ON QUASIMORPHISMS AND DISTORTION IN HOMEOMORPHISM GROUPS

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Abstract. Let M be a smooth compact oriented manifold, and $\mathrm{Homeo}_0(M,\mu)$ the group of homeomorphisms of M supported away from ∂M, which preserve a Borel probability measure μ induced by a volume form on M, and are isotopic to the identity. In this paper, we identify those Gambaudo-Ghys and Polterovich quasimorphisms $\Psi\colon \mathrm{Diff}_0(M,\mu) \to \mathbf{R}$ which extend C^0-continuously to $\mathrm{Homeo}_0(M,\mu)$ as quasimorphisms, and to $\mathrm{Homeo}_0(M)$ as group cochains whose differentials are semi-bounded cocycles.

We present several applications of this result which include unboundedness of certain bi-invariant metric on the commutator subgroup of $\mathrm{Homeo}_0(M,\mu)$, and conditions under which a homeomorphism in $\mathrm{Homeo}_0(M)$ is undistorted.

1. Introduction

Let M be a smooth compact oriented manifold equipped with a Borel probability measure μ defined by a volume form on M. Let $\mathrm{Homeo}_0(M,\mu)$ denote the group of measure preserving homeomorphisms of M which act by the identity homeomorphism on a neighborhood of ∂M and are isotopic to the identity. Let $P_n(M,z) = \pi_1(\mathrm{C}_n(M),z)$ be the pure braid group on M. Here $\mathrm{C}_n(M) \subset M^n$ denotes the space of ordered configurations of n points in M, equipped with the product $\mathrm{Homeo}_0(M)$ action. If $\dim(M) = 2$ then we let $n \in \mathbf{N}$ be an arbitrary number, and when $\dim(M) \geq 3$, we assume $n = 1$ throughout the paper. That is, for higher dimensional manifolds we consider only their fundamental groups $P_1(M,z) = \pi_1(M,z)$.

Let $\gamma\colon \mathrm{Homeo}_0(M) \times M \to P_n(M)$ be a measurable cocycle defined by

$$\gamma(f,x) = \left[\ell_{z,f(x)} * \{f_t(x)\} * \ell_{x,z}\right],$$

where $\{f_t\}$ is an isotopy from the identity to f and $\ell_{x,y}$ is a certain path in $\mathrm{C}_n(M)$ from x to y which is defined in Section 2. By convention, the concatenation of paths is read from right to left.

The main object of study in the present paper is a family of maps $\Psi\colon \mathrm{Homeo}_0(M) \to \mathbf{R}$ defined by

$$\Psi(f) = \int_{\mathrm{C}_n(M)} \varphi(\gamma(f,x))dx, \tag{1.1}$$

where $\varphi\colon P_n(M) \to \mathbf{R}$ is a homogeneous quasimorphism vanishing on the centre of $P_n(M)$. On $\mathrm{C}_n(M)$ we consider the product measure μ^n. The formula (1.1) was used by Gambaudo-Ghys to define non-trivial quasimorphisms on groups of area preserving diffeomorphisms of compact surfaces [19] and by Polterovich for volume preserving diffeomorphisms of compact manifolds of higher dimension [26]. It was further exploited by several authors in various configurations [5, 7, 9, 10, 27].

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In general, it is not well defined for homeomorphisms and we identify quasimorphisms $\varphi: P_n(M) \to \mathbf{R}$ for which it is. Here is our main result.

Theorem 1.1. *Let $\varphi: P_n(M) \to \mathbf{R}$ be a non-trivial homogeneous quasimorphism vanishing on the centre of $P_n(M)$. If M is a surface, we assume that M is closed of positive genus and φ vanishes on the subgroup $P_n(\Delta) \leq P_n(M)$ of braids supported in an interior of a full measure two-cell Δ, see Section 2 for definition. Then the map $\Psi: \mathrm{Homeo}_0(M) \to \mathbf{R}$ given by (1.1) is well defined. Moreover:*

(1) The differential $\delta\Psi$ is a semi-bounded 2-cocycle, that is

$$D(f) := \sup_g |\delta\Psi(f,g)| = \sup_g |\Psi(g) - \Psi(fg) + \Psi(f)| < \infty.$$

(2) The restriction of Ψ to the subgroup of homeomorphisms preserving the measure μ is a quasimorphism whose homogenization is C^0-continuous.

(3) If φ extends to an unbounded quasimorphism on the full braid group $B_n(M)$ then the homogeneous quasimorphism from the previous item is non-trivial.

Remark 1.2. Let Σ be a closed surface of a positive genus and $n > 1$. Recall that $B_n(\Sigma)/Z(B_n(\Sigma))$, where $Z(B_n(\Sigma))$ is the center of $B_n(\Sigma)$, is a non-reducible subgroup of the n punctured mapping class group MCG_n [22, Corollary 7.13] and obviously $P_n(\Delta) \leq \mathrm{MCG}_n$ is a reducible subgroup. It follows from Bestvina-Fujiwara construction [1, Theorem 12] that the space of quasimorphisms on $B_n(\Sigma)$ which vanish on $P_n(\Delta)$ is infinite-dimensional. Hence there are infinitely many linearly independent quasimorphisms $\varphi: B_n(\Sigma) \to \mathbf{R}$ which satisfy conditions of Theorem 1.1.

Let $q : \pi_1(\Sigma) \to \mathbf{R}$ be a quasimorphism. It defines a quasimorphism $\varphi : P_n(\Sigma) \to \mathbf{R}$ as follows. Let

$$i_* : P_n(\Sigma) \to (\pi_1(\Sigma))^n$$

be the homomorphism induced by the inclusion $i : \mathrm{C}_n(\Sigma) \to (\Sigma)^n$. We define $\varphi = q \circ p_i \circ i_*$, where

$$p_i : (\pi_1(\Sigma))^n \to \pi_1(\Sigma)$$

is the projection on the i-th factor. It is a well-known fact that the kernel of homomorphism i_* equals to the normal closure H_n of $P_n(\Delta)$ in the group $P_n(\Sigma)$ [20, Theorem 1], see also [2]. Thus every such φ vanishes on H_n.

Let us consider the case when $\Sigma = T$ is a torus. In this case the group H_n equals to the commutator subgroup $[P_n(T), P_n(T)]$, see [2]. Hence there are infinitely many linearly independent quasimorphisms on $P_n(T)$, that vanish on $P_n(\Delta)$, which are obviously different from each such φ, since in this case φ is a homomorphism. Let us discuss in more details the case $n = 2$. The commutator subgroup $[P_2(T), P_2(T)] \cong \mathbf{F}_2 = \langle a, b\rangle$ and $P_2(\Delta) \cong \mathbf{Z}$ is generated by the commutator $[a, b]$. Let $\mathrm{Q}(\mathbf{F}_2; \mathbf{Z}/2 \times \mathbf{Z}/2)$ be the subspace of quasimorphisms on $\mathbf{F}_2$ invariant under the action generated by inverting generators. Then every homogeneous quasimorphism $\phi \in \mathrm{Q}(\mathbf{F}_2; \mathbf{Z}/2 \times \mathbf{Z}/2)$ vanishes on $[a, b]$ and hence on $P_2(\Delta)$. Moreover, each such ϕ extends to $B_2(T)$, see [9, Proposition 2.8], hence by (3) in Theorem 1.1 Ψ is non-trivial. In particular, this gives an elementary construction of unbounded quasimorphisms on $\mathrm{Homeo}_0(T, \mu)$. Note that in this case, one needs to take $n > 1$, since for $n = 1$ we have $B_1(T) = Z(B_1(T)) \cong \mathbf{Z}^2$.

Let us discuss the remaining case when Σ is a closed hyperbolic surface. We would like to point out that there are infinitely many linearly independent quasimorphisms on $P_n(\Sigma)$, that vanish on $P_n(\Delta)$, which are different from each φ described above. Note that, there are infinitely many linearly independent quasimorphisms on $P_n(\Sigma)$ that vanish on $P_n(\Delta)$, but do not vanish on H_n. Indeed, by [1, Theorem 12] it is enough to show that the group H_n is a non-reducible subgroup of MCG_n. This can be shown as follows: Let C be a multicurve (a simplex in the curve complex). If it intersects non-trivially $\Delta \setminus z$, there is an element in $P_n(\Delta)$ which does no preserve C. Otherwise, if C lies outside of Δ, then since $P_n(\Sigma)$ is non-reducible, there exists $\alpha \in P_n(\Sigma)$ so that $\alpha(C) \neq C$. Note that one can choose α such that $\alpha(C)$ intersects $\Delta \setminus z$ non-trivially. Hence there exists $\beta \in P_n(\Delta)$ such that $\beta\alpha(C) \neq \alpha(C)$. It follows that $\alpha^{-1}\beta\alpha \in H_n$ and $\alpha^{-1}\beta\alpha(C) \neq C$. Thus H_n is a non-reducible subgroup of MCG_n.

Remark 1.3. Semi-boundedness of 2-cocycles was introduced by Gal-Kędra [16] where it was used to prove undistortedness of symplectic diffeomorphisms for certain symplectic manifolds. It was further generalized to the concept of p-boundedness in [17], which also introduced p-bounded cohomology of groups. Functions with semi-bounded differentials were recently used in [12] to study the L^p-geometry of the space of contractible loops on surfaces.

C^0-continuity. Let μ be the measure associated with a volume form on M. Given a homogeneous quasimorphism $\varphi\colon P_n(M) \to \mathbf{R}$ vanishing on the centre of $P_n(M)$, the formula (1.1) well defines a quasimorphism $\Psi\colon \mathrm{Diff}_0(M,\mu) \to \mathbf{R}$.

Corollary 1.4. *The homogenization* $\overline{\Psi}\colon \mathrm{Diff}_0(M,\mu) \to \mathbf{R}$ *has a unique* C^0*-continuous extension to* $\mathrm{Homeo}_0(M,\mu)$ *which is a homogeneous quasimorphism.*

Entropy. Let Σ be a closed orientable surface of positive genus. For certain quasimorphisms $\varphi\colon B_n(\Sigma) \to \mathbf{R}$ the restriction of Ψ to $\mathrm{Diff}_0(\Sigma,\mu)$ is bounded on the set Ent_0 of diffeomorphisms of zero entropy [10]. It follows from Corollary 1.4 that the extension

$$\widetilde{\overline{\Psi}}\colon \mathrm{Homeo}_0(\Sigma,\mu) \to \mathbf{R}$$

vanishes on the C^0-closure of Ent_0 in $\mathrm{Homeo}_0(\Sigma,\mu)$ and on all its conjugates.

Consider the set

$$S = \bigcup_{f \in \mathrm{Homeo}_0(\Sigma,\mu)} f^{-1} \cdot \overline{\mathrm{Ent}_0}^{\mathrm{Diff}} \cdot f,$$

which is the normal closure of the C^0-closure in $\mathrm{Homeo}_0(\Sigma,\mu)$ of the set of zero entropy measure preserving diffeomorphisms. Let $G_S \leq \mathrm{Homeo}_0(\Sigma,\mu)$ be the subgroup generated by S. Note that G_S is a large subgroup. For example, by simplicity of the kernel of the Calabi homomorphism on the Hamiltonian group, the kernel of the Flux homomorphism on $\mathrm{Diff}_0(\Sigma,\mu)$, it is easily seen that

$$\mathrm{Diff}_0(\Sigma,\mu) \leq G_S.$$

On the other hand, by [14, Theorem 1.10], the commutator group $[\mathrm{Homeo}_0(\Sigma,\mu), \mathrm{Homeo}_0(\Sigma,\mu)]$ is simple. This implies that

$$[\mathrm{Homeo}_0(\Sigma,\mu), \mathrm{Homeo}_0(\Sigma,\mu)] \leq G_S.$$

We have an immediate consequence of Corollary 1.4.

Corollary 1.5. *The diameter of the bi-invariant metric on G_S associated with the generating set S is infinite.* □

This corollary says that the set S is *small*, in the sense that for any positive integer $m \in \mathbf{N}$ there are elements of $\mathrm{Homeo}_0(\Sigma, \mu)$ which cannot be presented as product of up to m elements from S. It follows from [29] that S is contained in a closed subset with empty interior, so it is topologically small. The above corollary says that it is also algebraically small. Since entropy is not C^0-continuous, the relation between S and the set of measure preserving homeomorphisms of zero entropy is unclear.

It is also unclear how large the group G_S really is. Indeed, it contains all the elements which are currently known not to lie in $[\mathrm{Homeo}_0(\Sigma, \mu), \mathrm{Homeo}_0(\Sigma, \mu)]$. However, the abelianization $\mathrm{Homeo}_0(\Sigma, \mu)/[\mathrm{Homeo}_0(\Sigma, \mu), \mathrm{Homeo}_0(\Sigma, \mu)]$ is not yet completely determined and it is unclear what subgroup G_S defines in it.

Question 1. *Is it true that $G_S = \mathrm{Homeo}_0(\Sigma, \mu)$?*

Distortion. Recall that an element g of a finitely generated group Γ is undistorted if

$$\lim_{k \to \infty} \frac{|g^k|}{k} > 0,$$

where $|g|$ denotes the word norm with respect to any finite generating set of Γ. An element g of an arbitrary group G is undistorted if it is undistorted in every finitely generated subgroup $\Gamma \leq G$ containing it. Understanding distortion in groups of homeomorphisms is motivated by the topological version of the Zimmer conjecture.

Existence of functions on a group G whose differential is semi-bounded can be used to prove undistortedness of elements of G. The following observation almost immediately follows from [17, Proposition 4.2].

Theorem 1.6. *Let $\psi : G \to \mathbf{R}$ be a function such that $\delta\psi$ is semi-bounded. If*

$$\limsup_{k \to \infty} \frac{|\psi(f^k)|}{k} > 0$$

then f is undistorted.

Proof. It follows from Lemma 2.1 that $|f|_\psi = \sup_g |\psi(g) - \psi(fg)|$ is a pseudo-norm with respect to which ψ is Lipschitz which implies that

$$\limsup_{k \to \infty} \frac{|f^k|_\psi}{k} > 0. \tag{1.2}$$

Let $\Gamma \subseteq G$ be a finitely generated subgroup containing f and let $C = \max\{|s|_\psi \mid s \in S\}$, where $S \subseteq \Gamma$ is a finite generating set. Then

$$|f|_\psi = |s_1 \ldots s_m|_\psi \leq |s_1|_\psi + \cdots + |s_m|_\psi \leq C|f|_S$$

which, together with (1.2) implies that f is undistorted in Γ and hence in G. □

Corollary 1.7. *Let $f \in \mathrm{Homeo}_0(M)$. If*

$$\limsup_{k\to\infty} \frac{|\Psi(f^k)|}{k} > 0$$

then f is undistorted in $\mathrm{Homeo}_0(M)$. □

Verifying the hypothesis of the above corollary may not be easy in a concrete case. However, we tackle this problem by redefining Ψ in Theorem 1.1 as follows. Let point $z \in \mathrm{C}_n(M)$ be a base-point and let $\Psi_z\colon \mathrm{Homeo}_0(M) \to \mathbf{R}$ be given by

$$\Psi_z(f) = \varphi(\gamma(f,z)),$$

where $\varphi\colon P_n(M) \to \mathbf{R}$ is a quasimorphism vanishing on $P_n(\Delta)$ and on the centre of $P_n(M)$. We show in Section 4 that the cocycle $\delta\Psi_z$ is semi-bounded (Proposition 4.1).

Corollary 1.8. *Let $f \in \mathrm{Homeo}_0(M)$. If $z \in M$ is a fixed point of f such that $\varphi(\gamma(f,z)) > 0$ then f is undistorted in* $\mathrm{Homeo}_0(M)$.

Proof. Since z is a fixed point of f, we get that $\gamma(f^k,z) = \gamma(f,z)^k$. It follows that

$$\Psi_z(f^k) = \varphi(\gamma(f^k,z)) = k\Psi_z(f) > 0$$

and the statement follows from Theorem 1.6. □

Example 1.9. Let f be a homeomorphism of a closed hyperbolic manifold. Suppose $z \in M$ is a fixed point of f such that

$$\gamma(f,z) \in \pi_1(M,z)$$

is non-trivial. Then f is undistorted. Indeed, in this case there exists a homogeneous quasimorphism non-vanishing on $\gamma(f,z)$ and the statement follows from Corollary 1.7. ◊

1-bounded cohomology. Since Ψ is semi-bounded it is natural to ask whether it represents a non-trivial class in 1-bounded cohomology. The next corollary states that this is indeed the case.

Corollary 1.10. *Let M be a closed orientable surface of positive genus, or a higher dimensional manifold whose fundamental group modulo its centre admits a non-trivial homogeneous quasimorphism. Then*

$$\dim H^2_{(1)}(\mathrm{Homeo}_0(M),\mathbf{R}) = \infty.$$

Remark 1.11. The above statement for surfaces of positive genus also follows from the result of Bowden, Hensel and Webb [3, Theorem 1.4] who showed that $H^2_b(\mathrm{Homeo}_0(\Sigma),\mathbf{R})$ is infinite dimensional. Since the comparison homomorphism $H^*_b(G,\mathbf{R}) \to H^*_{(1)}(G,\mathbf{R})$ is injective, their result implies the above. However, we conjecture that for surfaces our 1-bounded classes are not represented by bounded cocycles. In other words, they don't come from quasimorphisms.

Standard cohomology. The formula (1.1) always defines a morphism between bounded cochains

$$C_b^*(P_n(\Sigma), \mathbf{R}) \to C_b^*(\mathrm{Homeo}_0(\Sigma, \mu), \mathbf{R}),$$

which, according to [11], descends to a homomorphism

$$\mathcal{G}: H_b^*(P_n(\Sigma), \mathbf{R}) \to H_b^*(\mathrm{Homeo}_0(\Sigma, \mu), \mathbf{R}).$$

We thus have a commutative diagram

$$\begin{array}{ccccc}
H_b^2(P_n(\Sigma); \mathbf{R}) & \xrightarrow{\mathcal{G}} & H_b^2(\mathrm{Homeo}_0(\Sigma, \mu), \mathbf{R}) & \xrightarrow{c} & H^2(\mathrm{Homeo}_0(\Sigma, \mu), \mathbf{R}) \\
\delta \uparrow & & \delta \uparrow & \nearrow_{0} & \\
Q(P_n(\Sigma)) & \dashrightarrow & Q(\mathrm{Homeo}_0(\Sigma, \mu)) & &
\end{array}$$

where the dashed arrow is not defined, and Q denotes the space of quasimorphisms on a group in question. However, when homeomorphisms are replaced by diffeomorphisms this arrow is well defined. In particular, for any quasimorphism φ the class

$$c(\mathcal{G}[\delta\varphi]) = 0$$

in $H^2(\mathrm{Diff}_0(\Sigma, \mu))$. For homeomorphism we get classes $\mathcal{G}[\delta\varphi]$ which are potentially nontrivial in $H^2(\mathrm{Homeo}_0(\Sigma, \mu), \mathbf{R})$.

Question 2. *Does there exists $\varphi: P_n(\Sigma) \to \mathbf{R}$ such that the class $c(\mathcal{G}[\delta\varphi])$ is non-trivial in the second cohomology group $H^2(\mathrm{Homeo}_0(\Sigma, \mu), \mathbf{R})$?*

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2. Preliminaries

Quasimorphisms. Let G be a group. A function $\varphi: G \to \mathbf{R}$ is called a quasimorphism if

$$D_\varphi := \sup_{g,h} |\delta\varphi(g, h)| = \sup_{g,h} |\varphi(h) - \varphi(gh) + \varphi(g)| < \infty.$$

In other words, $\delta\varphi$ is a bounded 2-cocycle on G with values in the trivial module $\mathbf{R}$. The constant D_φ is called the defect of φ. The differential $\delta\varphi$ of a quasimorphism represents a class of the second bounded cohomology $H_b^2(G, \mathbf{R})$. If the homogenization of φ is not a homomorphism then this class is non-trivial. A function $\psi: G \to \mathbf{R}$ is called homogeneous if

$$\psi(g^n) = n\psi(g)$$

for every $g \in G$ and every $n \in \mathbf{Z}$. A quasimorphism φ can be homogenized by defining

$$\overline{\varphi}(g) = \lim_{n\to\infty} \frac{\varphi(g^n)}{n}.$$

The above limit exists due to Fekete's Lemma and, moreover, $\sup_g |\overline{\varphi}(g) - \varphi(g)| \leq D_\varphi$. A homogeneous quasimorphism is constant on conjugacy classes. Proofs of the above facts are standard and can be found for example in [13].

Semi-bounded cohomology. A k-cochain $c\colon G^k \to \mathbf{R}$ is called p-bounded, where $p \leq k$, if for fixed $g_1, \ldots, g_{k-p} \in G$ the function

$$(g_{k-p+1}, \ldots, g_k) \mapsto c(g_1, \ldots, g_k)$$

is bounded. It follows that p-bounded cochains form subcomplexes and their homology, denoted by $H^*_{(p)}(G, \mathbf{R})$, is called p-bounded (real) cohomology of G. In particular, k-bounded cohomology is the bounded cohomology $H^k_{(k)}(G, \mathbf{R}) = H^k_b(G, \mathbf{R})$ and 0-bounded k-th cohomology is the ordinary one: $H^k_{(0)}(G, \mathbf{R}) = H^k(G, \mathbf{R})$. Since $(p+1)$-bounded cochains are p-bounded, there are comparison homomorphisms induced by inclusion of complexes

$$H^k_b(G, \mathbf{R}) \to H^k_{(p+1)}(G, \mathbf{R}) \to H^k_{(p)}(G, \mathbf{R}) \to H^k(G, \mathbf{R}).$$

We refer to [17] for a general discussion and here we only discuss 1-bounded 2-coboundaries. So, let $\psi\colon G \to \mathbf{R}$ be a function such that its coboundary is 1-bounded:

$$D(f) \coloneqq \sup_g |\psi(g) - \psi(fg) + \psi(f)| < \infty.$$

Lemma 2.1. *Let $\delta\psi$ be a 1-bounded cocycle. The formula*

$$|f|_\psi = \sup_g |\psi(g) - \psi(fg)|$$

defines a pseudo-norm on G. Moreover, ψ is 1-Lipschitz with respect to this norm.

Proof. Both the symmetry and triangle inequality is straightforward:

$$|f^{-1}|_\psi = \sup_g |\psi(g) - \psi(f^{-1}g)| = \sup_g |\psi(f^{-1}g) - \psi(f \cdot f^{-1}g)| = |f|_\psi.$$

$$\begin{aligned} |fg|_\psi &= \sup_h |\psi(h) - \psi(fgh)| \\ &\leq \sup_h |\psi(h) - \psi(gh)| + \sup_h |\psi(gh) - \psi(fgh)| = |f|_\psi + |g|_\psi. \end{aligned}$$

As well as the Lipschitz property:

$$|\psi(f) - \psi(g)| = |\psi(f) - \psi(f \cdot f^{-1}g)| \leq |f^{-1}g|_\psi = d_\psi(f, g),$$

and the proof follows. □

The main construction. Let d be an auxiliary Riemannian metric on M. It induces the supremum metric d_0 on $\mathrm{Homeo}_0(M)$ by

$$d_0(f,g) \coloneqq \sup_{x\in M} d(f(x),g(x)).$$

Since M is a smooth compact manifold it has a CW-decomposition with one top dimensional cell. Let $\Delta \subseteq M$ be the interior of that cell which is diffeomorphic to a ball in $\mathbf{R}^n$ and it is of full measure in M. Choose a fundamental domain in the universal cover of M so that the lift of the inclusion $\Delta \subseteq M$ is contained in it. Let $z = (z_1,\dots,z_n) \in \mathrm{C}_n(M)$ be the base-point, such that $z_i \in \Delta$ for each $i = 1,\dots,n$. Given two points $x, y \in \mathrm{C}_n(\Delta)$, let $\ell_{y,x}$ be a path from x to y consisting of n unit speed geodesics from x_i to y_i with respect to the Euclidean metric on Δ. This Euclidean metric is used to define geodesics $\ell_{y,x}$ and has nothing to do with the auxiliary metric chosen in the beginning of this section.

In case M is a surface and $n > 1$, then the geodesics between x_i and y_i may collide, and then $\ell_{x,y}$ is not defined. However, this happens only for a measure zero set, thus does not affect the integral from the definition of Ψ. For more details see [18, Section 3.2]. By convention, we read concatenation of paths from right to left. That is, $\ell_{z,y} * \ell_{y,x}$ is a path from x to y and then to z.

Let $f \in \mathrm{Homeo}_0(M)$ and let $x \in \mathrm{C}_n(\Delta)$ be a point such that $f(x) \in \mathrm{C}_n(\Delta)$. Let a braid $\gamma(f,x)$ in $P_n(M,z)$ be represented by

$$\gamma(f,x) = \left[\ell_{z,f(x)} * \{f_t(x)\} * \ell_{x,z}\right], \tag{2.1}$$

where $\{f_t\}$ is an isotopy from the identity to f.

The above braids are defined for points $x \in M$ from a set of full measure.

If $P_n(M)$ has no centre then the braid $\gamma(f,x)$ does not depend on the choice of an isotopy $\{f_t\}_{t=0}^1$ (see, e.g., [8, Section 2.2 (6),(7)]). Otherwise, it does depend and that is why we assume that the homogeneous quasimorphism φ from Theorem 1.1 vanishes on the centre and we need the following observation.

Lemma 2.2. *Let $\varphi\colon G \to \mathbf{R}$ be a homogeneous quasimorphism vanishing on a normal subgroup $H \le G$. Then*

$$\varphi(gh) = \varphi(g)$$

for all $g \in G$ and $h \in H$.

Proof. Observe that for any two elements $g, h \in G$ we have $(gh)^k = g^k h^k c_1 \dots c_k$, where c_i are conjugates of commutators of h. It follows that

$$\begin{aligned}
|\varphi(gh) - \varphi(g)| &= \frac{1}{k}\left|\varphi\left((gh)^k\right) - \varphi(g^k)\right| \\
&= \frac{1}{k}\left|\varphi\left(g^k h^k c_1 \dots c_k\right) - \varphi(g^k)\right| \\
&\le \frac{1}{k}\left(|\varphi(g^k) + \varphi(h^k c_1 \dots c_k) - \varphi(g^k)| + D_\varphi\right) = \frac{D_\varphi}{k}.
\end{aligned}$$

The last equality follows since H is normal and hence $h^k c_1 \dots c_k \in H$. Since $k \in \mathbf{N}$ is arbitrary the computation proves the statement. □

Let $f, g \in \mathrm{Homeo}_0(M)$ and let $x \in \mathrm{C}_n(\Delta)$ be such that $g(x), fg(x) \in \mathrm{C}_n(\Delta)$. Choosing the isotopy $\{(fg)_t\}$ to be the concatenation $\{f_t \circ g\} * \{g_t\}$ proves the following crucial fact.

Lemma 2.3. *The function* $\gamma \colon \mathrm{Homeo}_0(M) \times \mathrm{C}_n(\Delta) \to P_n(M)$ *is a cocycle. That is,*

$$\gamma(fg, x) = \gamma(f, g(x))\gamma(g, x)$$

for every $f, g \in \mathrm{Homeo}_0(M)$ *and almost every* $x \in \mathrm{C}_n(\Delta)$. □

3. Proofs

Proof of Theorem 1.1. If $n = 1$, it is known that $\varphi(\gamma(f, x))$ is bounded, thus the integral (1.1) is well defined [8]. Assume therefore that $M = \Sigma$ is a surface and n is arbitrary. Let $P_n(\Delta) = \pi_1(\mathrm{C}_n(\Delta))$. The inclusion $\Delta \subseteq \Sigma$ induces an inclusion $P_n(\Delta) \leq P_n(\Sigma)$, see [20, Theorem 1]. Let $f \in \mathrm{Homeo}_0(\Sigma)$ and let $\varphi \colon P_n(\Sigma) \to \mathbf{R}$ be a quasimorphism which vanishes on $P_n(\Delta)$ and on the centre of $P_n(\Sigma)$. Let $\mathrm{sys}(\Sigma, d)$ denote the systole of Σ. Suppose that f is C^0-small, in the sense, that $d_0(f, \mathrm{Id}) < \epsilon$ for ϵ chosen such that there exists an isotopy f_t, connecting the identity to f, such that

$$d_0(f_t, \mathrm{Id}) < \frac{1}{2n}\mathrm{sys}(\Sigma, d)$$

for all t. Existence of such an ϵ follows from local contractibility of $\mathrm{Homeo}_0(\Sigma)$ [15, Corollary 1.1]. For every x, we shall construct a decomposition $\gamma(f, x) = \alpha \cdot \beta$, where $\beta \in P_n(\Delta)$ and $\alpha \in P_n(\Sigma)$ is a braid from a certain fixed finite set.

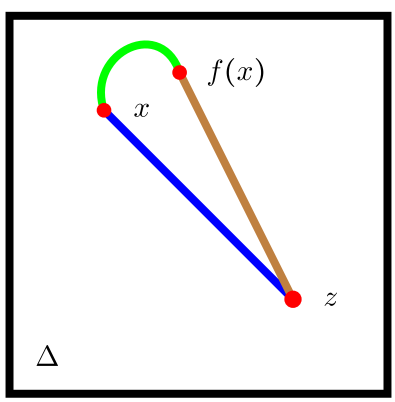

FIGURE 3.1. Isotopy inside Δ.

Let $x = (x_1, \dots, x_n)$. Assume first, that the evaluation $\{f_t(x_i)\}$ of an isotopy $\{f_t\}$ is contained in Δ for all $i = 1, \dots, n$. Then $\gamma(f, x) \in P_n(\Delta)$. In this case $\beta = \gamma(f, x)$ and $\alpha = e$. A simplified picture is presented in Figure 3.1. The points z, x and $f(x)$ should be seen as n-tuples, the blue and brown paths are a collection of geodesics between them and the green path is the union of images of the isotopy $\{f_t\}$ evaluated at x_i's which is possibly braided.

In general, since f is C^0-small, we can choose an isotopy f_t, such that for every i the path $\{f_t(x_i)\}$ is contained in a ball B_i centered in x_i, and of radius smaller than $\frac{1}{2n}\mathrm{sys}(\Sigma, d)$. Recall that

$$\gamma(f,x) = \left[\ell_{z,f(x)} * \{f_t(x)\} * \ell_{x,z}\right], \tag{3.1}$$

see Figure 3.2. Our goal is to move the green part inside $\mathrm{C}_n(\Delta)$ and modify the brown part in a controlled way. To this end, for each i we choose a point $y_i \in \Delta \cap B_i$ such that the unit speed geodesic segment $\sigma_{y_i,f(x_i)}$ connecting $f(x_i)$ to y_i lies entirely in B_i (in particular, we may take $y_i = x_i$). Note that some of these geodesic segments might intersect in the configuration space, but it happens only for measure zero set of points $x \in C_n(\Delta)$. Let $y = \{y_1, \ldots, y_n\}$, and let $\ell_{y,f(x)}$ denote the path in $\mathrm{C}_n(\Sigma)$ consisting of segments $\sigma_{y_i,f(x_i)}$. We have

$$\gamma(f,x) = \left[\ell_{z,f(x)} * \ell_{f(x),y} * \ell_{y,f(x)} * \{f_t(x)\} * \ell_{x,z}\right], \tag{3.2}$$

where $\ell_{f(x),y}$ denotes the time-reverse of $\ell_{y,f(x)}$.

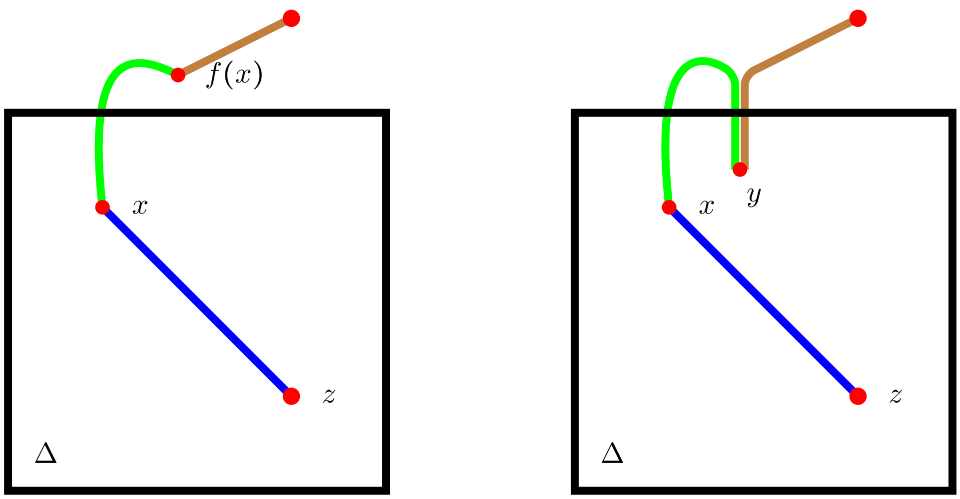

FIGURE 3.2. Back-and-forth.

The green part is contained in $\mathrm{C}_n(\bigcup B_i)$. Moreover, since each B_i has radius less than $\frac{1}{2n}\mathrm{sys}(\Sigma, d)$, every connected component of $\bigcup B_i$ has diameter less than $\mathrm{sys}(\Sigma)$ and is therefore contractible in Σ. Consequently, we can homotope the green part inside $\mathrm{C}_n(\Delta)$, keeping the endpoints fixed:

$$\gamma(f,x) = \left[\ell_{z,f(x)} * \ell_{f(x),y} * s * \ell_{x,z}\right], \tag{3.3}$$

where s denotes the green path after being pushed into $\mathrm{C}_n(\Delta)$, see Figure 3.3. The final step is to connect the endpoint of the green part to z, and return back. We have:

$$\gamma(f,x) = \left[\ell_{z,f(x)} * \ell_{f(x),y} * \ell_{y,z} * \ell_{z,y} * s * \ell_{x,z}\right]. \tag{3.4}$$

We define β to be the path obtained by concatenation of the blue and green paths, and the brown path comprise the braid α. If we vary f and x, only finitely many such braids α can arise, since

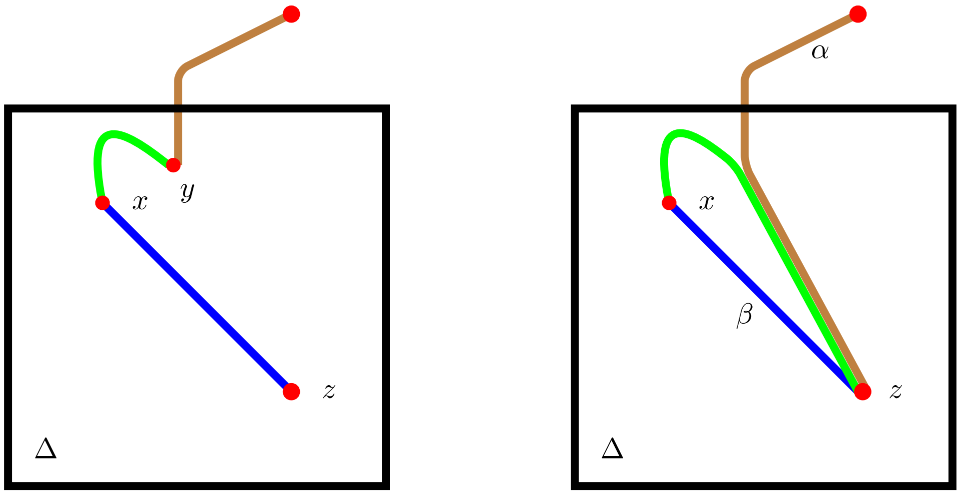

FIGURE 3.3. Push into Δ and back-and-forth again.

each strand of α is a concatenation of three geodesic segments, each of length less than the diameter of Σ. The quasimorphism φ vanishes on $P_n(\Delta)$ by hypothesis, hence the function

$$x \mapsto |\varphi(\gamma(f,x))| = |\varphi(\beta\alpha)| \leq |\varphi(\alpha)| + D_\varphi \tag{3.5}$$

is bounded and so the integral

$$\Psi(f) = \int_{\mathrm{C}_n(\Sigma)} \varphi(\gamma(f,x))dx$$

is well defined, i.e. $\varphi(\gamma(f,x))$ is a L^1-function, for a C^0-small f.

Let $f \in \mathrm{Homeo}_0(\Sigma)$ be an arbitrary homeomorphism and let $\{f_t\}$ be an isotopy from the identity to f, where $t \in [0,1]$. Let $0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$ be a partition of the interval $[0,1]$ such that

$$d_0(f_{t_i}^{-1} f_{t_{i+1}}, \mathrm{Id}) < r,$$

where $r < \frac{1}{2n}\mathrm{sys}(\Sigma, d)$. Of course,

$$f = f_{t_0} \circ f_{t_0}^{-1} f_{t_1} \circ f_{t_1}^{-1} f_{t_2} \circ \cdots \circ f_{t_{m-1}}^{-1} f_{t_m} = g_1 \ldots g_m,$$

where $g_i = f_{t_{i-1}}^{-1} f_{t_i}$. That is, f is a product of C^0-small homeomorphisms. We get

$$\begin{aligned}
\varphi(\gamma(f,x)) &= \varphi(\gamma(g_1 \ldots g_m, x)) \\
&= \varphi(\gamma(g_1, g_2 \ldots g_m x)\gamma(g_2, g_3 \ldots g_m x) \ldots \gamma(g_m, x)) \\
&= \varphi(\gamma(g_1, g_2 \ldots g_m x)) + \varphi(\gamma(g_2, g_3 \ldots g_m x)) + \cdots + \varphi(\gamma(g_m, x)) + d_1 + \cdots + d_{m-1},
\end{aligned}$$

which implies that $\varphi(\gamma(f,x))$ attains finitely many values. Here $|d_i| \leq D_\varphi$. As before, it follows that $\Psi(f)$ is well defined.

Conversely, assume that the integral (1.1) is well defined for all $f \in \mathrm{Homeo}_0(\Sigma)$. Suppose that $\varphi\colon P_n(\Sigma) \to \mathbf{R}$ does not vanish on a braid $\beta \in P_n(\Delta)$. Let $f_1 \in \mathrm{Homeo}_0(\Sigma)$ be a homeomorphism supported in a ball $B_1 \subset \Delta$ of area

$$0 < a < \frac{1}{10}\mu(\Delta)$$

such that

$$\gamma(f,x) = \beta^{10}$$

for $x \in B_1$ in the set of area $b > 0$. Let f_2 be a homeomorphism supported in a ball $B_2 \subset \Delta$ disjoint from B_1 and of area $a/2$ such that

$$\gamma(f,x) = \beta^{10^2}$$

for $x \in B_2$ in the set of area $b/2$. By iterating this construction and taking the limiting homeomorphism $f = f_1 \circ f_2 \circ f_3 \dots$ we get that the integral (1.1) diverges which contradicts our hypothesis. This shows the main statement of Theorem 1.1. We now prove the additional statements.

(1) Let $A(f) = \max\{\varphi(\gamma(f,x)) \mid x \in \mathrm{C}_n(M)\}$. It is a well defined number because $\varphi(\gamma(f,x))$ is bounded as explained in the first part of the proof. The following computations shows that $\delta\Psi$ is semi-bounded.

$$\begin{aligned}
|\delta\Psi(f,g)| &= |\Psi(g) - \Psi(fg) + \Psi(f)| \\
&\leq \int_{\mathrm{C}_n(M)} |\varphi(\gamma(g,x)) - \varphi(\gamma(fg,x)) + \varphi(\gamma(f,x))| dx \\
&\leq \int_{\mathrm{C}_n(M)} \left|\varphi(\gamma(g,x)) - \varphi(\gamma(f,gx)\gamma(g,x)) + \varphi(\gamma(f,x))\right| dx \\
&\leq \int_{\mathrm{C}_n(M)} \left(\left|\varphi(\gamma(g,x)) - \varphi(\gamma(f,gx)) - \varphi(\gamma(g,x)) + \varphi(\gamma(f,x))\right| + D_\varphi\right) dx \\
&\leq \int_{\mathrm{C}_n(M)} \left(\left|\varphi(\gamma(f,gx)) - \varphi(\gamma(f,x))\right| + D_\varphi\right) dx \\
&\leq \int_{\mathrm{C}_n(M)} (2A(f) + D_\varphi) dx < \infty.
\end{aligned}$$

(2) The quasimorphism property for the restriction of Ψ to area preserving homeomorphism is a standard computation as in [4]. The C^0-continuity of the homogenization $\overline{\Psi}$ follows from a theorem of Shtern [28, Theorem 1], which states that a homogeneous quasimorphism on a topological group is continuous if and only if it is bounded on a neighborhood of the identity. Indeed, if f is C^0-small then

$$|\varphi(\gamma(f,x))| = |\varphi(\beta\alpha)| \leq |\varphi(\alpha)| + D_\varphi,$$

which is bounded independently of f. Hence $\Psi(f)$ is bounded on a neighborhood of the identity. Since

$$\sup_f |\overline{\Psi}(f) - \Psi(f)| \leq D_\Psi,$$

we get that $\overline{\Psi}$ is also bounded on a neighborhood of the identity and its continuity follows from Shtern's theorem.

(3) Assume $n = 1$ and let γ be a braid such that $\overline{\varphi}(\gamma) \neq 0$. Let f be a point pushing homeomorphism f along the loop γ such that $\gamma(f,x) \in \{\gamma, e\}$ for all x away from a set of arbitrary small measure. This gives us $\overline{\Psi}(f) \neq 0$ (see, e.g., [10, Theorem 2.5]). If $n > 1$ and φ extends to the full braid group $B_n(\Sigma)$ then the unboundedness of Ψ is proven in [5] extending an argument of Ishida [21]. □

4. Distortion

Let $z \in \mathrm{C}_n(M)$ be the base-point and let $\Psi_z \colon \mathrm{Homeo}_0(M) \to \mathbf{R}$ be the composition

$$\Psi_z(f) = \varphi(\gamma(f,z)),$$

where $\varphi \colon P_n(M) \to \mathbf{R}$ is a homogeneous quasimorphism vanishing on the centre of $P_n(M)$ (recall that if the dimension of M is greater than 2 we assume $n = 1$). If M is a surface we additionally assume that φ vanishes on $P_n(\Delta)$. Recall that under this assumptions the function $x \mapsto |\varphi(\gamma(f,x))|$ is bounded, according to (3.5).

Proposition 4.1. *The differential $\delta\Psi_z$ is a semi-bounded cocycle.*

Proof. Let $B_f = \sup_x |\varphi(\gamma(f,x))|$. For $f, g \in \mathrm{Homeo}_0(M)$ we have

$$\begin{aligned}
|\delta\Psi_z(f,g)| &= |\varphi(\gamma(g,z)) - \varphi(\gamma(fg,z)) + \varphi(\gamma(f,z))| \\
&\le |\varphi(\gamma(g,z)) - \varphi(\gamma(f,g(z))) - \varphi(\gamma(g,z)) + \varphi(\gamma(f,z))| + D_\varphi \\
&= |\varphi(\gamma(f,g(z))) - \varphi(\gamma(f,z))| + D_\varphi \\
&= \begin{cases} |\varphi(\gamma(f,g(z)) - \varphi(\gamma(f,z))| + D_\varphi & \text{if } g(z) \neq z \\ D_\varphi & \text{if } g(z) = z. \end{cases} \\
&= 2B_f + D_\varphi.
\end{aligned}$$

This shows that $\delta\Psi$ is 1-bounded. ∎

As explained in the introduction, if

$$\limsup_{k\to\infty} \frac{|\Psi_z(f^k)|}{k} > 0$$

then f is undistorted in $\mathrm{Homeo}_0(M)$, according to Theorem 1.6. The following proposition is a direct application and its proof is straightforward.

Proposition 4.2. *Let $f \in \mathrm{Homeo}_0(M)$ and $x \in M$ be such that*

$$\gamma(f^{n_k}, x) = \gamma(f,x)^{n_k} \tag{4.1}$$

for an increasing sequence $\{n_k\}_{k\in\mathbf{N}}$ of positive integers. If $\varphi(\gamma(f,x)) > 0$ then f is undistorted in $\mathrm{Homeo}_0(M)$. ∎

Example 4.3. The hypothesis (4.1) of the above proposition is immediate if $x \in M$ is a fixed point of f. If not then there exists a homeomorphism $s \colon M \to M$ and its isotopy $\{s_t\}$ to the identity such that $t \mapsto s_t(f(x))$ is equal to $\ell_{x,f(x)}$. It follows that x is a fixed point of sf and

$$\gamma(sf,x) = \gamma(f,x).$$

Moreover,

$$\gamma\left((sf)^k, x\right) = \gamma(sf,x)^k = \gamma(f,x)^k.$$

If $\varphi(\gamma(f,x)) > 0$ then sf is undistorted, according to Proposition 4.2. ◊

The above example shows that the dynamics of distorted homeomorphisms is often very restricted, see also [25]. Let us discuss a family of examples. We say that a group G is quasi-residually real if for every element $g \in G$ there exists a homogeneous quasimorphism $\varphi: G \to \mathbf{R}$ such that $\varphi(g) \neq 0$. Examples of such groups include right-angled Artin groups, pure braid groups and many hyperbolic groups [6].

Corollary 4.4. *Let M be a closed manifold with quasi-residually real pure braid group $P_n(M)$ with trivial centre. Let $f \in \mathrm{Homeo}_0(M)$ be a distorted element. Then every fixed point $x \in M$ of f is contractible. That is, the loop $\{f_t(x)\}$ is contractible for every choice of isotopy from the identity to f. In particular, there is an isotopy from the identity to f fixing any finite tuple $(x_1, \dots, x_n)$ of fixed points of f.*

Proof. Let $\mathrm{Homeo}_0(M, x_1, \dots, x_n) = \{g \in \mathrm{Homeo}_0(M) \mid g(x_i) = x_i,\ i = 1, \dots, n\}$. Consider the evaluation fibration

$$\mathrm{Homeo}_0(M, x_1, \dots, x_n) \to \mathrm{Homeo}_0(M) \xrightarrow{\mathrm{ev}} \mathrm{C}_n(M),$$

where $\mathrm{ev}(g) = g(x_1, \dots, x_n)$. Since $P_n(M) = \pi_1(\mathrm{C}_n(M, (x_1, \dots, x_n)))$ has trivial centre and the group $\mathrm{Homeo}_0(M)$ is connected, the connecting homomorphism

$$\partial: P_n(M) \to \pi_0(\mathrm{Homeo}_0(M, x_1, \dots, x_n))$$

is an isomorphism of groups. Since f is distorted and $P_n(M)$ is quasi-residually real, the element $\gamma(f, (x_1, \dots, x_n))$ is trivial. This implies that f belongs to the connected component of the identity of $\mathrm{Homeo}_0(M, (x_1, \dots, x_n))$. □

Remark 4.5. In the above corollary we allow $n > 1$ for higher dimensional manifolds. In such case $P_n(M) = \pi_1(M)^n$.

Remark 4.6. If we drop the hypothesis about trivial centre of $P_n(M)$ then the element $\gamma(f, x)$ is not well defined and it depends on the choice of isotopy $\{f_t\}$ from the identity to f. However, the conclusion in this case is that the loop $\{f_t(x_1, \dots, x_n)\}$ represents a central element of $P_n(M)$.

A homeomorphism f of a manifold M is called *recurrent* if there exists an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that

$$\lim_{k \to \infty} d_0(f^{n_k}, \mathrm{Id}) = 0.$$

In other words, arbitrary large powers of f are arbitrarily close to the identity. For example, if f is periodic then it is obviously recurrent. Another example is an irrational rotation of the circle. The only recurrent diffeomorphisms of surfaces of higher genus are periodic, according to [23].

Proposition 4.7. *Let $f \in \mathrm{Homeo}_0(M)$ be a recurrent homeomorphism of a closed manifold. Suppose that $\pi_1(M, z)$ has trivial centre. If $z \in M$ is a fixed point of f, then $\gamma(f, z) \in \pi_1(M, z)$ is torsion.*

Proof. Since z is a fixed point of f, we have $\gamma(f^k, z) = \gamma(f, z)^k$. On the other hand, since f is recurrent, there exists $k_0 \in \mathbf{N}$ such that f^{k_0} is C^0-close to the identity. Hence $\gamma(f^{k_0}, z) = 1$. □

Militon proved that every recurrent *diffeomorphism* of a closed manifold M is distorted [24]. The recurrence of a diffeomorphism is meant here with respect to the C^∞-topology.

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