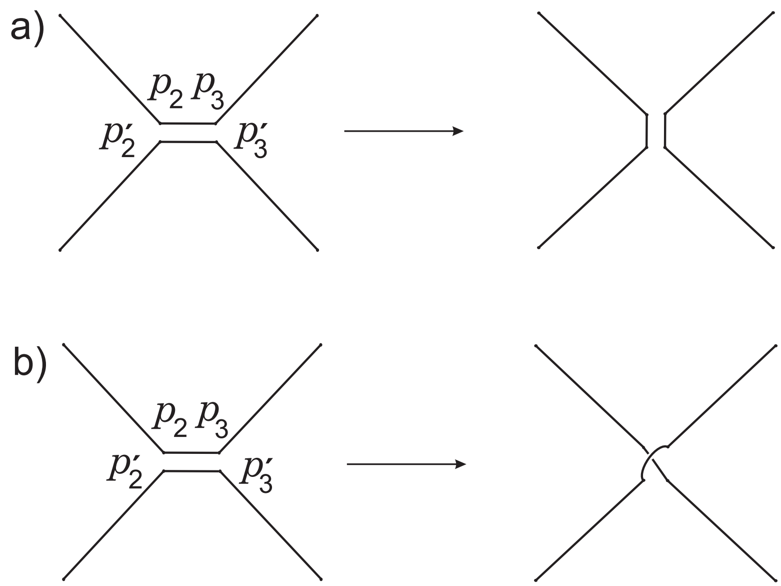


FIGURE 6. Saddle move and crossing move

$[p_2'(t), p_3'(t)]$ in each section U_t of U with the line segment $[p_2(t), p_3'(t)]$ and a circle arc in the half-space $\{z \geq 0\}$, with the ends at $p_2'(t)$ and $p_3(t)$, orthogonal to the plane $\{z = 0\}$. Let $C(X'_{K,i})$ be the surface obtained after this move (see Figure 6b). This operation is called *crossing move.*

Lemma 3.5. *If the surfaces $X'_{K,i}$ and $X'_{K,j}$ are ambient bi-Lipschitz equivalent, then the results of either the saddle move or the crossing move applied to these two surfaces are ambient bi-Lipschitz equivalent.*

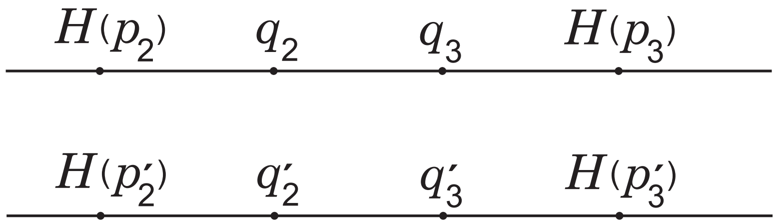


FIGURE 7. The images by the map H

Proof: Let $\widetilde{H} : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$ be a germ of a bi-Lipschitz map, such that $\widetilde{H}(X'_{K,j}) = X'_{K,i}$. By Valette's Theorem (see [7]) there exist another germ of a bi-Lipschitz map $H : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$, such that $H(X'_{K,j}) = X'_{K,i}$ and, moreover, H maps the spheres centered at the origin to the spheres centered at the origin. Let $p_1(t)$, $p_2(t)$, $p'_2(t)$, $p'_3(t)$ be the points described in the proof of Theorem 3.2 for the surface $X'_{K,j}$, and let $q_1(t)$, $q_2(t)$, $q'_2(t)$, $q'_3(t)$ be the corresponding points for $X'_{K,i}$. Note that the pairs of arcs (γ_1, γ_2), such that $tord_{outer}(\gamma_1, \gamma_2) = \beta_2$ and $tord_{inner}(\gamma_1, \gamma_2) = 1$ must be mapped to the pairs of arcs $(\widetilde{\gamma}_1, \widetilde{\gamma}_2)$ such that $tord_{outer}(\widetilde{\gamma}_1, \widetilde{\gamma}_2) = \beta_2$ and $tord_{inner}(\widetilde{\gamma}_1, \widetilde{\gamma}_2) = 1$. Thus there exist a pair of Hölder triangles T_1 and T_2 in $X'_{K,i}$ such that their union is ambient bi-Lipschitz equivalent to the union over $t \geq 0$ of the segments connecting $(x = -t^{\beta_1},\ y = t^{\beta_2})$ with $(x = t^{\beta_1},\ y = t^{\beta_2})$ and $(x = -t^{\beta_1},\ y = -t^{\beta_2})$ with $(x = t^{\beta_1},\ y = -t^{\beta_2})$ in the xy-planes $\{t = \text{const},\ z = 0\}$ (see Figure 7).

Then the homeomorphism H can be replaced by another homeomorphism $H' : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$, such that $H'(p_2(t)) = q_2(t)$, $H'(p_3(t)) = q_3(t)$, $H'(p'_2(t)) = q'_2(t)$, $H'(p'_3(t)) = q'_3(t)$. That is why $H'(S(X'_{K,j})) = S(X'_{K,i})$ and $H'(C(X'_{K,j})) = C(X'_{K,i})$. This proves the lemma.

Proof of the Theorem: The result of the saddle move of $X'_{K,j}$ is a surface such that the tangent link of it is the union of two copies of the knot K. As the linking number of the two copies is equal to the number of complete twists, $X'_{K,j}$ and $X'_{K,i}$ are not ambient bi-Lipschitz equivalent.

Note that the topology of the tangent link of $X'_{K,j}$ does not depend on j. The tangent link is formed by two copies of K pinched at one point. This concludes the proof of Theorem .

The next statement is a modification of the Universality Theorem.

Theorem 3.6. *For any two knots K and L, there exists a germ of a semialgebraic one-bridge surface X_{KL} such that:*

1. The link of X_{KL} at zero is isotopic to L.

2. For a fixed knot K all surface germs X_{KL} have isotopic tangent links. In particular, surface germs X_{K_1L} and X_{K_2L} are ambient bi-Lipschitz equivalent only if the knots K_1 and K_2 are isotopic.

Proof: Consider the surface X_K, constructed in the proof of Theorem 3.1 for the knot K. Take an arc $\gamma \subset X_K$ passing through the origin and situated "far away" from the set W defined in the proof of Theorem 3.1 (i.e., $tord_{outer}(\gamma', \gamma) = 1$ for any $\gamma' \subset W$). Let $V(\gamma)$ be a conical

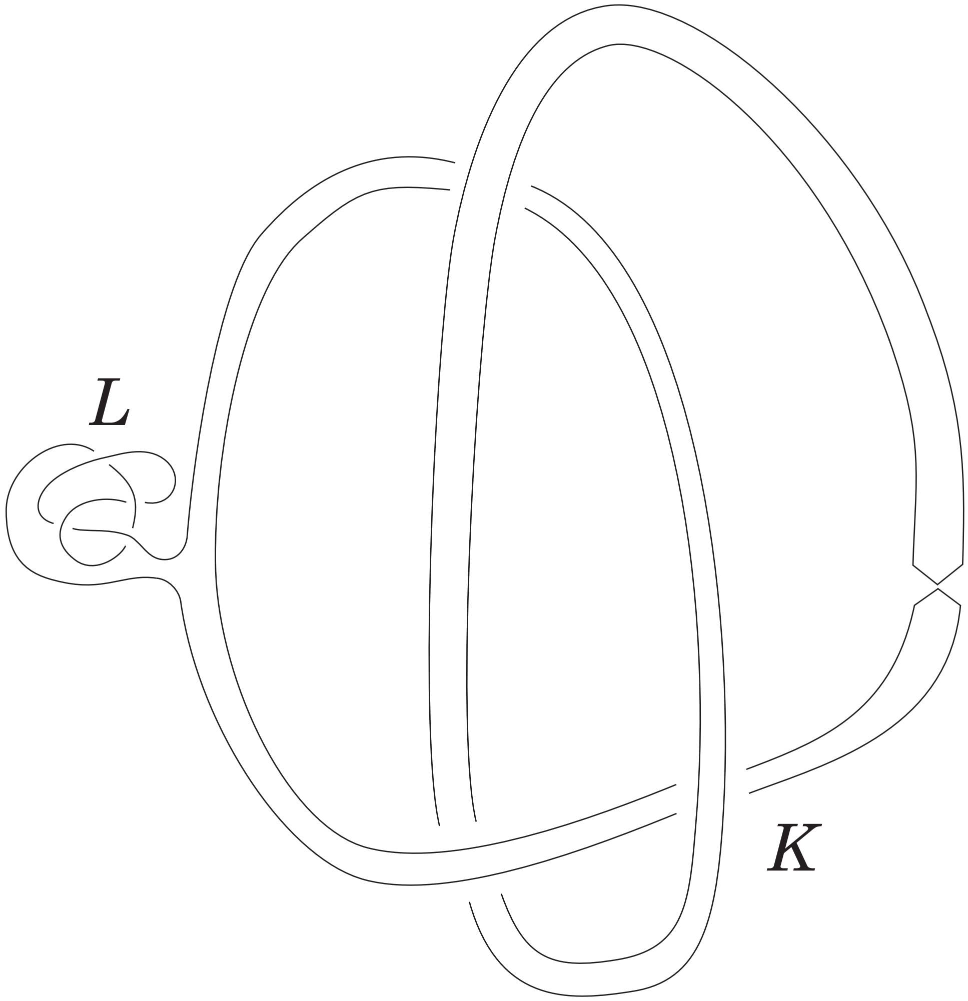


FIGURE 8. Construction of X_{KL}

neighbourhood of γ in $\mathbb{R}^4$. One can embed a singular surface Z_L inside $V(\gamma)$ in such a way that the link of Z_L is ambient topologically equivalent to L.

On each level t, consider connected sum of the link of X_K and the link of Z_L. The result of this move is called X_{KL}. (see Figure 8).

1. Since X_K has a trivial link, the connected sum is isotopic to L.

2. The proof of the fact that the surfaces X_{KL} are outer bi-Lipschitz equivalent for L, being fixed is the same as the all X_K are outer bi-Lipschitz equivalent in the proof of Theorem 2.1.

3. Notice, that X_{KL} is a one-bridge surface and that is why the tangent link at the origin is the two knots, pinched at one point. One of the knots is K and another one is the connected some of K and L. Another knot is sotopic to K. This proves 3. □

The next result is another modification of the Universality Theorem. In contrast to the previous results the metric structure is more complicated then one-bridge.

Theorem 3.7. *For any two knots K and L, there exists a germ of a semialgebraic surface $X_{KL}^{\alpha\beta}$ such that:*

1. The link of $X_{KL}^{\alpha\beta}$ at zero is isotopic to L.

2. For any fixed α and β, all surface germs $X_{KL}^{\alpha\beta}$ are outer bi-Lipschitz equivalent.

3. For a knot K, being fixed the tangent link of $X_{KL}^{\alpha\beta}$ is isotopic to K.

Proof: take $1 \leq \alpha$. Let $F_K \subset S^3$ be a smooth semialgebraic embedded surface diffeomorphic to $S^1 \times [0,1]$, such that the boundary of F_K contains two components $\widetilde{K}$ and $\widetilde{K}'$ isotopic to the same knot K. Let (w,v) be the coordinates in F_K. Let $\widetilde{Y}_K$ be the cone over F_K and let $\widetilde{X}_K$ be the cone over the boundary of F_K. Then (w,v,t) are coordinates in $\widetilde{Y}_K$, where t is the distance to the origin. Let $\widetilde{Y}_K^\alpha$ be a subset of $\widetilde{Y}_K$, defined as follows:
$\widetilde{Y}_K^\alpha = ((w,v,t) :(0 \leq v \leq t^\alpha))$. The set $\widetilde{Y}_K^\beta$ is called *α-saturation* of $F_K \subset S^3$.

Notice that the tangent link of $\widetilde{Y}_K^\alpha$ is isotopic to K.

Take a slice S_K of F_K defined by the coordinates $((w,v,t) :(w_0 - r \leq w \leq w_0 + r))$ for some w_0 and for sufficiently small r. Consider the *α-saturation* of S_K defined as follows: $((w,v,t) :(w_0 - r \leq w \leq w_0 + r), (0 \leq v \leq t^\alpha))$. This *$\alpha$-saturation* of S_K may be removed and can be replaced by $W^{\alpha|beta}$ exactly in the same way, as we did in the proof of the Universality Theorem.

Let $Y_K^{\alpha\beta}$ be the set, obtained by this operation and let $X_K^{\alpha\beta}$ be the boundary of $Y_K^{\alpha\beta}$.

Take an arc $\gamma \subset X_K^\beta$ passing through the origin and situated "far away" from the set W^β defined above (i.e., $tord_{outer}(\gamma',\gamma) = 1$ for any $\gamma' \subset W^\alpha$). Let $V_\beta(\gamma)$ be a β-horn like neighbourhood of γ in $\mathbb{R}^4$. One can embed a singular surface Z_L inside $V_\beta(\gamma)$ in such a way that the link of Z_L is ambient topologically equivalent to L.

On each level t, consider connected sum of the link of $X_K^{\alpha\beta}$ and the link of Z_L. The result of this move is called $X_{KL}^{\alpha\beta}$. (see Figure 8).

1. Since $X_K^{\alpha\beta}$ has a trivial link, the connected sum is isotopic to L.

3. The proof of the fact that the surfaces X_{KL} are outer bi-Lipschitz equivalent for L, being fixed is the same as the all $X_K^{\alpha\beta}$ are outer bi-Lipschitz equivalent in the proof of Theorem 2.1.

3. From the other hand, since Z_L is a subset of a β-horn neighbourhood of γ, it corresponds to a single point in the tangent link. That is why the tangent link of $X_{KL}^{\alpha\beta}$ is the same as the tangent cone of $X_K^{\alpha\beta}$,i.e. isotopic to K. □

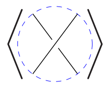
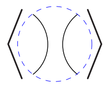
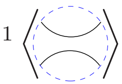
4. Knot Invariants

In this section we are going to work with one bridge germs. The following proposition is a direct consequence of 3.5 and the Theorem of Valette ([7]).

Proposition 4.1. *Let X and Y be two one-bridge germs. If the germs are ambient bi-Lipschitz equivalent, then the links at the origin of the germs are isotopic. Moreover, the links of the origin of $S(X)$ and $S(Y)$ are isotopic and the links at the origin of $C(X)$ and $C(Y)$ are isotopic.*

Let us first recall the definition of the Jones polynomial $J(L)$ of a link L via Kauffmann bracket polynomial $\langle D_L \rangle$, where D_L is a link diagram of L. Kauffmann bracket polynomial [5] is a polynomial in a variable A which is uniquely determined by the following properties:

(1) Kauffmann bracket on the trivial diagram equals one, i.e., $\langle O \rangle = 1$

(2) Skein relation  $= A$  $+ A^{-1}$ 

(3) For any link diagram $D_{L'}$ we have $\langle O \cup D_{L'} \rangle = (-A^2 - A^{-2}) \langle D_{L'} \rangle$

The Jones polynomial of an oriented link L can be defined as

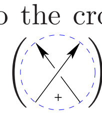
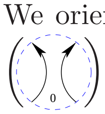
$$J(L) = (-A^3)^{-\omega(D_L)} \langle D_L \rangle ,$$

after the substitution $A = t^{-\frac{1}{4}}$. Here $\omega(D_L)$ is the writhe number of the diagram D_L, i.e., the number of positive crossings minus the number of negative crossings in D_L.

Proposition 4.2. *Let X be a one bridge surface such that the link of X at the origin is a knot K. Let $C(X)$ be a germ obtained from X by the crossing move. Let $K_{C(X)}$ be the link at the origin of $C(X)$. Let Y be a one-bridge germ such that the link at the origin of Y is the same knot K as the link at the origin of X. Let $S(Y)$ be a germ obtained from Y by the saddle move. Suppose that Y is such that the link at the origin of the surface $S(Y)$ is a 2-component link $L_{S(Y)}$. If the Jones polynomial $J(K_{C(X)})$ of the knot $K_{C(X)}$ satisfies*

$$J(K_{C(X)}) \neq -t^{\frac{1}{2}} J(L_{S(Y)}) + (-1)^{\omega(D_{K'}) - \omega(D_K)} t^{\frac{3(\omega(D_{K'}) - \omega(D_K)) + 1}{4}} J(K),$$

then the links at origin of X and Y are not isotopic.

Proof. Let D_K be a diagram of a knot K determined by X and let $D_{K'}$ be a diagram of a knot $K_{C(X)}$ which is determined by $C(X)$. Let us orient D_K in an arbitrary way. We orient $D_{K'}$ so that the the intersection, corresponding to the crossing move (see Figure 6) on the diagram (see Figure 9) is positive, i.e., it looks like . Let $S(X)$ be a germ of a surface obtained from X by a saddle move. Let D_L be the corresponding diagram of the characteristic link $L_{S(X)}$. We orient D_L such that the part, corresponding to the saddle move (see Figure 9) looks like . Before the substitution $A = t^{-\frac{1}{4}}$ we have

$$\langle D_{K'} \rangle = (-A^3)^{\omega(D_{K'})} J(K_{C(X)}) \qquad \langle D_L \rangle = (-A^3)^{\omega(D_L)} J(L_{S(X)}).$$

Now it follows from the condition (2) of the Kauffmann bracket that

$$(-A^3)^{\omega(D_{K'})} J(K_{C(X)}) = A(-A^3)^{\omega(D_L)} J(L_{S(X)}) + A^{-1}(-A^3)^{\omega(D_K)} J(K).$$

Using the fact that $\omega(D_{K'}) = \omega(D_L) + 1$ and after the substitution $A = t^{-\frac{1}{4}}$ we get

$$J(K_{C(X)}) = -t^{\frac{1}{2}} J(L_{S(X)}) + (-1)^{\omega(D_{K'}) - \omega(D_K)} t^{\frac{3(\omega(D_{K'}) - \omega(D_K)) + 1}{4}} J(K). \tag{1}$$

Recall that Lemma 3.5 implies that if the link $L_{S(X)}$ is not isotopic to the link $L_{S(Y)}$, then X and Y are not ambient bi-Lipschitz equivalent. Hence if $J(L_{S(X)}) \neq J(L_{S(Y)})$, then X and Y are not ambient bi-Lipschitz equivalent. Now equality (1) yields the proof of the proposition. $\square$

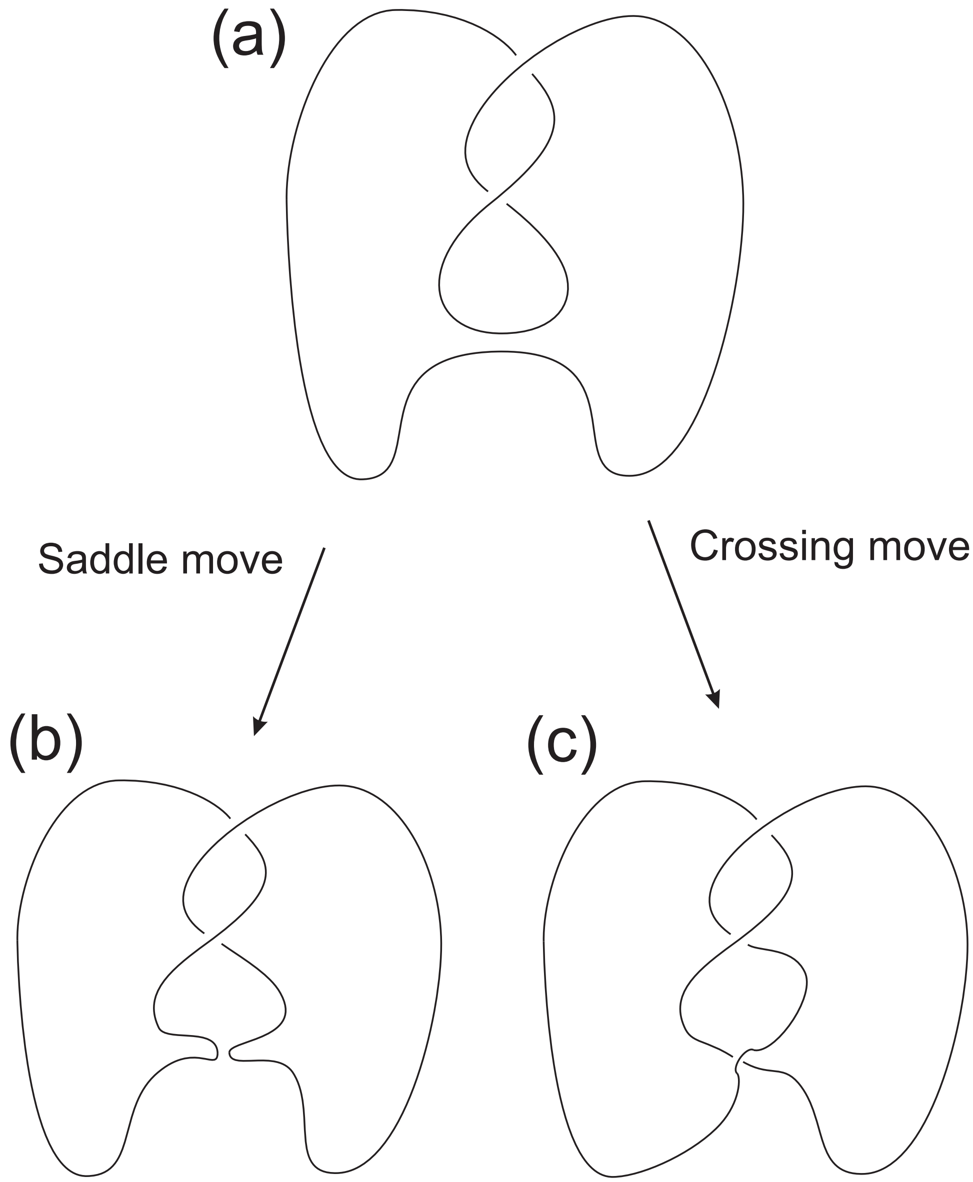


FIGURE 9. The Saddle move and the Crossing move

Corollary 4.3. *If K is a trivial knot and $L_{S(Y)}$ is $(2,2m)$-torus link $L(2,2m)$, where m is a non-negative integer, then we get the following closed formula: If the Jones polynomial $J(K_{C(X)})$ of the knot $K_{C(X)}$ satisfies*

$$J(K_{C(X)}) \neq t^m + t^{m+2}\left(\frac{1+t^{2m-1}}{1+t}\right) + (-1)^{\omega(D_{K'})-\omega(D_K)}\, t^{\frac{3(\omega(D_{K'})-\omega(D_K))+1}{4}}, \tag{2}$$

then X and Y are not ambient bi-Lipschitz equivalent.

Proof. Recall that for each n the Jones polynomial of the torus knot $K(2,2n+1)$ equals to

$$J(K(2,2n+1)) = t^n \frac{1-t^3-t^{2n+2}+t^{2n+3}}{1-t^2},$$

see e.g. [4]. The skein relation for the Jones polynomial together with the above equality yield

$$J(L(2,2m)) = -t^{\frac{2m-1}{2}} - t^{\frac{2m+3}{2}}\left(\frac{1+t^{2m-1}}{1+t}\right). \tag{3}$$

Noting that if K is a trivial knot, then its Jones polynomial $J(K) = 1$, and applying equalities (1) and (3) we obtain the proof of the corollary. □

Remark 4.4. The above theorem has two advantages: it has a computational value, and as its immediate corollary we obtain the main result of Birbrair-Gabrielov [2, Theorem 4.1]. Let us illustrate this on the following example. Let X be such that it determines a knot diagram D_K which has no intersections, and after the crossing move the diagram $D_{K'}$ has exactly one positive intersection. It follows that $\omega(D_{K'}) - \omega(D_K) = 1$, and $J(K_{C(X)}) = 1$ since $K_{C(X)}$ is a trivial knot. Let Y be such that it determines a trivial knot diagram presented in Figure 9a. The diagram of the link $L_{S(Y)}$ is presented in Figure 9b. Note that it is a $(2,2)$-torus link (Hopf link). The diagram of the knot $K_{C(Y)}$ is presented in Figure 9c. Note that it is a trefoil knot. Noting that $m = 1$ the right hand side of equation (2) equals to t^3. Hence $J(K_{C(X)}) \neq t^3$ and thus X and Y are not ambient bi-Lipschitz equivalent.

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