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ON QUASI-MORPHISMS FROM KNOT AND BRAID INVARIANTS

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ABSTRACT

We study quasi-morphisms on the groups P_n of pure braids on n strings and on the group $\mathcal{D}$ of compactly supported area-preserving diffeomorphisms of an open two-dimensional disk. We show that it is possible to build quasi-morphisms on P_n by using knot invariants which satisfy some special properties. In particular, we study quasi-morphisms which come from knot Floer homology and Khovanov-type homology. We then discuss possible variations of the Gambaudo–Ghys construction, using the above quasi-morphisms on P_n to build quasi-morphisms on the group $\mathcal{D}$ of diffeomorphisms of a 2-disk.

Keywords: Quasi-morphisms; braid groups; knot concordance invariants; area-preserving diffeomorphisms.

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1. Introduction

Real-valued quasi-morphisms are known to be a helpful tool in the study of algebraic structure of non-Abelian groups, especially the ones that admit a few or no (linearly independent) real-valued homomorphisms. Recall that a *quasi-morphism* on a group G is a function $\varphi : G \to \mathbb{R}$ which satisfies the homomorphism equation up to a bounded error: there exists $K_\varphi > 0$ such that

$$|\varphi(ab) - \varphi(a) - \varphi(b)| \leq K_\varphi$$

for all $a, b \in G$. A quasi-morphism φ is called *homogeneous* if $\varphi(a^m) = m\varphi(a)$ for all $a \in G$ and $m \in \mathbb{Z}$. Any quasi-morphism φ can be *homogenized*: setting

$$\widetilde{\varphi}(a) := \lim_{k \to +\infty} \varphi(a^k)/k \tag{1.1}$$

we get a homogeneous (possibly trivial) quasi-morphism $\widetilde{\varphi}$.

In [13], Gambaudo and Ghys showed that a signature invariant of links in S^3 defines quasi-morphisms on the full braid groups B_n on n strings. These quasi-morphisms on B_n are constructed in the following way: close up a braid to a link in the standard way and take the value of the signature invariant on that link. In this paper, we are motivated by the following question: "What happens if one plugs in a general knot/link invariant in this construction?" In this paper, we give a sufficient condition which is satisfied by some already known knot invariants. More specifically we show that any homomorphism from the concordance group of knots in S^3 to the reals, which is bounded in some canonical norm on this group, defines a quasi-morphism on B_n. Known knot invariants which satisfy the above condition *do not produce new quasi-morphisms* on B_n. However, this condition *may possibly* lead to new examples of quasi-morphisms.

We consider three specific remarkable knot/link invariants: the Rasmussen link invariant s [5, 32], which comes from a Khovanov-type theory, the Ozsvath–Szabo knot invariant τ [27], which comes from the knot Floer homology, and the classical signature link invariant "sign" [20, 33]. In [2] Baader has shown that the Rasmussen link invariant s defines a quasi-morphism on B_n. We show that the homogenization of this quasi-morphism is equal to the classical linking number homomorphism lk on B_n. We also show that the situation with the Ozsvath–Szabo knot invariant τ is similar: it defines a quasi-morphism on B_n and its homogenization is again the linking number homomorphism divided by 2. In addition, we show that the homogenization of an induced signature quasi-morphism on B_n and lk coincide on alternating braids. We also present an inequality which connects s, τ and the braid index of a knot.

Further, we discuss the group $\mathcal{D}$ of compactly supported area-preserving diffeomorphisms of the open unit disk in the Euclidean plane. The group $\mathcal{D}$ admits a unique (continuous, in the proper sense) homomorphism to the reals — the famous Calabi homomorphism (see [3, 8, 12]). At the same time $\mathcal{D}$ is known to admit many (linearly independent) homogeneous quasi-morphisms (see [4, 6, 13]). In this work we consider a particular geometric construction of such quasi-morphisms, essentially contained in [13], which produces quasi-morphisms on $\mathcal{D}$ from quasi-morphisms on the pure braid groups P_n. We discuss the computation of the quasi-morphisms on $\mathcal{D}$, obtained by this construction, on diffeomorphisms generated by time-independent (compactly supported) Hamiltonians. For a generic Hamiltonian H of this sort we present the result of the computation in terms of the Reeb graph of H and the integral of the push-forward of H to the graph against a certain signed measure on the graph. This result enables us to show that the Calabi homomorphism and the quasi-morphism on $\mathcal{D}$ induced by the signature invariant of n-component links are asymptotically equivalent, as $n \to \infty$, on the flows generated by time-independent (compactly supported) Hamiltonians.

Plan of the paper. In Sec. 2, we formulate sufficient conditions for a knot invariant to yield a quasi-morphism on B_n. In Sec. 3, we give examples of such knot invariants and provide properties of homogeneous quasi-morphisms, defined by them, on B_n. In Sec. 4, we discuss the Gambaudo–Ghys construction, which produces homogeneous quasi-morphisms on $\mathcal{D}$ from homogeneous quasi-morphisms on P_n. We define a set of generic autonomous Hamiltonians, and discuss the computation of the induced quasi-morphisms on $\mathcal{D}$ on the elements generated by these Hamiltonians. At the end we discuss the asymptotic behavior of the induced signature quasi-morphism.

2. Quasi-Morphisms on Braid Groups Defined by Knot Invariants

It is shown in [16] that the full braid group B_n admits infinitely many linearly independent homogeneous quasi-morphisms for every integer $n > 2$. However, none of these quasi-morphisms are constructed geometrically. In [13], Gambaudo and Ghys gave an explicit geometric construction of a family of quasi-morphisms $\mathbf{sign}_n$ on groups B_n defined as follows:

$$\mathbf{sign}_n(\alpha) := \mathrm{sign}(\widehat{\alpha}),$$

where "sign" is a signature link invariant (see Sec. 3) and $\widehat{\alpha}$ is the link in S^3 which is obtained in the natural way from α (see Fig. 1). In this section, we show that knot invariants of certain type define quasi-morphisms on B_n in a similar way.

Remark 2.1. Another family of quasi-morphisms on B_n (one for each n) was constructed recently by Malyutin in [22]. These quasi-morphisms are constructed using different methods, in particular they are not defined by knot/link invariants, and will not be discussed in this paper.

Let us recall some useful notions from knot theory. Let K be a knot in S^3. The *four-ball genus* $g_4(K)$ of K is the minimal genus of an oriented surface with boundary which is smoothly embedded in $\mathbb{D}^4$ such that the image of its boundary under this embedding is the knot $K \subset S^3 = \partial\mathbb{D}^4$. We denote by K^* the mirror image of K, and by $-K$ the same knot K with the reversed orientation. For any knot K', the knot $K\#K'$ represents the connected sum of K and K'. A knot K is

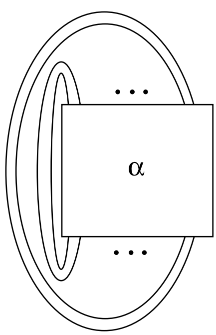

Fig. 1. Closure $\widehat{\alpha}$ of a braid α.

called *slice* if $g_4(K) = 0$. We say that K is concordant to K', if there exists a smooth embedding $\iota : S^1 \times [0,1] \hookrightarrow S^3 \times [0,1]$ such that $\iota(S^1 \times \{0\}) = K$ and $\iota(S^1 \times \{1\}) = K'$. We denote by $\mathrm{Conc}(S^3)$ the Abelian group whose elements are equivalence classes of knots in S^3, and the multiplication is the connected sum operation. It is easy to see that the multiplication is well-defined. The following lemma is a well-known fact in knot theory, see [20, 33].

Lemma 2.2. *For any knot K the knot $K \# -K^*$ is a slice knot.*

Main Theorem. *One of the ways to find a quasi-morphism on B_n is to find an $\mathbb{R}$-valued invariant I of (isotopy classes of) links in S^3 so that*

$$|I(\widehat{\alpha\beta}) - I(\widehat{\alpha}) - I(\widehat{\beta})| \leq K_I,$$

where $K_I \geq 0$ depends only on I. We will now describe certain ways of closing braids into knots.

Lemma 2.3. *Let $\beta \in B_n$. Then there exists a braid $\alpha_\beta \in B_n$ which satisfies the following properties*:

(1) *The closure of $\alpha_\beta\beta$ is a knot.*
(2) *The closure of α_β is a k-component unlink for some $1 \leq k \leq n$.*

We will say that such a braid α_β is a *completing braid for β*. Let $\{\sigma_i\}_{i=1}^{n-1}$ be the standard (Artin) generators of B_n.

Proof. Let $\beta \in B_n$. Then there exists a braid α such that $\widehat{\alpha\beta}$ is a knot and $\widehat{\alpha}$ is a k-component link for some $1 \leq k \leq n$. We write α as a product of generators $\sigma_i^{\pm 1}$ and perform a sequence of crossing changes (i.e. replacing *some* of the generators in the product by their inverses) until we are left with a braid α_β such that $\widehat{\alpha\beta} = \widehat{\alpha_\beta\beta}$ is a knot and $\widehat{\alpha_\beta}$ is the k-component unlink. □

Definition 2.4. Let I be a real-valued knot invariant. Let us fix some choices of completing braids α_β for every $\beta \in B_n$. Define a function

$$\widehat{I} : B_n \to \mathbb{R},$$

by $\widehat{I}(\beta) := I(\widehat{\alpha_\beta\beta})$.

We will show that under certain conditions $\widehat{I}$ is a quasi-morphism.

Remark 2.5. Note that our function $\widehat{I} : B_n \to \mathbb{R}$ is slightly different from an analogous quasi-morphism defined by Gambaudo–Ghys. Their definition $(\widehat{\mathbf{I}}(\beta) := I(\widehat{\beta}))$ requires I to be a real-valued *link* invariant, but in our definition we require I to be only a real-valued *knot* invariant. For example the Ozsvath–Szabo τ invariant is defined only for knots. In case I is a real-valued link invariant defining a quasi-morphism $\widehat{I}$, the quasi-morphisms $\widehat{I}$ and $\widehat{\mathbf{I}}$ differ by a constant which depends only on n, hence their homogenizations are equal.

Theorem 2.6. *Suppose that a real-valued knot invariant I defines a homomorphism*

$$I : \mathrm{Conc}(S^3) \to \mathbb{R},$$

such that $|I(K)| \leq c_1 g_4(K)$, where c_1 is a real positive constant independent of K. Then $\widehat{I}$ is a quasi-morphism on B_n. Moreover, for a different set of completing braids α_β for every $\beta \in B_n$ we get a (possibly different) quasi-morphism on B_n such that the absolute value of its difference with $\widehat{I}$ is bounded from above by a constant depending only on n and therefore the homogenizations of the two quasi-morphisms are equal.

Proof. Take any $\beta, \gamma \in B_n$ and let $\alpha_\beta, \alpha_\gamma, \alpha_{\beta\gamma}$ be the chosen completing braids for β, γ, and $\beta\gamma$.

Lemma 2.7. *There exists a cobordism S between the knots*

$$(-\widehat{\alpha_{\beta\gamma}\beta\gamma})^* \# (\widehat{\alpha_\beta\beta} \# \widehat{\alpha_\gamma\gamma}) \quad \text{and} \quad (-\widehat{\alpha_{\beta\gamma}\beta\gamma})^* \# \widehat{\alpha_{\beta\gamma}\beta\gamma}$$

such that $\chi(S) \geq -6n$.

Proof. If T is a cobordism between two links L and L', we will write $L \overset{T}{\sim} L'$.

By an observation of Baader (see [2, Sec. 4]) for any braids $\mu, \nu \in B_n$

$$\widehat{\mu\nu} \overset{T}{\sim} \widehat{\mu} \sqcup \widehat{\nu},$$

where $\chi(T) = -n$. Also note that since $\widehat{\alpha_{\beta\gamma}}, \widehat{\alpha_\beta}, \widehat{\alpha_\gamma}$ are unlinks with no more than n components, we have

$$\widehat{\alpha_{\beta\gamma}} \overset{S_1}{\sim} \widehat{\alpha_\beta} \sqcup \widehat{\alpha_\gamma},$$

where $\chi(S_1) \geq 1 - 2n$. Therefore

$$\widehat{\alpha_{\beta\gamma}\beta\gamma} \overset{S_2}{\sim} \widehat{\alpha_{\beta\gamma}} \sqcup \widehat{\beta} \sqcup \widehat{\gamma} \overset{S_3}{\sim} \widehat{\alpha_\beta} \sqcup \widehat{\alpha_\gamma} \sqcup \widehat{\beta} \sqcup \widehat{\gamma} \overset{S_4}{\sim} \widehat{\alpha_\beta\beta} \sqcup \widehat{\alpha_\gamma\gamma} \overset{S_5}{\sim} \widehat{\alpha_\beta\beta} \# \widehat{\alpha_\gamma\gamma},$$

where $\chi(S_2) = -2n$, $\chi(S_3) \geq 1 - 2n$ (the cobordism S_3 is the disjoint union of S_1 and the trivial cobordism over $\widehat{\beta} \sqcup \widehat{\gamma}$ given by a disjoint union of cylinders), $\chi(S_4) = -2n$ and $\chi(S_5) = -1$ (S_5 is a saddle cobordism between the disjoint union of two knots and their connected sum). Thus

$$\widehat{\alpha_{\beta\gamma}\beta\gamma} \overset{S_6}{\sim} \widehat{\alpha_\beta\beta} \# \widehat{\alpha_\gamma\gamma}$$

and hence

$$(-\widehat{\alpha_{\beta\gamma}\beta\gamma})^* \# \widehat{\alpha_{\beta\gamma}\beta\gamma} \overset{S_7}{\sim} (-\widehat{\alpha_{\beta\gamma}\beta\gamma})^* \# (\widehat{\alpha_\beta\beta} \# \widehat{\alpha_\gamma\gamma}),$$

where

$$\chi(S_6) = \chi(S_7) = \chi(S_2) + \chi(S_3) + \chi(S_4) + \chi(S_5) \geq -6n,$$

as required. □

Let us now finish the proof of the theorem. Lemma 2.2 implies that the knot

$$(-\widehat{\alpha_{\beta\gamma}\beta\gamma}^{*})\#\widehat{\alpha_{\beta\gamma}\beta\gamma}$$

is a slice knot. Therefore using Lemma 2.7, we get

$$g_4(-(\widehat{\alpha_{\beta\gamma}\beta\gamma}^{*})\#(\widehat{\alpha_\beta\beta}\#\widehat{\alpha_\gamma\gamma})) \leq 3n.$$

This yields

$$|I(-(\widehat{\alpha_{\beta\gamma}\beta\gamma}^{*})\#(\widehat{\alpha_\beta\beta}\#\widehat{\alpha_\gamma\gamma}))| \leq c_1 \cdot 3n.$$

Applying the equalities

$$I(-K^*) = -I(K), \quad I(K\#K') = I(K) + I(K'), \tag{2.1}$$

we get

$$|\widehat{I}(\beta\gamma) - \widehat{I}(\beta) - \widehat{I}(\gamma)| \leq 3c_1 n,$$

which means that $\widehat{I}$ is a quasi-morphism.

Finally note that for any two choices α_β and α'_β of completing braids for $\beta \in B_n$ one can show, similarly to the proof of Lemma 2.7, that

$$g_4((\widehat{\alpha_\beta\beta})\# - (\widehat{\alpha'_\beta\beta})^*) \leq c_2 n,$$

for some positive constant c_2 independent of $\beta, \alpha_\beta, \alpha'_\beta$. Hence by Eq. (2.1) we get

$$|I(\widehat{\alpha_\beta\beta}) - I(\widehat{\alpha'_\beta\beta})| = |I((\widehat{\alpha_\beta\beta})\# - (\widehat{\alpha'_\beta\beta})^*)| \leq c_1 \cdot c_2 n.$$

Thus different choices of completing braids yield (possibly different) quasi-morphisms on B_n whose difference is bounded in absolute value from above by a constant depending only on n and therefore their homogenizations are equal. □

Remark 2.8. Note that g_4 defines a semi-norm on $\mathrm{Conc}(S^3)$ and Theorem 2.6 can be reformulated as follows: each element of $\mathrm{Hom}(\mathrm{Conc}(S^3), \mathbb{R})$, which is Lipshitz with respect to the semi-norm defines a quasi-morphism on B_n.

3. Examples, Properties, and Applications

In this section, we discuss the following knot/link invariants: link ω-signatures, the Rasmussen link invariant s and the Ozsvath–Szabo knot invariant τ.

First, let us recall a few definitions and notations concerning braids and quasi-morphisms. The *braid length* $l(\gamma)$ of $\gamma \in B_n$ is the length of the shortest word representing γ with respect to the generators $\sigma_1, \ldots, \sigma_{n-1}$. Throughout the paper, the induced homogeneous quasi-morphism obtained by the homogenization of a quasi-morphism φ (see (1.1)) will be denoted by $\widetilde{\varphi}$.

3.1. *Signature and ω-signature quasi-morphisms*

Let L be an oriented link in S^3, then there exists an oriented surface Σ_L with boundary L. It is called a *Seifert surface* of L. We choose a basis $\{b_1, \ldots, b_{2g+|L|-1}\}$ in $H_1(\Sigma_L, \mathbb{Z})$ and define a symmetric bilinear form on $H_1(\Sigma_L, \mathbb{Z})$ as follows:

$$\Omega(b_i, b_j) = \mathrm{lk}(b_i, b_j^+) + \mathrm{lk}(b_j, b_i^+),$$

where lk is the linking number and b_i^+ is a push-off of the curve, which represents b_i in Σ_L, from Σ_L along the positive normal direction to Σ_L. Tensoring by $\mathbb{R}$, we get a symmetric bilinear form on $H_1(\Sigma_L, \mathbb{R})$. The signature of this form is independent of the choices of Σ_L and the basis of $H_1(\Sigma_L, \mathbb{Z})$ (see [20, 26]). Thus it is an invariant of L and is denoted by $\mathrm{sign}(L)$.

For any complex number $\omega \neq 1$ and link L there exists the ω-signature link invariant $\mathrm{sign}_\omega(L)$, such that $\mathrm{sign}_{-1}(L) = \mathrm{sign}(L)$. It is defined as follows. We tensor the bilinear form Ω by $\mathbb{C}$, and we get a bilinear form on $H_1(\Sigma_L, \mathbb{C})$. The signature of the following hermitian form

$$\Omega_\omega(b_i, b_j) = (1 - \omega)\Omega(b_i, \overline{b_j}) + (1 - \overline{\omega})\Omega(\overline{b_j}, b_i)$$

on $H_1(\Sigma_L, \mathbb{C})$ is independent of the choice of the Seifert surface Σ_L and the basis for $H_1(\Sigma_L, \mathbb{Z})$ (see [20]). Thus $\mathrm{sign}_\omega(L)$ is a link invariant for each $\omega \neq 1$.

In [14], Gambaudo and Ghys showed that sign_ω defines a quasi-morphism on B_n. Alternatively, sign_ω satisfies conditions of Theorem 2.6 for each ω (see [20]). Hence we have the following.

Corollary 3.1. *Both* $\widehat{\mathrm{sign}} : B_n \to \mathbb{R}$ *and* $\widehat{\mathrm{sign}}_\omega : B_n \to \mathbb{R}$ *are quasi-morphisms.*

The induced homogeneous quasi-morphisms on B_n are denoted by $\widetilde{\mathrm{sign}}$ and $\widetilde{\mathrm{sign}}_\omega$ respectively. Following [13] we denote by $\mathrm{lk} : B_n \to \mathbb{Z}$ the unique (up to the multiplication by a constant) homomorphism from B_n to $\mathbb{Z}$ by setting $\mathrm{lk}(\sigma_i^{\pm 1}) = \pm 1$. Let us recall the following definition.

Definition 3.2. A link diagram is called *alternating* if the crossings alternate under, over, under, over, and so on as one travels along each component of the link. A link is called *alternating* if it has an alternating diagram. A braid $\alpha \in B_n$ is called *alternating* if its closure $\widehat{\alpha}$ is an alternating link diagram.

Proposition 3.3. *Let* $\gamma \in B_n$ *be an alternating braid, then*

$$\widetilde{\mathrm{sign}}(\gamma) = \mathrm{lk}(\gamma).$$

The proof of this proposition will be given in the next subsection.

Remark 3.4. One can easily show that $\widetilde{\mathrm{sign}}$ is not a homomorphism for each $n > 2$.

3.2. *Rasmussen quasi-morphism*

The Rasmussen invariant $s(K)$ of a knot K in S^3 was discovered by Jacob Rasmussen in 2004 (see [32]). It comes from the Lee theory [19] which is closely related to the Khovanov homology [18]. This is a very powerful knot invariant which was used by Rasmussen in [32] to give a first combinatorial proof of the Milnor conjecture. This invariant was extended to links in [5].

Before we list the properties of s let us recall a notion of the Seifert algorithm. Let D_L be a diagram of an oriented link L. Let us smoothen each crossing in D_L as shown in Fig. 2. The resulting smoothed diagram $\widehat{D}_L$ consists of oriented simple closed curves, which are called the *Seifert circles*. Thus $\widehat{D}_L$ is the boundary of a union of disjoint disks. We join these disks together with half-twisted strips corresponding to the crossings in the diagram. This yields an oriented surface bounded by L. If this surface is disconnected, then we connect its components by the connected sum operation. The above algorithm is called the *Seifert algorithm*.

We denote by K_+ and K_- the knots which differ by a single crossing change: from a positive crossing in K_+ to a negative one in K_- (positive/negative crossings are shown on the left/right in Fig. 2). Now we list eight properties of the Rasmussen link invariant s. The first seven were proved in [32] and the last one was proved in [2].

$$s(K) = s(-K), \tag{3.1}$$

$$s(K\#K') = s(K) + s(K'), \tag{3.2}$$

$$s(K^*) = -s(K), \tag{3.3}$$

$$|s(K)| \leq 2g_4(K), \tag{3.4}$$

$$s(K_-) \leq s(K_+) \leq s(K_-) + 2. \tag{3.5}$$

For an alternating knot K we have

$$s(K) = \text{sign}(K). \tag{3.6}$$

For a positive knot K we have

$$s(K) = w(D_K) - o(D_K) + 1, \tag{3.7}$$

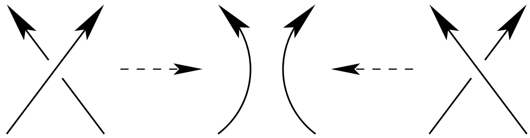

Fig. 2. Smoothing of a crossing.

where D_K is some positive diagram of K, $w(D_K)$ is the number of positive crossings in D_K minus the number of negative crossings in D_K, and $o(D_K)$ is the number of Seifert circles. For any knot K we have

$$1 + w(D_K) - o(D_K) \leq s(K) \leq -1 + w(D_K) + o(D_K), \tag{3.8}$$

where D_K is any diagram of K. It is shown in [2] that (3.8) follows from (3.3), (3.5) and (3.7).

In [2] Baader proved that the Rasmussen link invariant s induces a quasi-morphism on the braid group B_n. Alternatively, note that properties (3.1)–(3.4) imply that s satisfies conditions of Theorem 2.6 and hence $\widehat{s}: B_n \to \mathbb{R}$ is a quasi-morphism.

Theorem 3.5. *Suppose that* $\widehat{I}: B_n \to \mathbb{R}$ *is a quasi-morphism defined by a real-valued knot invariant* I, *which satisfies property* (3.8), *where* s *is substituted by* I. *Then* $\widetilde{I} = \mathrm{lk}$.

Proof. Take $\beta \in B_n$. Then for all $p > 0$ there exists a completing braid α_p in B_n, such that $\widehat{\alpha_p \beta^p}$ is a knot and $|\,\mathrm{lk}(\alpha_p)| \leq M(n)$, where $M(n)$ is some real positive constant which depends only on n. For example, if $\beta \in P_n$, then we can take $\alpha_p = \sigma_1 \cdot \ldots \cdot \sigma_{n-1}$ for all $p \in \mathbb{N}$. Note that

$$w(D_{\widehat{\alpha_p \beta^p}}) = \mathrm{lk}(\alpha_p \beta^p) \quad \text{and} \quad o(D_{\widehat{\alpha_p \beta^p}}) = n.$$

The second equation follows from the fact that for any chosen orientation on the knot $\widehat{\alpha_p \beta^p}$ all strands in $\alpha_p \beta^p$ are oriented from the bottom to the top, or from the top to the bottom (this depends on the orientation) and hence the number of Seifert circles is n. Property (3.8) yields

$$1 + \mathrm{lk}(\alpha_p \beta^p) - n \leq I(\widehat{\alpha_p \beta^p}) \leq -1 + \mathrm{lk}(\alpha_p \beta^p) + n.$$

Hence

$$\lim_{p\to\infty} \frac{1}{p}(1 + \mathrm{lk}(\alpha_p \beta^p) - n) \leq \lim_{p\to\infty} \frac{1}{p} I(\widehat{\alpha_p \beta^p}) \leq \lim_{p\to\infty} \frac{1}{p}(-1 + \mathrm{lk}(\alpha_p \beta^p) + n).$$

Therefore $\widetilde{I}(\beta) = \lim_{p\to\infty} \frac{1}{p} \widehat{I}(\beta^p) = \lim_{p\to\infty} \frac{1}{p} I(\widehat{\alpha_p \beta^p}) = \mathrm{lk}(\beta)$. □

Corollary 3.6. *Each* $\alpha \in B_n$ *satisfies* $\widetilde{s}(\alpha) = \mathrm{lk}(\alpha)$.

Now we are ready to prove Proposition 3.3.

Proof of Proposition 3.3. For each $p \in \mathbb{N}$ the braid γ^p is alternating and there exists an alternating braid α_p such that $l(\alpha_p) < M(n)$, where l is the braid length, $\widehat{\alpha_p \gamma^p}$ is an alternating knot and $M(n)$ is some real positive constant which depends only on n. Hence

$$\widetilde{\text{sign}}(\gamma) = \lim_{p\to\infty} \frac{\text{sign}(\widehat{\alpha_p\gamma^p})}{p} = \lim_{p\to\infty} \frac{s(\widehat{\alpha_p\gamma^p})}{p} = \lim_{p\to\infty} \frac{s(\widehat{\gamma^p})}{p} = \widetilde{s}(\gamma) = \text{lk}(\gamma),$$

where the second equality follows from (3.6). □

In [2] Baader defined the following quasi-morphism:

$$s - \text{lk} + n - 1 : B_n \to \mathbb{R},$$

where $s(\beta) := s(\widehat{\beta})$ (Gambaudo–Ghys definition). This quasi-morphism maps all positive braids to zero. Therefore it descends to a quasi-morphism on $B_n/\langle\Delta_n\rangle$, where $\Delta_n = (\sigma_1\sigma_2\cdot\ldots\cdot\sigma_{n-1})^n$. In [2] Baader asked whether this quasi-morphism is bounded for $n > 2$. An immediate corollary of Theorem 3.5 gives a positive answer to this question.

Corollary 3.7. *The quasi-morphism* $s - \text{lk} + n - 1 : B_n \to \mathbb{R}$ *is bounded.*

3.3. *Connection between τ, s and the braid index*

In [27] Ozsvath and Szabo defined a knot invariant τ which comes from the knot Floer homology [23, 28, 31]. This invariant satisfies in particular the following properties:

$$\tau(K^*) = -\tau(K), \tag{3.9}$$

$$\tau(K\#K') = \tau(K) + \tau(K'), \tag{3.10}$$

$$\tau(-K) = \tau(K), \tag{3.11}$$

$$|\tau(K)| \leq g_4(K), \tag{3.12}$$

$$0 \leq \tau(K_+) - \tau(K_-) \leq 1, \tag{3.13}$$

$$\tau(\widehat{\alpha}) = \frac{\text{lk}(\alpha) - n + 1}{2}, \tag{3.14}$$

where $\alpha \in B_n$ is a positive braid (i.e. it may be written as a product of positive powers of standard generators $\sigma_1, \ldots, \sigma_{n-1}$ of B_n). The first five properties were proved in [27], and the last one was proved in [21]. The invariants s and 2τ share similar properties and coincide on positive and alternating knots. It was conjectured by Rasmussen [32] that they are equal. This conjecture was disproved by Hedden and Ording [17]. In this subsection we show that $\widetilde{s} = 2\widetilde{\tau}$, and that the absolute value of the difference between 2τ and s is bounded by twice the braid index of the knot.

Lemma 3.8. *Let* $\beta \in B_n$ *such that* $\widehat{\beta}$ *is a knot. Then*

$$\text{lk}(\beta) - n + 1 \leq 2\tau(\widehat{\beta}) \leq \text{lk}(\beta) + n - 1. \tag{3.15}$$

Proof. The proof of the lower bound in (3.15) is exactly the same as in [34], where s is substituted by 2τ. We present it for the reader's convenience. Let K be any knot in S^3. Recall that

$$0 \le 2\tau(K_+) - 2\tau(K_-) \le 2, \tag{3.16}$$

$$\tau(K^*) = -\tau(K). \tag{3.17}$$

Let $\alpha \in B_n$ be any positive braid such that $\widehat{\alpha}$ is a knot. It follows from property (3.14) that the following equality holds:

$$2\tau(\widehat{\alpha}) = \mathrm{lk}(\alpha) - n + 1. \tag{3.18}$$

It means that the left inequality in (3.15) is then an equality. When a positive crossing of $\widehat{\beta}$ is changed into a negative one, the number $\mathrm{lk}(\beta) - n + 1$ decreases by 2, while $2\tau(\widehat{\beta})$ decreases by at most 2, because of (3.16). Hence, the left inequality in (3.15) is preserved. The upper bound in (3.15) is true, because of (3.17) combined with the lower bound. □

Corollary 3.9. *The knot invariant τ defines a quasi-morphism on B_n and*

$$2\widetilde{\tau} = \mathrm{lk}\,.$$

Proof. Note that properties (3.9)–(3.12) imply that τ satisfy conditions of Theorem 2.6, and hence $\widehat{\tau}\colon B_n \to \mathbb{R}$ is a quasi-morphism. The second statement follows from Lemma 3.8 and the proof of Theorem 3.5. □

Recall that every knot K in $\mathbb{R}^3$ can be presented as a closure of some braid in B_n. The braid index of K is the minimal such n. It is denoted by $\mathrm{br}(K)$.

Theorem 3.10. *Let $\widehat{s}$ and $\widehat{\tau}$ be the quasi-morphisms on B_n which are induced from Rasmussen and Ozsvath–Szabo knot invariants. Then for every $\beta \in B_n$ the following inequality holds*

$$|\widehat{s}(\beta) - 2\widehat{\tau}(\beta)| \le 2(n-1).$$

Proof. As we explained before both s and 2τ satisfy property (3.15). It means that

$$|\widehat{s}(\beta) - \mathrm{lk}(\alpha_\beta\beta)| \le n-1 \quad \text{and} \quad |2\widehat{\tau}(\beta) - \mathrm{lk}(\alpha_\beta\beta)| \le n-1.$$

Thus by the triangle inequality we have

$$|\widehat{s}(\beta) - 2\widehat{\tau}(\beta)| \le 2(n-1).$$ □

Corollary 3.11. *For every knot K the following inequality holds*:

$$|s(K) - 2\tau(K)| \le 2(\mathrm{br}(K) - 1).$$

Problem 1. Does there exist a real-valued knot invariant I, independent from the ω-signatures, τ and s, such that I satisfies the conditions of Theorem 2.6, and such that the induced homogeneous quasi-morphism $\widetilde{I}\colon B_n \to \mathbb{R}$ is non-trivial?

4. Induced Quasi-Morphisms on $\mathcal{D}$

4.1. *Gambaudo–Ghys construction*

Recall that $\mathcal{D} := \mathrm{Diff}^\infty(\mathbb{D}^2, \partial\mathbb{D}^2, \mathrm{area})$ is the group of compactly supported area-preserving diffeomorphisms of the open unit disk $\mathbb{D}^2$. In this subsection we discuss the construction, essentially contained in [13], which takes a homogeneous quasi-morphism on P_n and produces from it a quasi-morphism on $\mathcal{D}$.

Denote by X_n the space of all *ordered* n-tuples of distinct points in $\mathbb{D}^2$. Let us fix a base point $\overline{z} = (z_1, \ldots, z_n) \in X_n$ and let $\overline{x} = (x_1, \ldots, x_n)$ be any other point in X_n. It is a well-known fact that $\mathcal{D}$ is path-connected (see [24]). Take $g \in \mathcal{D}$ and any path ψ_t, $0 \leq t \leq 1$, in $\mathcal{D}$ between Id and g. Connect $\overline{z}$ to $\overline{x}$ by a straight line in $(\mathbb{D}^2)^n$, then act on $\overline{x}$ with the path ψ_t, and then connect $g(\overline{x})$ to $\overline{z}$ by the straight line in $(\mathbb{D}^2)^n$. We get a loop in $(\mathbb{D}^2)^n$. More specifically it looks as follows. Connect z_i to x_i by straight lines $\mathfrak{l}_{1,i} : [0, \frac{1}{3}] \to \mathbb{D}^2$ in the disk, then act with the path ψ_{3t-1}, $\frac{1}{3} \leq t \leq \frac{2}{3}$, on each x_i, and then connect $g(x_i)$ to z_i by straight lines $\mathfrak{l}_{2,i} : [\frac{2}{3}, 1] \to \mathbb{D}^2$ in the disk, for all $1 \leq i \leq n$. Note that for almost all n-tuples of different points $x_1, \ldots, x_n$ in the disk the concatenations of the paths $\mathfrak{l}_{1,i} : [0, \frac{1}{3}] \to \mathbb{D}^2$, $\psi_{3t-1} : [\frac{1}{3}, \frac{2}{3}] \to \mathbb{D}^2$ and $\mathfrak{l}_{2,i} : [\frac{2}{3}, 1] \to \mathbb{D}^2$, $i = 1, \ldots, n$, yield a loop in X_n. The homotopy type of this loop is an element in P_n (here P_n is identified with the fundamental group $\pi_1(X_n, \overline{z})$). This element is independent of the choice of ψ_t because $\mathcal{D}$ is contractible (see [11, 35]), it will be denoted by $\gamma(g; \overline{x})$. Let $\widetilde{\varphi}_n$ be a homogeneous quasi-morphism on P_n. Denote $d\overline{x} := dx_1 \cdot \ldots \cdot dx_n$ and set

$$\Phi(g) = \int_{X_n} \widetilde{\varphi}_n(\gamma(g; \overline{x})) d\overline{x}. \tag{4.1}$$

Lemma 4.1 (cf. [13]). *The function Φ is well defined. It is a quasi-morphism on $\mathcal{D}$.*

As an immediate corollary we get that the formula

$$\widetilde{\Phi}(g) = \lim_{p \to \infty} \frac{1}{p} \int_{X_n} \widetilde{\varphi}_n(\gamma(g^p; \overline{x})) d\overline{x}$$

yields a homogeneous quasi-morphism $\widetilde{\Phi} : \mathcal{D} \to \mathbb{R}$.

Proof. Let $g \in \mathcal{D}$. For any isotopy ψ_t, $0 \leq t \leq 1$, in $\mathcal{D}$ between Id and g, any $\overline{x} \in X_n$ and any $1 \leq i, j \leq n, i \neq j$ denote

$$l_{i,j}(t) := \frac{\psi_t(x_i) - \psi_t(x_j)}{\|\psi_t(x_i) - \psi_t(x_j)\|} : [0, 1] \to S^1 \quad \text{and} \quad L_{i,j}(\overline{x}) := \frac{1}{2\pi} \int_0^1 \left\| \frac{\partial}{\partial t}(l_{i,j}(t)) \right\| dt,$$

where $\| \cdot \|$ is the Euclidean norm. Note that $L_{i,j}(\overline{x})$ is the length of the path $l_{i,j}(t)$ divided by 2π. It follows that $L_{i,j}(\overline{x}) + 4$ is an upper bound for the number of times the string i turns around the string j in the positive direction plus the number of times the string i turns around the string j in the negative direction in the braid $\gamma(g;\overline{x})$. Thus we get the following inequality

$$\sum_{i<j}^{n} 2(L_{i,j}(\overline{x})+4) \geq l(\gamma(g;\overline{x})). \tag{4.2}$$

Take any finite generating set $\mathcal{S}$ of P_n. Note that for any homogeneous quasi-morphism $\widetilde{\varphi}_n : P_n \to \mathbb{R}$ one has

$$|\widetilde{\varphi}_n(\gamma)| \leq \left(D_{\widetilde{\varphi}_n} + \max_{\xi\in\mathcal{S}} |\widetilde{\varphi}_n(\xi)|\right) l_{\mathcal{S}}(\gamma), \tag{4.3}$$

where $l_{\mathcal{S}}(\gamma)$ is the length of a word γ with respect to $\mathcal{S}$. Recall that $l(\gamma)$ is the length of γ with respect to the set $\{\sigma_i\}_{i=1}^{n-1}$. It follows from [9, Corollary 24] that there exist two positive constants $K_{1,\mathcal{S}}$ and $K_{2,\mathcal{S}}$, which are independent of γ, such that

$$l_{\mathcal{S}}(\gamma) \leq K_{1,\mathcal{S}} \cdot l(\gamma) + K_{2,\mathcal{S}}.$$

It follows from (4.3) that

$$|\widetilde{\varphi}_n(\gamma(g;\overline{x}))| \leq N_1 l(\gamma(g;\overline{x})) + N_2, \tag{4.4}$$

where $N_1 = K_{1,\mathcal{S}}(D_{\widetilde{\varphi}_n} + \max_{\xi\in\mathcal{S}} |\widetilde{\varphi}_n(\xi)|)$ and $N_2 = K_{2,\mathcal{S}}(D_{\widetilde{\varphi}_n} + \max_{\xi\in\mathcal{S}} |\widetilde{\varphi}_n(\xi)|)$. Inequalities (4.2) and (4.4) yield the following inequality:

$$|\widetilde{\varphi}_n(\gamma(g;\overline{x}))| \leq 2N_1 \left(\sum_{i<j}^{n} L_{i,j}(\overline{x}) + 4\right) + N_2.$$

It follows that

$$|\Phi(g)| \leq 2N_1 \cdot \mathrm{vol}((\mathbb{D}^2)^{n-2}) \left(\sum_{i<j}^{n} \int_{\mathbb{D}^2\times\mathbb{D}^2} L_{i,j}(\overline{x}) dx_i dx_j\right) + (4N_1 n(n+1) + N_2)\,\mathrm{vol}((\mathbb{D}^2)^n).$$

It follows from [15, Lemma 1] that the integral in the above inequality is well defined and hence $|\Phi(g)| < \infty$. The proof of the fact that Φ is a quasi-morphism is exactly the same as in [13]. □

From now on the homogeneous quasi-morphism on $\mathcal{D}$ induced by $\widetilde{\varphi}_n : P_n \to \mathbb{R}$ will be denoted by $\widetilde{\Phi}$. We denote by $\mathcal{C} : \mathcal{D} \to \mathbb{R}$ the celebrated Calabi homomorphism (see [3, 8], cf. [12]).

Remark 4.2. It follows from the interpretation of $\mathcal{C}$ in [12] that the homogeneous quasi-morphisms on $\mathcal{D}$ induced by the homogeneous Rasmussen and Ozsvath–Szabo quasi-morphisms $\widetilde{s}$ and $\widetilde{\tau}$ are equal to $\mathcal{C}$ multiplied by $2n(n-1)\pi^{n-1}$ and by $n(n-1)\pi^{n-1}$, respectively.

4.2. *Generic autonomous Hamiltonians*

Denote the space of autonomous compactly supported Hamiltonians $H : \mathbb{D}^2 \to \mathbb{R}$ by $\mathcal{H}$. In this subsection we define the notion of a Morse-type Hamiltonian. It follows from [25, Theorem 2.7] that Morse-type Hamiltonians form a C^1-dense subset of $\mathcal{H}$. For any Hamiltonian H in this subset we present a calculation of $\widetilde{\Phi}$ on the time-one flow of H (Theorem 4.5). Our result is presented in terms of the integral of the push-forward of H to its Reeb graph against a certain signed measure on the graph. Similar formulas on other surfaces were established in [10, 29, 30]. This result enables us to relate the asymptotic behavior of an induced signature quasi-morphism with the Calabi homomorphism (Theorem 4.11).

Definition 4.3. We say that a function $H \in \mathcal{H}$ is of *Morse-type* if:

(1) There exists a connected open neighborhood U of $\partial\mathbb{D}^2$, such that $\partial\overline{U}\backslash\partial\mathbb{D}^2$ is a smooth simple curve, $H|_{\overline{U}} \equiv 0$ and H has no degenerate critical points in $\mathbb{D}^2\backslash\overline{U}$.
(2) There exists an open set $V \supset \overline{U}$ such that $H|_{V\backslash\overline{U}}$ has no critical points.
(3) The inequality $H(x) \neq H(y)$ holds for each two non-degenerate different critical points x and y.

Definition 4.4. *A charged tree* T is a finite tree equipped with a signed measure, such that each edge has a total finite measure.

Let $\widetilde{\varphi}_n : P_n \to \mathbb{R}$ be any homogeneous quasi-morphism. Here we explain how we associate to $(H, \widetilde{\varphi})$ a charged tree (T, μ). Let $H \in \mathcal{H}$ be a Morse-type function with l critical points in $\mathbb{D}^2\backslash\overline{U}$, where U is as in Definition 4.3, and let $c = \partial\overline{U}\backslash\partial\mathbb{D}^2$.

Step 1. Let us recall the notion of angle-action symplectic coordinates. We remove from $\mathbb{D}^2$ all singular level curves inside $\mathbb{D}^2\backslash\overline{U}$, the curve c and $\partial\mathbb{D}^2$. We get l open annuli A_i (a punctured disk is also viewed as an annulus) and an open annulus $A_{l+1} = U^\circ$. Each $\mathbb{D}^2\backslash A_i$ has two connected components. We denote by CA_i the component which does not contain $\partial\mathbb{D}^2$ and by

$$a_i := \partial(CA_i) \cap \partial\overline{A}_i, \quad R_i := \frac{\text{area}(A_i)}{2\pi}, \quad CR_i := \frac{\text{area}(CA_i)}{2\pi}.$$

By the Liouville–Arnold theorem (see [1]) on each one of the annuli A_i $(1 \le i \le l)$, there exist so-called angle-action symplectic coordinates (θ_i, J_i), $\theta_i \in [0, 2\pi]$, and a C^∞-function

$$\hbar_i : [CR_i, CR_i + R_i] \to \mathbb{R},$$

such that on each level curve c_i in A_i one has $H|_{c_i} = \hbar_i(J_i)$, where the coordinate J_i along c_i is equal to the sum of CR_i and the area, divided by 2π, of the annulus bounded by c_i and a_i. Let h_1 be the time-one Hamiltonian flow generated by H. In these coordinates h_1 moves points on each level curve with a constant speed

$$\hbar_i' = \frac{\partial \hbar_i}{\partial J_i}.$$

Let p, q be the coordinates on $\mathbb{R}^2$ and note that $H|_{A_{l+1}} = 0$. Then

$$\int_{\mathbb{D}^2} H(p,q)dpdq = \sum_{i=1}^{l} \int_{A_i} H(p,q)dpdq = 2\pi \sum_{i=1}^{l} \int_{CR_i}^{CR_i+R_i} \hbar_i(J_i)dJ_i.$$

Step 2. Let H be a Morse-type function with l critical points in $\mathbb{D}^2 \backslash \overline{U}$. Define the equivalence relation $\sim$ on points on $\mathbb{D}^2$ by $x \sim y$ whenever x, y are in the same connected component of a level set of H. The Reeb graph T corresponding to H is the quotient of $\mathbb{D}^2$ by the relation $\sim$. The edges of T come from the annuli in $\mathbb{D}^2$ fibered by level loops, the valency-one vertices correspond to the min/max critical points of H and to $\partial\mathbb{D}^2$, and the valency-three vertices correspond to the saddle critical points of H. In our case T is a rooted tree with the root being the vertex of T corresponding to the domain $\overline{U} \supset \partial\mathbb{D}^2$. Each edge e_j in T corresponds to an annulus A_j for each $0 \leq j \leq l$. A Reeb graph is called *simple* if it has only two vertices and one edge.

Step 3. The action coordinate on any of the annuli induces a coordinate J_j on the corresponding edge e_j of T. Thus each Morse-type Hamiltonian H descends to a function $\hbar : T \to \mathbb{R}$, where $\hbar = \hbar_j$ and $\hbar' = \hbar'_j$ on each open edge e_j. Here $j \in \{0, \ldots, l\}$. For a homogeneous quasi-morphism $\widetilde{\varphi}_n : P_n \to \mathbb{R}$ and $0 \leq j \leq l$ we define a signed measure μ on T by setting

$$d\mu(J_j) := (2\pi)^n \sum_{i=2}^{n} \widetilde{\varphi}_n(\eta_{i,n}) i \binom{n}{i} (J_j)^{i-1} \left(\frac{1}{2} - J_j\right)^{n-i} dJ_j,$$

where J_j is the coordinate on each edge e_j and $\eta_{i,n}$ is the pure braid in P_n shown in Fig. 3. Note that all of these braids commute with each other.

Theorem 4.5. *Let $H \in \mathcal{H}$ be a Morse-type function, $\widetilde{\varphi}_n : B_n \to \mathbb{R}$ be a homogeneous quasi-morphism, and $\widetilde{\Phi}$ be the corresponding homogeneous quasi-morphism on $\mathcal{D}$. Then*

$$\widetilde{\Phi}(h_1) = \int_T \hbar' d\mu, \tag{4.5}$$

where h_1 is the time-one Hamiltonian flow generated by H.

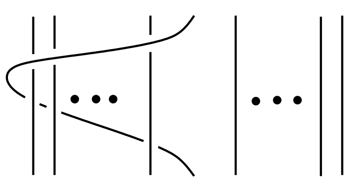

Fig. 3. Braid $\eta_{i,n}$.

Proof. [Sketch of the proof] Let l be the number of isolated critical points of H. If $l = 1$ then the Reeb graph T which corresponds to H is simple, and Theorem 4.5 is just [13, Lemma 5.2]. Let $l > 1$. Take a sequence of non-negative integers $n_1, \dots, n_{l+1}$ such that $n = n_1 + \cdots + n_{l+1}$. Let $\overline{x} \in X_n$ such that $\overline{x} = (x_1, \dots, x_{n_1}, x_{n_1+1}, \dots, x_{n_1+n_2}, \dots, x_{n_1+n_2+\cdots+n_l}, \dots, x_n)$ where x_i is in A_j if $n_1 + n_2 + \cdots + n_{j-1} + 1 \leq i \leq n_1 + n_2 + \cdots + n_j$ for each $1 \leq i \leq n$ and $1 \leq j \leq l+1$. We also require that for each $1 \leq j \leq l$ and $x_i, x_k \in A_j$ such that $i < k$ we have $H(x_i) < H(x_k)$. Then for every $p \in \mathbb{N}$

$$\gamma(h_1^p, \overline{x}) = \alpha_{1,p}\gamma_{1,p} \cdot \dots \cdot \gamma_{l+1,p}\alpha_{2,p},$$

where $|\widetilde{\varphi}_n(\alpha_{1,p})|$ and $|\widetilde{\varphi}_n(\alpha_{2,p})|$ are bounded by a constant independent of $\overline{x}$ and p, each braid $\gamma_{j,p}$ depends up to conjugation only on p and points $x_{n_1+n_2+\cdots+n_j+1}, \dots, x_{n_1+n_2+\cdots+n_j+n_{j+1}}$. All of $\gamma_{j,p}$ commute with each other and up to conjugation are elements of the Abelian group which is generated by the braids $\eta_{i,n}$. Now we apply angle-action coordinates in each annuli A_j and proceed as in the proof of [13, Lemma 5.2]. Summing over all such $n_1, \dots, n_{l+1}$, $\overline{x} \in X_n$ and multiplying by $n!$ yields Eq. (4.5). The interested reader may find a complete proof in [7]. □

Let $\theta \in [0, 1]$ and denote by $\omega(\theta) := e^{2\pi i\theta}$. The motivation for the next theorem is explained in Problem 2.

Theorem 4.6. *Let AP_n be the Abelian subgroup of P_n generated by $\eta_{i,n}$, and denote by*

$$V_{AP} := \{\widetilde{\varphi}|_{AP_n} : AP_n \to \mathbb{R} \text{ where } \widetilde{\varphi} \text{ is a homogeneous quasi-morphism on } B_n\}.$$

Then

$$\mathcal{B} = \{\widetilde{\text{sign}}_{\omega(\frac{1}{2})}, 2\widetilde{\text{sign}}_{\omega(\frac{1}{2})} - 3\widetilde{\text{sign}}_{\omega(\frac{1}{3})}, \dots, (n-1)\widetilde{\text{sign}}_{\omega(\frac{1}{n-1})} - n\widetilde{\text{sign}}_{\omega(\frac{1}{n})}\}$$

is a basis for V_{AP}.

Proof. If $n = 2$, then $\widetilde{\text{sign}}_{\omega(\frac{1}{2})}(\eta_{2,2}) = 2$, and the proof follows. Let $n > 2$.

Lemma 4.7. *Let $\theta \in [0, 1] \cap \mathbb{Q}$. Then*:

$$\widetilde{\text{sign}}_{\omega(\theta)}(\eta_{i,n}) = \begin{cases} 4(i-1)\theta & \text{if } 0 \leq \theta \leq \dfrac{1}{i}, \\ 4(l-1)(1-\theta) & \text{if } \dfrac{l-1}{i} \leq \theta \leq \dfrac{l-1}{i-1},\ 2 \leq l \leq i-1, \\ 4(i-l)\theta & \text{if } \dfrac{l-1}{i-1} \leq \theta \leq \dfrac{l}{i},\ 2 \leq l \leq i-1, \\ 4(i-1)(1-\theta) & \text{if } \dfrac{i-1}{i} \leq \theta \leq 1. \end{cases}$$

Proof. Recall that the torus link $K(p,q)$ has a braid representation $(\sigma_1 \cdot \ldots \cdot \sigma_{p-1})^q$. Note that

$$\widetilde{\operatorname{sign}}_{\omega(\theta)}(K(n,n)) = \widetilde{\operatorname{sign}}_{\omega(\theta)}(\eta_{2,n} \cdot \ldots \cdot \eta_{n,n}) = \sum_{i=2}^{n} \widetilde{\operatorname{sign}}_{\omega(\theta)}(\eta_{i,n}).$$

Hence for each $i \le n$ we have

$$\widetilde{\operatorname{sign}}_{\omega(\theta)}(\eta_{i,n}) = \widetilde{\operatorname{sign}}_{\omega(\theta)}(K(i,i)) - \widetilde{\operatorname{sign}}_{\omega(\theta)}(K(i-1,i-1)). \tag{4.6}$$

In [14], Gambaudo and Ghys proved the following proposition.

Proposition 4.8 ([14, Proposition 5.2]). *Let* $\theta \in [0,1] \cap \mathbb{Q}$. *Then*

$$\widetilde{\operatorname{sign}}_{\omega(\theta)}(\sigma_1 \cdot \ldots \cdot \sigma_{n-1}) = 2(n-2l+1)\theta + \frac{2l(l-1)}{n}$$

for $\frac{l-1}{n} \le \theta \le \frac{l}{n}$, *where* $1 \le l \le n$.

An immediate consequence of Proposition 4.8 is that for $1 \le l \le n$ and $\frac{l-1}{n} \le \theta \le \frac{l}{n}$ we have

$$\widetilde{\operatorname{sign}}_{\omega(\theta)}(K(n,n)) = 2n(n-2l+1)\theta + 2l(l-1). \tag{4.7}$$

A simple calculation combined with equalities (4.6) and (4.7) yields the proof of this lemma. □

Let us finish the proof of the theorem. It follows from Lemma 4.7 that for each $3 \le i \le n$ and for each $2 \le j \le i-1$ we have

$$((i-1)\widetilde{\operatorname{sign}}_{\omega(\frac{1}{i-1})} - i\widetilde{\operatorname{sign}}_{\omega(\frac{1}{i})})(\eta_{i,i}) = -4,$$
$$((i-1)\widetilde{\operatorname{sign}}_{\omega(\frac{1}{i-1})} - i\widetilde{\operatorname{sign}}_{\omega(\frac{1}{i})})(\eta_{j,i}) = 0.$$

Thus $\mathcal{B}$ is a basis for V_{AP}.

Problem 2. The following problem is motivated by a question posed to us by Polterovich. Denote by $\mathcal{D}_{\text{aut}} \subset \mathcal{D}$ the set of area-preserving diffeomorphisms generated by autonomous Hamiltonians. Observe that, by Theorems 4.5 and 4.6, one can find a linear combination of homogeneous quasi-morphisms $\widetilde{\operatorname{sign}}_\omega$ on B_n so that the corresponding homogeneous quasi-morphism $\widetilde{\Phi} : \mathcal{D} \to \mathbb{R}$ vanishes on $\mathcal{D}_{\text{aut}}$. On the other hand, using a construction from [13] (completely different from the one described in this paper) one can construct more homogeneous quasi-morphisms $\widetilde{\Psi}_\alpha$ on $\mathcal{D}$ that vanish on $\mathcal{D}_{\text{aut}}$. It would be interesting to check whether the quasi-morphism $\widetilde{\Phi}$ is non-trivial and, if so, whether it is a linear combination of the quasi-morphisms $\widetilde{\Psi}_\alpha$. So far we have not been able to compute or estimate the value of *any* of the Gambaudo–Ghys quasi-morphisms coming from quasi-morphisms on B_n ($n > 2$) on *any* area-preserving diffeomorphism that is not generated by a flow preserving a foliation.

4.3. *Asymptotic behavior of an induced signature quasi-morphism*

Recall that $\widetilde{\mathrm{sign}}$ is a homogeneous quasi-morphism on B_n. The induced family of homogeneous quasi-morphisms on $\mathcal{D}$ is denoted by $\widetilde{\mathrm{Sign}}_{n,\mathbb{D}^2}$. In [13] Gambaudo and Ghys proved that all of them are non-trivial and linearly independent.

Proposition 4.9. *For each $n \geq 2$ we have*

$$\widetilde{\mathrm{Sign}}_{n,\mathbb{D}^2}(h_1) = n\pi^n \int_T (1 + 4(n-1)J - (1-4J)^{n-1})\hbar'(J)dJ,$$

where h_1 is the time-one Hamiltonian flow defined by a Morse-type Hamiltonian H.

Proof. In [13] Gambaudo and Ghys showed that

$$\widetilde{\mathrm{sign}}(\eta_{i,n}) = \begin{cases} i & \text{if } i \text{ is even,} \\ i-1 & \text{if } i \text{ is odd.} \end{cases}$$

A simple computation yields

$$\sum_{i=2}^{n} \widetilde{\mathrm{sign}}_n(\eta_{i,n}) i \binom{n}{i} J^{i-1} \left(\frac{1}{2} - J\right)^{n-i}$$
$$= \frac{n}{2}\left(\left(\frac{1}{2}\right)^{n-1} + 4(n-1)J\left(\frac{1}{2}\right)^{n-1}\right) - \frac{n}{2}\left(\frac{1}{2} - 2J\right)^{n-1}.$$

Now the proof follows immediately from Theorem 4.5. □

The following theorem is used in the proof of the main result in this subsection (Theorem 4.11). The proof relies on the estimates from [15, Theorem 1]. It is technical and may be found in [7, Theorem 3.3.5].

Theorem 4.10 ([7]). *Let $H \in \mathcal{H}$ and $\{H_k\}_{k=1}^{\infty}$ be a sequence of functions such that each $H_k \in \mathcal{H}$ and $H_k \xrightarrow[k\to\infty]{} H$ in C^1-topology. Let h_1 and $h_{1,k}$ be the time-one Hamiltonian flows generated by H and H_k, respectively. Let $\widetilde{\varphi}_n : B_n \to \mathbb{R}$ be a homogeneous quasi-morphism, and $\widetilde{\Phi}$ be the corresponding homogeneous quasi-morphism on $\mathcal{D}$. Then*

$$\lim_{k\to\infty} \widetilde{\Phi}(h_{1,k}) = \widetilde{\Phi}(h_1).$$

Now we are ready to prove our main theorem in this subsection.

Theorem 4.11. *For each $h_1 \in \mathcal{D}$ generated by an autonomous Hamiltonian H we have*

$$\lim_{n\to\infty} \frac{\widetilde{\mathrm{Sign}}_{n,\mathbb{D}^2}(h_1)}{\pi^{n-1}n(n-1)} = \mathcal{C}(h_1),$$

where $\mathcal{C}$ is the Calabi homomorphism.

Proof. **Step 1.** Suppose that h_1 is generated by a Morse-type Hamiltonian H. It follows from Proposition 4.9 that

$$\begin{aligned}\lim_{n\to\infty} & \frac{\widetilde{\mathrm{Sign}}_{n,\mathbb{D}^2}(h_1)}{\pi^{n-1}n(n-1)} \\ &= 4\pi\int_T J\hbar'(J)dJ + \pi\lim_{n\to\infty}\left(\int_T \hbar'(J)dJ/(n-1)\right. \\ &\quad \left. - \int_T (1-4J)^{n-1}\hbar'(J)dJ/(n-1)\right).\end{aligned}$$

The limit of the first integral in the equation above equals to zero, hence

$$\begin{aligned}\lim_{n\to\infty} & \frac{\widetilde{\mathrm{Sign}}_{n,\mathbb{D}^2}(h_1)}{\pi^{n-1}n(n-1)} \\ &= 4\pi\int_T J\hbar'(J)dJ - \lim_{n\to\infty}\pi\left(\int_T (1-4J)^{n-1}\hbar'(J)dJ/(n-1)\right).\end{aligned}$$

Note that J is an action coordinate on each edge of T. This yields

$$0 \le J \le \frac{\mathrm{area}(\mathbb{D}^2)}{2\pi} = \frac{\pi}{2\pi} = \frac{1}{2}.$$

It follows that $|1-4J| \le 1$ and hence

$$\begin{aligned}0 &\le \left|\lim_{n\to\infty}\pi\left(\int_T (1-4J)^{n-1}\hbar'(J)dJ/(n-1)\right)\right| \\ &\le \lim_{n\to\infty}\pi\left(\int_T |\hbar'(J)|dJ/(n-1)\right) = 0.\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\frac{\widetilde{\mathrm{Sign}}_{n,\mathbb{D}^2}(h_1)}{\pi^{n-1}n(n-1)} = 4\pi\int_T J\hbar'(J)dJ = -4\pi\int_T \hbar(J)dJ = \mathcal{C}(h_1).$$

The second equality is just integration by parts, and the proof of the third equality is exactly the same as the proof of [12, Proposition 2.2].

Step 2. Suppose that h_1 is generated by any $H \in \mathcal{H}$. Then by [25, Theorem 2.7] there exists a sequence $\{H_k\}_{k=1}^{\infty}$ of Morse-type functions, which converge in C^1-topology to H. Now the proof follows from Theorem 4.10. □

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