

The Schwarz-Milnor lemma for braids and area-preserving diffeomorphisms

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Abstract

We prove a number of new results on the large-scale geometry of the L^p -metrics on the group of area-preserving diffeomorphisms of each orientable surface. Our proofs use in a key way the Fulton-MacPherson type compactification of the configuration space of n points on the surface due to Axelrod-Singer and Kontsevich. This allows us to apply the Schwarz-Milnor lemma to configuration spaces, a natural approach which we carry out successfully for the first time. As sample results, we prove that all right-angled Artin groups admit quasi-isometric embeddings into the group of area-preserving diffeomorphisms endowed with the L^p -metric, and that all Gambaudo-Ghys quasi-morphisms on this metric group coming from the braid group on n strands are Lipschitz. This was conjectured to hold, yet proven only for low values of n and the genus g of the surface.

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1 Introduction and main results

1.1 Introduction

The L^2 -length of a path of volume-preserving diffeomorphisms, which describes a time-dependent flow of an ideal incompressible fluid, corresponds to the hydrodynamic action of the flow in the same way as the length of a path in a Riemannian manifold corresponds to its energy (cf. [50]). Indeed, it is the length of this path with respect to the formal right-invariant Riemannian metric on the group \mathcal{G} of volume preserving diffeomorphisms introduced by Arnol'd in [1]. The L^1 -length of the same path has a dynamical interpretation as the average length of a trajectory of a point under the flow.

Therefore, following the principle of least action, it makes sense to consider the infimum of the lengths of paths connecting two fixed volume-preserving diffeomorphisms. This gives rise to a right-invariant

distance function (metric) on \mathcal{G} . Taking the identity transformation as the initial point, Arnol'd observes that a path whose L^2 -length is minimal (and equal to the distance) necessarily solves the Euler equation of an ideal incompressible fluid.

It follows from works of Ebin and Marsden [21] that for diffeomorphisms in \mathcal{G} that are C^2 -close to the identity, the infimum is indeed achieved. Further, more global results on the corresponding Riemannian exponential map were obtained in [19],[51] (see also [20]). In [49, 50] Shnirel'man showed, among a number of surprising facts about this subject, that in the case of the ball of dimension 3, the diameter of the L^2 -metric is bounded. This result is conjectured to hold for all compact simply connected manifolds of dimension 3 or larger (see [23, 38, 2]), while its analogue in the non-simply-connected case is false [23, 7]. Furthermore, Shnirel'man has conjectured that for compact manifolds of dimension 2, the L^2 -diameter is infinite.

Shnirel'man's conjecture, and its analogues for L^p -metrics, with $p \geq 1$ are by now proven. It follows from results of Eliashberg and Ratiu [23] that on compact surfaces (possibly with boundary) other than T^2 and S^2 , Shnirel'man's conjecture holds for all $p \geq 1$. Their arguments rely on the Calabi homomorphism Cal [16] from the compactly supported Hamiltonian group $\text{Ham}_c(M, \sigma)$ to the real numbers in the case of a surface M with non-empty boundary (σ is the area form), and on non-trivial first cohomology combined with trivial center of the fundamental group in the closed case. For the two-torus T^2 Shnirel'man's conjecture holds by [14, Appendix A]. Finally, the case of S^2 was settled in [14] by means of differential forms on the configuration space related to the cross-ratio map. In [43] the second author gave a new uniform proof of Shnirel'man's conjecture for all compact surfaces.

The methods that were used to prove Shnirel'man's conjecture are two-dimensional in nature, and have to do with braiding of trajectories of time-dependent two-dimensional Hamiltonian flows (in extended phase space). Indeed, Shnirel'man has proposed to use relative rotation numbers to bound from below the L^2 -lengths of two-dimensional Hamiltonian paths in [50]. This direction is related to the method of [23] by a theorem of Fathi [24] and Gambaudo and Ghys [27] (see also [48, 31, 33]). This theorem shows that the Calabi homomorphism is proportional to the relative rotation number of the trajectories of two distinct points in the two-disc \mathbb{D} under a Hamiltonian flow, averaged over the configuration space of ordered pairs of distinct points (x_1, x_2) in the two-disc.

This line of research was notably pursued in [29], and further in [4], [18], [7], [9], [39], [14] obtaining quasi-isometric and bi-Lipschitz embeddings of various groups (right-angled Artin groups and additive groups of finite-dimensional real vector spaces) into $\text{Ham}_c(\mathbb{D}^2, dx \wedge dy)$, into $\ker(\text{Cal}) \subset \text{Ham}_c(\mathbb{D}^2, dx \wedge dy)$, and into $\text{Ham}(S^2, \sigma)$ endowed with their respective L^p -metrics (see [10] for similar embedding results on manifolds with a sufficiently complicated fundamental group). In all cases, the key technical estimate is an upper bound, via the L^p -length of an isotopy of volume-preserving diffeomorphisms, of the average, over all points in a configuration space of the manifold, of the word length in the fundamental group of the trace of the point under the induced isotopy (closed up to a loop by a system of short paths on the configuration space).

Such estimates were initially produced by means of analyzing relative rotation numbers of pairs of braids, or quadruples as in [14]. However, in the case of a single braid, as observed by Polterovich, a simpler estimate is possible via the Schwarz-Milnor lemma¹ [22, 54, 44]. Indeed, according to this result, the universal cover of compact proper length space, in this case our surface, is quasi-isometric to its fundamental group. In this way one can control the word length of trajectories in terms of their lengths, which allows one to prove the required upper bound. A similar estimate in the case of braids on more than one strand is not as readily available, because the configuration space $X_n(M)$ of n points on M is not a compact metric space. The arguments of the second author in [43] rely crucially on a new complete metric on $X_n(M)$. However, with this metric $X_n(M)$ still can not be considered to be a compact metric space from the point of view of the Schwarz-Milnor lemma.

¹“Schwarz” is a transliteration from Russian; alternative spellings “Švarc”, “Schwartz” sometimes appear in the literature.

In this paper we show how to successfully carry out the strategy of the Schwarz-Milnor lemma for configuration spaces. While hints of a similar approach can be discerned in [26] in the special case of the two-disc and of double collisions, it was not known earlier to be applicable in the general context discussed herein. Specifically, we consider a metric on $X_n(M)$ coming from a natural compactification $\bar{X}_n(M)$ thereof, which is a compact geodesic metric space. This compactification is equivariant under the action of the diffeomorphism group $\text{Diff}(M)$ on $X_n(M)$ in the sense that the action extends naturally to the compactification². Furthermore, the map $\pi_1(X_n(M)) \rightarrow \pi_1(\bar{X}_n(M))$ induced by the inclusion is an isomorphism. Hence we may apply the Schwarz-Milnor lemma to $\bar{X}_n(M)$. This, and further comparison to the metric from [43] allows us to prove our main estimate.

The compactification we use was introduced by Axelrod-Singer in [3] and by Kontsevich in [40, 41], inspired by the Fulton-MacPherson compactification in algebraic geometry [25]. The compactification $\bar{X}_n(M)$ is roughly speaking a certain positive oriented blow-up of M^n along its multi-diagonals. For instance, one part of its codimension 1 boundary stratum is identified with the disjoint union of spaces of the form $N_1(D_{ij}(M))|_{D_{ij}^0(M)}$ where N_1 is the unit normal bundle, $D_{ij}(M) \subset M^n$ is the submanifold of points (x_1, \dots, x_n) where $x_i = x_j$ and $D_{ij}^0(M) \subset D_{ij}(M)$ is the open dense subset where $x_k \neq x_i = x_j$ for all $k \in \{1, \dots, n\} \setminus \{i, j\}$. Note that this is an S^1 -bundle. Intuitively, from a physical perspective, this means that one resolves a double collision of points by recording the collision point and the direction in which they have collided. Other parts of the codimension 1 stratum correspond to simple k -tuple collisions, and correspond to S^{2k-3} -bundles normal to k -diagonals. Higher strata correspond to more complicated collisions modeled by suitable graphs. It will, however, be technically most convenient for us to use a model of this compactification recently constructed directly as a subspace of a Euclidean space by Sinha [52]. We describe this construction in Section 2 below.

Finally, we observe that lower bounds on the average word length can often be provided by quasimorphisms - functions that are additive with respect to the group multiplication - up to an error which is uniformly bounded (as a function of two variables). The quasimorphisms we consider here were introduced and studied by Gambaudo and Ghys in the beautiful paper [28] (see also [8, 45, 46, 47, 6, 14, 13]). These quasimorphisms essentially appear from invariants of braids traced out by the action of a Hamiltonian path on an ordered n -tuple of distinct points in the surface (suitably closed up), averaged over the configuration space $X_n(M)$ of n -tuples of distinct points on the surface M . As one of our results, we prove that all homogeneous Gambaudo-Ghys quasi-morphisms are Lipschitz in the L^p -metric for all $p \geq 1$. This subsumes all previous results in this direction and provides a maximally general result. It also contrasts a recent result of Khanevsky according to which none of these quasi-morphisms are continuous in Hofer's metric [37].

We prove further stronger results on the large-scale geometry on the L^p -metric on $\mathcal{G} = \text{Diff}_{c,0}(M, \sigma)$ for a compact surface (M, σ) with an area form. In particular, we provide bi-Lipschitz group monomorphisms of \mathbb{R}^m endowed with the standard (say Euclidean) metric into (\mathcal{G}, d_{L^p}) for each positive integer m and each $p \geq 1$. Finally, our methods combined with an argument of Kim-Koberda [39] (cf. Crisp-Wiest [18], Benaim-Gambaudo [4]) show the existence of quasi-isometric group monomorphisms from each right-angled Artin group to (\mathcal{G}, d_{L^p}) for each $p \geq 1$. We note that this was previously known only for \mathbb{D}^2 and S^2 [39, 14].

Let $M_1 \hookrightarrow M_2$ be a measure preserving embedding of surfaces. It is an open question if the induced Lipschitz monomorphism $\text{Diff}_0(M_1, \sigma_1) \hookrightarrow \text{Diff}_0(M_2, \sigma_2)$ is a quasi-isometric embedding. Note that our results provide a partial positive answer to this problem, see Remark 1.3.5.

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²It is curious to note that the same is *not* true for the action of $\text{Homeo}(M)$ as was recently proven [42].

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1.2 Preliminaries

1.2.1 The L^p -metric

Let M denote a smooth oriented manifold without boundary that is either closed, or $M = X \setminus \partial X$ for a compact manifold X . Let M be endowed with a Riemannian metric g and smooth measure μ (given by an orientation on M and volume form, which in our case of a surface M is an area form σ). We require that g and μ extend continuously to X in the second case. Finally denote by

$$\mathcal{G} = \text{Diff}_{c,0}(M, \mu)$$

the identity component of the group of compactly supported diffeomorphisms of M preserving the smooth measure μ . In other words, if $M = X \setminus \partial X$, it is the identity component of the group of measure preserving diffeomorphisms of X fixing point-wise a neighbourhood of ∂X .

Fix $p \geq 1$. For a smooth isotopy $\{\phi_t\}_{t \in [0,1]}$, from $\phi_0 = \mathbf{1}$ to $\phi_1 = \phi$, we define the L^p -length by

$$l_p(\{\phi_t\}) = \int_0^1 \left(\frac{1}{\text{vol}(M, \mu)} \cdot \int_M |X_t|^p d\mu \right)^{\frac{1}{p}} dt,$$

where $X_t = \frac{d}{dt}|_{t'} \phi_{t'} \circ \phi_t^{-1}$ is the time-dependent vector field generating the isotopy $\{\phi_t\}$, and $|X_t|$ is its length with respect to the Riemannian structure on M . As is easily seen by a displacement argument, the L^p -length functional determines a non-degenerate norm on \mathcal{G} by the formula

$$d_p(\mathbf{1}, \phi) = \inf l_p(\{\phi_t\}).$$

This in turn defines a right-invariant metric on \mathcal{G} by the formula

$$d_p(\phi_0, \phi_1) = d_p(\mathbf{1}, \phi_1 \phi_0^{-1}).$$

Remark 1.2.1. Consider the case $p = 1$. It is easy to see that the L^1 -length of an isotopy is equal to the average Riemannian length of the trajectory $\{\phi_t(x)\}_{t \in [0,1]}$ (over $x \in M$, with respect to μ). Moreover for each $p \geq 1$, by Jensen's (or Hölder's) inequality, we have

$$l_p(\{\phi_t\}) \geq l_1(\{\phi_t\}).$$

Denote by $\tilde{\mathbf{1}}$ the identity element of the universal cover $\tilde{\mathcal{G}}$ of \mathcal{G} . Similarly one has the L^p -pseudo-norm (that induces the right-invariant L^p -pseudo-metric) on $\tilde{\mathcal{G}}$, defined for $\tilde{\phi} \in \tilde{\mathcal{G}}$ as

$$d_p(\tilde{\mathbf{1}}, \tilde{\phi}) = \inf l_p(\{\phi_t\}),$$

where the infimum is taken over all paths $\{\phi_t\}$ in the class of $\tilde{\phi}$. Clearly $d_p(\mathbf{1}, \phi) = \inf d_p(\tilde{\mathbf{1}}, \tilde{\phi})$, where the infimum runs over all $\tilde{\phi} \in \tilde{\mathcal{G}}$ that map to ϕ under the natural epimorphism $\tilde{\mathcal{G}} \rightarrow \mathcal{G}$.

Up to bi-Lipschitz equivalence of metrics (d and d' are equivalent if $\frac{1}{C}d \leq d' \leq Cd$ for a certain constant $C > 0$) the L^p -metric on \mathcal{G} (and its pseudo-metric analogue on $\tilde{\mathcal{G}}$) is independent of the choice Riemannian structure and of the volume form μ on M . In particular, the question of boundedness or unboundedness of the L^p -metric enjoys the same invariance property.

Terminology: For a positive integer n , we use $A, B, C > 0$ as generic notation for positive constants that depend only on M, μ, g and n .

1.2.2 Quasimorphisms

Some of our results have to do with the notion of a quasimorphism. Quasimorphisms are a helpful tool for the study of non-abelian groups, especially those that admit few homomorphisms to \mathbb{R} . A quasimorphism $r : G \rightarrow \mathbb{R}$ on a group G is a real-valued function that satisfies

$$r(xy) = r(x) + r(y) + b_r(x, y),$$

for a function $b_r : G \times G \rightarrow \mathbb{R}$ that is uniformly bounded:

$$\delta(r) := \sup_{G \times G} |b_r| < \infty.$$

A quasimorphism $\bar{r} : G \rightarrow \mathbb{R}$ is called *homogeneous* if $\bar{r}(x^k) = k\bar{r}(x)$ for all $x \in G$ and $k \in \mathbb{Z}$. In this case, it is additive on each pair $x, y \in G$ of commuting elements: $\bar{r}(xy) = \bar{r}(x) + \bar{r}(y)$ if $xy = yx$.

For each quasimorphism $r : G \rightarrow \mathbb{R}$ there exists a unique homogeneous quasimorphism \bar{r} that differs from r by a bounded function:

$$\sup_G |\bar{r} - r| < \infty.$$

It is called the *homogenization* of r and satisfies

$$\bar{r}(x) = \lim_{n \rightarrow \infty} \frac{r(x^n)}{n}.$$

Denote by $Q(G)$ the real vector space of homogeneous quasimorphisms on G .

For a finitely-generated group G , with finite symmetric generating set S , define the word norm $|\cdot|_S : G \rightarrow \mathbb{Z}_{\geq 0}$ by

$$|g|_S = \min\{k \mid g = s_1 \cdots s_k, \forall 1 \leq j \leq k, s_j \in S\}$$

for $g \in G$. This is a norm on G , and as such it induces a right-invariant metric $d_S : G \times G \rightarrow \mathbb{Z}_{\geq 0}$ by $d_S(f, g) = |gf^{-1}|_S$. This metric is called the word metric. In this setting, any quasimorphism $r : G \rightarrow \mathbb{R}$ is controlled by the word norm. Indeed, for all $g \in G$,

$$|r(g)| \leq \left(\delta(r) + \max_{s \in S} |r(s)| \right) \cdot |g|_S.$$

When a specific symmetric generating set S for G can be fixed, we will usually denote $|\cdot|_S$ by $|\cdot|_G$.

We refer to [17] for more information about quasimorphisms.

1.2.3 Configuration spaces and braid groups

For a manifold M , which in this paper is usually of dimension 2, the configuration space $X_n(M) \subset M^n$ of n -tuples of points on M is defined as

$$X_n(M) = \{(x_1, \dots, x_n) \mid x_i \neq x_j, 1 \leq i < j \leq n\}.$$

That is

$$X_n(M) = M^n \setminus \bigcup_{1 \leq i < j \leq n} D_{ij}$$

where for $1 \leq i < j \leq n$, the partial diagonal $D_{ij} \subset M^n$ is defined as $D_{ij} = \{(x_1, \dots, x_n) \mid x_i = x_j\}$. Note that D_{ij} is a submanifold of M^n of codimension $\dim M$.

Finally, if $\dim M = 2$, we define the pure braid group of M as

$$P_n(M) = \pi_1(X_n(M)).$$

Noting that the symmetric group S_n on n elements acts freely on $X_n(M)$, we form the quotient $C_n(M) = X_n(M)/S_n$ and define the full braid group of M as

$$B_n(M) = \pi_1(C_n(M)).$$

We note that $P_n(M)$ and $B_n(M)$ enter the exact sequence $1 \rightarrow P_n(M) \rightarrow B_n(M) \rightarrow S_n \rightarrow 1$. In particular $P_n(M)$ is a normal subgroup of $B_n(M)$ of finite index. We refer to [36] for further information about braid groups.

1.2.4 Short paths and the Gambaudo-Ghys construction

Let M be a compact oriented surface. Given a real valued quasimorphism r on $P_n(M) = \pi_1(X_n(M), q)$ for a fixed basepoint $q \in X_n(M)$ there is a natural way to construct a real valued quasimorphism on the universal cover $\tilde{\mathcal{G}}$ of the group $\mathcal{G} = \text{Diff}_{c,0}(M, \sigma)$ of area preserving diffeomorphisms of the surface M . We shall see that in the case of $M \neq T^2$ this induces a quasimorphism on \mathcal{G} itself, because the fundamental group of \mathcal{G} is finite. The same is true for $M = T^2$ where we consider the group $\mathcal{G} = \text{Ham}(M, \sigma)$ of Hamiltonian diffeomorphisms instead. This is not a restrictive condition from the viewpoint of large-scale geometry, since by a small modification of [14, Proposition A.1], the inclusion $(\text{Ham}(T^2, \sigma), d_{L^p}) \hookrightarrow (\text{Diff}_0(T^2, \sigma), d_{L^p})$ is a quasi-isometry for all $p \geq 1$.

The construction is carried out by the following steps (cf. [28, 45, 6]).

1. For all $x \in X_n(M) \setminus Z$, with Z a closed negligible subset (e.g. a union of submanifolds of positive codimension) choose a smooth path $\gamma(x) : [0, 1] \rightarrow X_n(M)$ between the basepoint $q \in X_n(M)$ and x . Make this choice continuous in $X_n(M) \setminus Z$. We first choose a system of paths on M itself. Then we consider the induced coordinate-wise paths in M^n , and pick Z to ensure that these induced paths actually lie in $X_n(M)$. After choosing the system of paths $\{\gamma(x)\}_{x \in X_n(M) \setminus Z}$ we extend it measurably to all $x \in X_n(M)$ (obviously, no numerical values computed in the paper will depend on this extension). We call the resulting choice a "system of short paths".
2. Given a path $\{\phi_t\}_{t \in [0,1]}$ in \mathcal{G} starting at Id , and a point $x \in X_n(M)$ consider the path $\{\phi_t \cdot x\}$, to which we then catenate the corresponding short paths. That is consider the loop

$$\lambda(x, \{\phi_t\}) := \gamma(x) \# \{\phi_t \cdot x\} \# \gamma(x)^{-1}$$

in $X_n(M)$ based at q , where $^{-1}$ denotes time reversal. Hence we obtain for each $x \in X_n(M) \setminus Z \cup (\phi_1)^{-1}(Z)$ an element $[\lambda(x, \{\phi_t\})] \in \pi_1(X_n(M), q)$ (or rather for each $x \in X_n(M)$ after the measurable extension in Step 1).

3. Consequently applying the quasimorphism $r : \pi_1(X_n(M), q) \rightarrow \mathbb{R}$ we obtain a measurable function $f : X_n(M) \rightarrow \mathbb{R}$. Namely $f(x) = r([\lambda(x, \{\phi_t\})])$. The quasimorphism Φ on $\tilde{\mathcal{G}}$ is defined by

$$\Phi([\{\phi_t\}]) = \int_{X_n(M)} f d\mu^{\otimes n}.$$

It is immediate to see that this function is well-defined by topological reasons. The quasimorphism property follows by the quasimorphism property of r combined with finiteness of volume. The fact that the function f is absolutely integrable can be shown to hold a-priori by a reduction to the case of the disc. We note, however, that by Tonelli's theorem this fact follows as a by-product of the proof of our main theorem, and therefore requires no additional proof.

4. Of course this quasimorphism can be homogenized, to obtain a homogeneous quasimorphism $\bar{\Phi}$.

Remark 1.2.2. If $M \neq S^2$ and $M \neq T^2$, then $\pi_1(\mathcal{G}) = 0$ and for $M = T^2$ the same is true for $\mathcal{G} = \text{Ham}(M, \sigma)$. In the case $M = S^2$, by the result of Smale [53] $\pi_1(\mathcal{G}) = \mathbb{Z}/2\mathbb{Z}$. Hence the quasimorphisms descend to quasimorphisms on \mathcal{G} , e.g. by minimizing over the two-element fibers of the projection $\tilde{\mathcal{G}} \rightarrow \mathcal{G}$. For $\bar{\Phi}$, the situation is easier since by homogeneity it vanishes on $\pi_1(\mathcal{G}) \subset Z(\tilde{\mathcal{G}})$, and therefore

depends only on the image in \mathcal{G} of an element in $\tilde{\mathcal{G}}$. We keep the same notations for the induced quasimorphisms.

1.3 Main results

Recall that we work with the group $\mathcal{G} = \text{Diff}_{c,0}(M, \sigma)$ for (M, σ) a compact oriented surface with an area form. Our main technical result is the following.

Theorem 1. *For an isotopy $\bar{\phi} = \{\phi_t\}$ in \mathcal{G} , the average word norm of a trajectory $\lambda(x, \bar{\phi})$ is controlled by the L^1 -length of $\bar{\phi}$:*

$$W(\bar{\phi}) = \int_{X_n(M) \setminus Z \cup (\phi_1)^{-1}(Z)} |\lambda(x, \bar{\phi})|_{P_n(M)} d\mu^{\otimes n}(x) \leq A \cdot l_1(\bar{\phi}) + B,$$

for certain constants $A, B > 0$.

Remark 1.3.1. Note that $W(\bar{\phi})$ depends only on the class $\tilde{\phi} = [\bar{\phi}] \in \tilde{\mathcal{G}}$ of $\bar{\phi}$ in the universal cover $\tilde{\mathcal{G}}$ of \mathcal{G} . Hence for all compact oriented surfaces other than S^2 and T^2 , $W(\bar{\phi})$ does not depend on the chosen isotopy $\bar{\phi}$, but only on the diffeomorphism ϕ_1 . This is because \mathcal{G} is simply connected.

Theorem 1 has a number of consequences concerning the large-scale geometry of the L^1 -metric on \mathcal{G} . Firstly, as any quasimorphism on a finitely generated group is controlled by the word norm, we immediately obtain the following statement.

Corollary 1. *The homogenization $\bar{\Phi}$ of each Gambaudo-Ghys quasimorphism Φ satisfies*

$$|\bar{\Phi}(\phi)| \leq C \cdot d_1(\phi, 1).$$

In particular, Corollary 1 implies that all Gambaudo-Ghys quasimorphisms are continuous in the L^1 -metric (and hence in the L^p -metric, see Remark 1.3.4), a fact which was known so far only for the genus zero case (see [14, 7]) and for the higher genus case only when one considers Gambaudo-Ghys quasimorphisms coming from the fundamental group $P_1(M)$, see [7]. Note that none of these quasimorphisms are continuous in the Hofer metric by a recent result by Khanevsky [37].

By a theorem of Ishida [34] in the genus zero case and its generalisation to any compact oriented surface [12, Theorem 2.2] (see also Brandenbursky-Kedra-Shelukhin [11] in the genus one case, and Brandenbursky [8] in the higher genus case), the image of the map $Q(P_n(M)) \xrightarrow{GG} Q(\mathcal{G})$ is infinite dimensional for $n \geq 4$. Thus $Q(\mathcal{G})$ is an infinite-dimensional vector space. Hence by Corollary 1 we obtain in particular the following.

Corollary 2. *The L^1 -diameter of \mathcal{G} is infinite.*

Let Ent^k be the set of products of at most k entropy zero diffeomorphisms. Theorem 1 in [12] and Corollary 1 imply the following.

Corollary 3. *For each positive integer k , the complement in \mathcal{G} of the set Ent^k contains a ball of any arbitrarily large radius in the L^1 -metric. In particular, the set of non-autonomous diffeomorphisms contains a ball of any arbitrarily large radius in the L^1 -metric.*

In what follows, we apply an argument of Kim-Koberda [39] (cf. Benaim-Gambaudo [4] and Crisp-Wiest [18]), and use Theorem 1 in order to obtain the following statement, which generalizes an answer to a question of Kapovich [35] in the case of S^2 [14] to arbitrary compact oriented surfaces.

Corollary 4. *The metric group (\mathcal{G}, d_1) admits a quasi-isometric group embedding from each right-angled Artin group endowed with the word metric.*

Proof. Assume first that $M = \Sigma_g$ a closed surface of a positive genus. The case of S^2 was already done in [14] and the case of surfaces with boundary will be discussed in the end of the proof. Let $n \in \mathbb{N}$. Let $D \subset \Sigma_g$ be a smoothly embedded open disc and z_1, \dots, z_n disjoint points in D . For each i , let $D_i \subset D$ be an embedded open disc centered at z_i such that each short geodesic between two points in D_i lies in D_i , and $D_i \cap D_j = \emptyset$ for each $i \neq j$. We denote by $\text{Diff}(D; D_1, \dots, D_n) < \mathcal{G}$ the group which consists of all diffeomorphisms in \mathcal{G} which are compactly supported in D and act by identity on each D_i . This subgroup is equipped with the L^1 -metric coming from \mathcal{G} .

Lemma 1. *Let $n > 4$. We identify P_n with the pure mapping class group of the disc D punctured at z_1, \dots, z_n . Then the inclusion $P_n \rightarrow P_n(\Sigma_g)$ is a quasi-isometric embedding.*

Proof. Since P_n is quasi-isometric to B_n and $P_n(\Sigma_g)$ is quasi-isometric to $B_n(\Sigma_g)$, it is enough to show that the inclusion $B_n \rightarrow B_n(\Sigma_g)$ is a quasi-isometric embedding, where B_n is identified with the mapping class group of the disc D punctured at z_1, \dots, z_n . By results of Goldberg [30] (c.f. Birman [5]) we have that B_n is a subgroup of $B_n(\Sigma_g)$. In addition, the composition

$$B_n \xrightarrow{i} B_n(\Sigma_g) \xrightarrow{F} MCG_{g,n}$$

is injective, where i is the inclusion, the map $F : B_n(\Sigma_g) \rightarrow MCG_{g,n}$ is the point pushing map and $MCG_{g,n}$ is the mapping class group of Σ_g punctured at z_1, \dots, z_n . Moreover, by a result of Hamenstadt [32, Theorem 2] the map $F \circ i : B_n \hookrightarrow MCG_{g,n}$ is a quasi-isometric embedding for $n > 4$. Since F is Lipschitz, it follows that i is a quasi-isometric embedding (see [10, Lemma 2.1]). \square

Let us return to the proof of the corollary. Consider the homomorphism

$$H : \text{Diff}(D; D_1, \dots, D_n) \rightarrow P_n < P_n(\Sigma_g),$$

where $H(\phi)$ is an element in $P_n(\Sigma_g)$ represented by the path $(\phi_t(z_1), \dots, \phi_t(z_n))$ in the configuration space, and $\{\phi_t\}$ is any isotopy in \mathcal{G} between the identity ϕ_0 and $\phi := \phi_1$.

It follows from the results of Kim-Koberda [39] (c.f. Crisp-Wiest [18] and Benaim-Gambaudo [4]) that for each RAAG Γ there exists n , which can be assumed to satisfy $n > 4$, and an embedding of Γ into $\text{Diff}(D; D_1, \dots, D_n)$ whose composition with H is a quasi-isometric embedding of Γ into P_n . By Lemma 1, it gives as well a quasi-isometric embedding of Γ into $P_n(\Sigma_g)$. Then in order to obtain a quasi-isometric embedding of Γ into \mathcal{G} it is enough to show that the map H is large-scale Lipschitz whenever $n > 4$.

This fact follows from our main result, Theorem 1. Let us prove it. Set

$$X(D_1, \dots, D_n) := \{(x_1, \dots, x_n) \in X_n(\Sigma_g) \mid x_i \in D_i\}.$$

Let $\{\phi_t\}$ be any isotopy in \mathcal{G} between the identity $\phi_0 = \mathbf{1}$ and $\phi := \phi_1$. It follows from Theorem 1 that there exist positive constants A and B which depend only on n such that

$$\int_{X(D_1, \dots, D_n)} |[\lambda(x, \phi)]|_{P_n(\Sigma_g)} d\mu^{\otimes n}(x) \leq A \cdot l_1(\phi) + B.$$

Note that for each $x \in X(D_1, \dots, D_n)$ we have $[\lambda(x, \phi)] = H(\phi)$. Thus

$$\int_{X(D_1, \dots, D_n)} |[\lambda(x, \phi)]|_{P_n(\Sigma_g)} d\mu^{\otimes n}(x) = \text{vol}(X(D_1, \dots, D_n)) \cdot |H(\phi)|_{P_n(\Sigma_g)},$$

and we obtain the result, i.e., H is large-scale Lipschitz:

$$|H(\phi)|_{P_n(\Sigma_g)} \leq \frac{A}{\text{vol}(X(D_1, \dots, D_n))} \cdot l_1(\phi) + \frac{B}{\text{vol}(X(D_1, \dots, D_n))}.$$

Assume now that M_1 is a surface with boundary. We can embed M_1 in a closed surface $M_2 = \Sigma_g$ such that the embedding is area preserving. This induces a monomorphism $\iota: \text{Diff}_{c,0}(M_1, \sigma_1) \hookrightarrow \text{Diff}_{c,0}(M_2, \sigma_2)$ (note that elements of $\text{Diff}_{c,0}(M_1, \sigma_1)$ fix point-wise the neighbourhood of the boundary, so they can be extended by the identity). Since ι is Lipschitz, and the embedding of Γ to $\text{Diff}_{c,0}(M_2, \sigma_2)$ can be arranged to factor through $\text{Diff}_{c,0}(M_1, \sigma_1)$, we get a quasi-isometric embedding to $\text{Diff}_{c,0}(M_1, \sigma_1)$ (see [10, Lemma 2.1]). \square

Remark 1.3.2. We note that Corollary 4 implies Corollary 2, providing the latter with a proof that does not use quasimorphisms.

Furthermore, note that Corollary 4 implies that for each $k \in \mathbb{N}$ there is a quasi-isometric embedding $i_k: \mathbb{Z}^k \rightarrow (\mathcal{G}, d_1)$. Moreover, there exists a k -tuple of autonomous Hamiltonian flows (one-parameter subgroups) $\{\{\phi_i^t\}_{t \in \mathbb{R}}\}_{1 \leq i \leq k}$ which have disjoint supports, and $i_k(e_j) = \phi_j^1$ for each $1 \leq j \leq k$, where $e_j = (0, \dots, 1, \dots, 0)$ and 1 lies in the j -th entry. The above k -tuple of flows defines a homomorphism $j_k: \mathbb{R}^k \rightarrow (\mathcal{G}, d_1)$ such that $i_k = j_k|_{\mathbb{Z}^k}$. Since \mathbb{R}^k is quasi-isometric to \mathbb{Z}^k , we obtain the following statement:

Corollary 5. *The metric group (\mathcal{G}, d_1) admits a quasi-isometric embedding from (\mathbb{R}^k, d) where d is any metric on \mathbb{R}^k induced by a vector-space norm.*

The following lemma allows us to prove that the quasi-isometric embeddings of $\mathbb{Z}^k, \mathbb{R}^k$ from Corollary 5 are in fact bi-Lipschitz embeddings.

Lemma 2. *Let (G, d) be a metric group and let $A \subset V$ be a subgroup of a normed linear space. Then each Lipschitz quasi-isometric embedding $j: A \rightarrow G$ of metric groups is a bi-Lipschitz embedding. If A is discrete and finitely generated then each homomorphism $j: A \rightarrow G$ is Lipschitz.*

Proof. Let $j: A \rightarrow G$ be a homomorphism. If A is discrete and finitely generated, then $A \cong \mathbb{Z}^l$ for $l \in \mathbb{Z}_{\geq 0}$, the norm being equivalent to the standard norm on \mathbb{Z}^l . The upper bound $d(j(x), \mathbf{1}) \leq C|x|$ for all $x \in A$ is now immediate. Hence j is Lipschitz.

Suppose now that $A \subset V$ is a subgroup and $j: A \rightarrow G$ is a Lipschitz quasi-isometric embedding. Let $C \geq 1$ be such that

$$\frac{1}{C}|x| - B \leq d(j(x), \mathbf{1}) \leq C|x| \tag{1}$$

for a constant $B \geq 0$ and all $x \in A$. We claim that the inequality (1) holds with $B = 0$. Indeed, consider (1) for x^m where $m \in \mathbb{Z}_{>0}$. We have

$$\frac{1}{C}m|x| - B \leq d(j(x^m), \mathbf{1}) = d(j(x)^m, \mathbf{1}) \leq m \cdot d(j(x), \mathbf{1}),$$

where the last inequality is due to the right-invariance of d . Dividing by m yields

$$\frac{1}{C}|x| - B/m \leq d(j(x), \mathbf{1}),$$

which finishes the proof by taking limits as $m \rightarrow \infty$. \square

This immediately implies the following strengthening of Corollary 5, since j_k is evidently Lipschitz.

Corollary 6. *The metric group (\mathcal{G}, d_1) admits a bi-Lipschitz embedding from (\mathbb{R}^k, d) where d is any metric on \mathbb{R}^k induced by a vector-space norm.*

Remark 1.3.3. It should be possible to prove Corollary 6 by using quasimorphisms, as in [14].

Remark 1.3.4. Let $p \geq 1$. Note that since, by Jensen's (or Hölder's) inequality,

$$d_1 \leq d_p,$$

all the above results for d_1 continue to hold for d_p .

Recall that if M has boundary then we assume that elements of $\text{Diff}_{c,0}(M, \sigma_1)$ fix point-wise an open neighbourhood of ∂M . Thus if $(M_1, \sigma_1) \hookrightarrow (M_2, \sigma_2)$ is a measure preserving embedding of manifolds, then extending a diffeomorphism on M_1 by the identity to a diffeomorphism of M_2 gives a well defined monomorphism $\iota: \text{Diff}_{c,0}(M_1, \sigma_1) \hookrightarrow \text{Diff}_{c,0}(M_2, \sigma_2)$.

We finish this section with a question.

Question 1. Let $(M_1, \sigma_1) \hookrightarrow (M_2, \sigma_2)$ be a measure preserving embedding of surfaces. Is the monomorphism $\iota: \text{Diff}_{c,0}(M_1, \sigma_1) \hookrightarrow \text{Diff}_{c,0}(M_2, \sigma_2)$ a quasi-isometric embedding?

Remark 1.3.5. Note that this question is motivated by our results since the statement holds for the compositions of ι with the embeddings of right-angled Artin groups and (\mathbb{R}^k, d) from the proofs of Corollary 4 and Corollary 6.

1.4 Outline of the proof

In order to show Theorem 1, we define a Riemannian metric g on $X_n(M)$ such that d_g , the geodesic metric on $X_n(M)$ induced by g , extends to the geodesic metric on the compactification $\overline{X}_n(M)$. This allows us to use the Schwarz-Milnor lemma.

More precisely, Theorem 1 is a consequence of the following two propositions (we defer the proofs to Section 2).

Proposition 1.1. *Let λ be a piecewise C^1 loop in $X_n(M)$ based at q . Let S be a finite generating set of $P_n(M)$. The word norm of the class $[\lambda] \in \pi_1(X_n(M), q) \cong P_n(M)$ with respect to S satisfies*

$$|[\lambda]|_{P_n(M)} \leq A_0 \cdot l_g(\lambda) + B_0,$$

for constants $A_0, B_0 > 0$ depending only on S and n .

The next proposition says that the average length of loops $\lambda(x, \bar{\phi})$ is controlled by $l_1(\bar{\phi})$.

Proposition 1.2. *Let $\bar{\phi} = \{\phi_t\}$ be an isotopy in \mathcal{G} such that $\phi_0 = 1$. There exist constants $A_1, B_1 > 0$ depending only on n , such that*

$$\int_{X_n(M) \setminus Z \cup (\phi_1)^{-1}(Z)} l_g(\lambda(x, \bar{\phi})) d\mu^{\otimes n}(x) \leq A_1 \cdot l_1(\bar{\phi}) + B_1.$$

To show this proposition, we first compare a metric g to an auxiliary metric g_0 . The metric g_0 does not extend to a metric on the compactification $\overline{X}_n(M)$, but it is relatively easy to show [43, Lemma 5.2], that the inequality from Proposition 1.2 holds for g_0 .

2 Proofs.

2.1 Compactification of the configuration space

In this section M is a compact m -dimensional manifold. Below we describe the compactification of $X_n(M)$ using an embedding of the configuration space into a high-dimensional Euclidean space. We follow closely the construction given in [52].

Let us start with describing a family of maps on $X_n(\mathbb{R}^d)$. Let $\pi_{ij}: X_n(\mathbb{R}^d) \rightarrow S^{d-1}$ be defined by the formula

$$\pi_{ij}(x) = \frac{x_i - x_j}{|x_i - x_j|},$$

where $0 \leq i < j \leq n$ and $x = (x_1, \dots, x_n) \in X_n(\mathbb{R}^d)$. Let $s_{ijk}: X_n(\mathbb{R}^d) \rightarrow [0, \infty]$ be defined by the formula

$$s_{ijk}(x) = \frac{|x_i - x_j|}{|x_i - x_k|},$$

where $0 \leq i < j < k \leq n$ and $[0, \infty]$ is the one point compactification of $[0, \infty)$. Let $\iota: X_n(\mathbb{R}^d) \hookrightarrow (\mathbb{R}^d)^n$ be the standard inclusion and let $A_n[\mathbb{R}^d] = (\mathbb{R}^d)^n \times (S^{n-1})^{\binom{n}{2}} \times [0, \infty]^{\binom{n}{3}}$. We consider the embedding of $X_n(\mathbb{R}^d)$ to the ambient space $A_n[\mathbb{R}^d]$ given by the product map

$$\alpha_n = \iota \times (\pi_{ij}) \times (s_{ijk}): X_n(\mathbb{R}^d) \rightarrow A_n[\mathbb{R}^d].$$

Suppose M is a submanifold of \mathbb{R}^d . Let $\overline{X}_n(M)$ be the closure of the image $\alpha_n(X_n(M))$ in $A_n[\mathbb{R}^d]$. Then $\overline{X}_n(M)$ is a manifold with boundary, and the inclusion $X_n(M) \hookrightarrow \overline{X}_n(M)$ is a homotopy equivalence [52, Theorem 4.4 and Corollary 4.5].

2.2 Two metrics on $X_n(M)$ and proof of Proposition 1.1

On $[0, \infty]$ we introduce the structure of a smooth manifold by a 1-map atlas $e^{-x}: [0, \infty] \rightarrow [0, 1]$. Let g_{exp} be the Riemannian metric on $[0, \infty]$ given by the pull-back of the Euclidean metric from $[0, 1]$. In particular, we have that $|\partial_x|_{g_{exp}} = e^{-x}$, where ∂_x is a standard 'unit' vector field on $[0, \infty)$.

Let euc be the metric on $A_n[\mathbb{R}^d]$ given by the product of standard metrics on \mathbb{R}^d , S^{d-1} and $([0, \infty], g_{exp})$. The first metric on $X_n(\mathbb{R}^d)$ we want to consider is defined to be the pull-back of euc to $X_n(\mathbb{R}^d)$ by the map α_n :

$$g = \alpha_n^*(euc).$$

Now, since we regard M as a submanifold of \mathbb{R}^d , g induces a Riemannian metric on $X_n(M)$.

The second metric is defined as follows. Let $x = (x_1, \dots, x_n) \in X_n(\mathbb{R}^d)$. Let $d(x)$ denote the minimal distance between the points in x , that is:

$$d(x) = \min\{|x_i - x_j| : i \neq j\},$$

where $|x - y|$ denotes the standard Euclidean distance between vectors $x, y \in \mathbb{R}^d$.

Note that on $X_n(\mathbb{R}^d)$ we have the Euclidean metric restricted from $(\mathbb{R}^d)^n$. We rescale this metric by the factor $\frac{1}{d}$, i.e. we define a metric g_0 on $X_n(\mathbb{R}^d)$ by

$$|v|_{g_0} = \frac{|v|}{d(x)},$$

where $v \in T_x(X_n(\mathbb{R}^d)) = (\mathbb{R}^d)^n$ and $|v|$ is the Euclidean length of $v \in (\mathbb{R}^d)^n$. One should note that $d(x)$, and consequently g_0 , is continuous, but not differentiable. A manifold with such a metric is called a C^0 -Riemannian manifold. On a C^0 -Riemannian manifold one defines the lengths of paths and a geodesic metric in the same way as in the smooth case.

Again, g_0 restricts to a C^0 -Riemannian metric on $X_n(M)$.

Proof of Proposition 1.1. We need to show that

$$||[\lambda]||_{P_n(M)} \leq A_0 \cdot l_g(\lambda) + B_0,$$

where λ be a piecewise C^1 loop in $X_n(M)$ based at q , $[\lambda] \in \pi_1(X_n(M), q) \cong P_n(M)$ and $||[\lambda]||_{P_n(M)}$ is the word norm of $[\lambda]$.

By [52, Theorem 4.4] $\overline{X}_n(M)$ is a manifold with corners and $X_n(M)$ is its interior. Moreover, the coordinate charts on $\overline{X}_n(M)$ are defined in such a way that the embedding of $\overline{X}_n(M)$ into $A_n[\mathbb{R}^d]$ is smooth. Thus one can restrict the metric *euc* to $\overline{X}_n(M)$. In other words, $g = \alpha_n^*(\textit{euc})$ extends to the compactification $\overline{X}_n(M)$. Now the proposition follows directly from the Schwarz-Milnor lemma [22, 54, 44]. More precisely, consider the following formulation of this result [15, Proposition 8.19].

Lemma 3. *Let a group Γ act properly and discontinuously by isometries on a proper length space X . If the action is cocompact, then Γ is finitely generated and for any choice of base-point $x_0 \in X$, the map $\Gamma \rightarrow X$, $h \mapsto h \cdot x_0$, is a quasi-isometry.*

Denote by g_c the metric on $\overline{X}_n(M)$ obtained from *euc*. We apply Lemma 3 to $X = \widetilde{\overline{X}}_n(M)$, $\Gamma = \pi_1(\overline{X}_n(M), \pi(x_0)) \cong \pi_1(X_n(M), \pi(x_0))$, and the length space structure on X being induced by the lift $\widetilde{g}_c = \pi^*g_c$ to X of g_c on $K = X/\Gamma = \overline{X}_n(M)$ by the natural projection map $\pi : X \rightarrow K$. Note that $d_{\widetilde{g}_c}(x_0, h \cdot x_0) \leq l_{g_c}(\lambda)$ where λ is any C^1 loop in $X_n(M)$ based at $\pi(x_0)$ representing the element h . Moreover, since g_c is an extension of g , we have $l_{g_c}(\lambda) = l_g(\lambda)$. \square

2.3 Proof of Proposition 1.2

We begin with the proof of the following technical result.

Lemma 4. *There exists $C > 0$ depending only on n , such that $|v|_g \leq C \cdot |v|_{g_0}$ for every $v \in T(X_n(M))$.*

Proof. Since α_n is a product map, in order to get a bound on the norm $|v|_g$, we need to bound the norms of the vectors $D\pi_{ij}(v)$ and $Ds_{ijk}(v)$. Let $x = (x_1, \dots, x_n) \in X_n(M)$ and let $v = (v_1, \dots, v_n) \in T_x(X_n(M))$. There exists $\epsilon > 0$ such that $x + tv \in X_n(\mathbb{R}^d)$ for every $t < \epsilon$. Note that even though the path $\{x + tv\}_{t \in [0, \epsilon]}$ is not contained in $X_n(M)$, it still represents the tangent vector $v \in T_x(X_n(M)) \subset T_x(X_n(\mathbb{R}^d))$. We have

$$\begin{aligned} D\pi_{ij}(v) &= \frac{d}{dt}\bigg|_{t=0} \pi_{ij}(x + tv) \\ &= \frac{d}{dt}\bigg|_{t=0} \frac{x_i - x_j + t(v_i - v_j)}{|x_i - x_j + t(v_i - v_j)|} \\ &= \frac{v_i - v_j}{|x_i - x_j|} - \frac{(x_i - x_j)\langle v_i - v_j, x_i - x_j \rangle}{|x_i - x_j|^3}. \end{aligned}$$

By definition $D\pi_{ij}(v)$ is a vector in \mathbb{R}^d tangent to a $(d-1)$ -dimensional sphere at point $\pi_{ij}(x)$. By $|D\pi_{ij}(v)|$ we denote its Euclidean length.

Applying the inequalities $\langle v_i - v_j, x_i - x_j \rangle \leq |v_i - v_j||x_i - x_j|$ and $|v_i - v_j| \leq |v_i| + |v_j|$, we obtain:

$$\begin{aligned} |D\pi_{ij}(v)| &\leq 2 \frac{1}{|x_i - x_j|} (|v_i| + |v_j|) \\ &\leq \frac{2}{d(x)} (|v_i| + |v_j|) \\ &\leq \frac{2}{d(x)} \left(\sum_i |v_i| \right). \end{aligned}$$

Similarly, for s_{ijk} we compute (already in the map):

$$\begin{aligned} De^{-s_{ijk}}(v) &= \frac{d}{dt}\Big|_{t=0} e^{-s_{ijk}(x+tv)} \\ &= \frac{d}{dt}\Big|_{t=0} e^{-\frac{|x_i-x_j+t(v_i-v_j)|}{|x_i-x_k+t(v_i-v_k)|}} \\ &= \left[\frac{\langle v_i-v_j, x_i-x_j \rangle}{|x_i-x_j||x_i-x_k|} - \frac{|x_i-x_j|\langle x_i-x_k, v_i-v_k \rangle}{|x_i-x_k|^3} \right] e^{-\frac{|x_i-x_j|}{|x_i-x_k|}}. \end{aligned}$$

By definition $|Ds_{ijk}(v)|_{g_{exp}} = |De^{-s_{ijk}}(v)|$. Using similar inequalities as before we get:

$$|Ds_{ijk}(v)|_{g_{exp}} \leq \left[\frac{|v_i|+|v_j|}{|x_i-x_k|} + \frac{(|v_i|+|v_k|)|x_i-x_j|}{|x_i-x_k|^2} \right] e^{-\frac{|x_i-x_j|}{|x_i-x_k|}}.$$

Applying the inequality $e^{-x} \leq \frac{1}{x}$ to the last term we obtain:

$$\begin{aligned} |Ds_{ijk}(v)|_{g_{exp}} &\leq \left[\frac{|v_i|+|v_j|}{|x_i-x_k|} + \frac{(|v_i|+|v_k|)|x_i-x_j|}{|x_i-x_k|^2} \right] \frac{|x_i-x_k|}{|x_i-x_j|} \\ &= \frac{|v_i|+|v_j|}{|x_i-x_j|} + \frac{|v_i|+|v_k|}{|x_i-x_k|} \\ &\leq \frac{1}{d(x)}(2|v_i|+|v_j|+|v_k|) \\ &\leq \frac{2}{d(x)}\left(\sum_i |v_i|\right). \end{aligned}$$

Finally, since $\sum_i |v_i| \leq \sqrt{n}|v|$, we get:

$$\begin{aligned} |D\pi_{ij}(v)| &\leq \frac{\sqrt{n}}{d(x)}|v|, \\ |Ds_{ijk}(v)|_{g_{exp}} &\leq \frac{2\sqrt{n}}{d(x)}|v|. \end{aligned}$$

We assume that M is compact, therefore there exists $A > 0$ such that $d(x) \leq A$ for every $x \in X_n(M)$. Now we can bound the pulled-back metric $|\cdot|_g$:

$$\begin{aligned} |v|_g^2 &= |D\iota(v)|^2 + \sum_{i,j} |D\pi_{ij}(v)|^2 + \sum_{i,j,k} |Ds_{ijk}(v)|_{g_{exp}}^2 \\ &\leq \frac{A^2}{d(x)^2}|v|^2 + \sum_{i,j} \frac{n}{d(x)^2}|v|^2 + \sum_{i,j,k} \frac{4n}{d(x)^2}|v|^2 \\ &= C \frac{|v|^2}{d(x)^2} = C \cdot |v|_{g_0}^2, \end{aligned}$$

for $C = A^2 + n\binom{n}{2} + 4n\binom{n}{3}$. \square

Remark 2.3.1. In the proof of Lemma 4, the inequality $e^{-x} < \frac{1}{x}$ is used to show the bound on the length of Ds_{ijk} . Indeed, the choice of the function $e^{-x}: [0, \infty] \rightarrow [0, 1]$ to define a smooth structure and the metric on $[0, \infty]$ is not completely arbitrary. For a different identification, e.g. $\ln\left(\frac{x+\epsilon}{x+1}\right): [0, \infty] \rightarrow [0, 1]$, we get a different smooth structure on $[0, \infty]$ (in the sense, that the identity map is not smooth) and a metric which is not equivalent to g_{exp} . Then after pull-back by α_n we get a metric on $X_n(M)$ which is not equivalent to g and for this metric Lemma 4 might not hold.

Proof of Proposition 1.2. We need to show that

$$\int_{X_n(M) \setminus Z \cup (\phi_1)^{-1}(Z)} l_g(\lambda(x, \bar{\phi})) d\mu^{\otimes n}(x) \leq A_1 \cdot l_1(\bar{\phi}) + B_1,$$

where $\bar{\phi} = \{\phi_t\}$ is an isotopy in \mathcal{G} such that $\phi_0 = 1$.

A similar inequality was proven in [43] for a metric which is equivalent to g_0 . Let us first describe this metric. By d_M denote the geodesic metric on M induced by the restriction of the standard Riemannian metric on \mathbb{R}^d . Let $x \in X_n(M)$ and $v \in T_x(X_n(M))$. We define

$$|v|_{g_b} = \frac{|v|}{d_M(x)},$$

where $d_M(x) = \min\{d_M(x_i, x_j) : i \neq j\}$ and $|v|$ is the Euclidean length of v seen as a vector in $(\mathbb{R}^d)^n$. The difference between the function d used to define g_0 and d_M is that in d we measure the distance between points x_i and x_j in the ambient space \mathbb{R}^d and in d_M inside M . Since M is compact, clearly d and d_M are equivalent and consequently g_b and g_0 are equivalent.

It follows from [43, Lemma 5.2] that

$$\int_{X_n(M) \setminus Z \cup (\phi_1)^{-1}(Z)} l_{g_b}(\{\phi_t \cdot x\}) d\mu^{\otimes n}(x) \leq C' \cdot l_1(\bar{\phi}).$$

Recall that $\{\phi_t \cdot x\}$ is a path in $X_n(M)$ (ϕ_t acts on $x \in X_n(M)$ component-wise). Since g_0 and g_b are equivalent, this inequality holds as well for g_0 (with a possibly different constant).

Let us now focus on the metric g . Since the geodesic metric d_g defined by g extends to the compactification of $X_n(M)$, the diameter of $(X_n(M), d_g)$ is finite. We can choose the system of short paths $\gamma(x)$ such that $l_g(\gamma(x)) \leq D$ for some $D > 0$ and every $x \in X_n(M) \setminus Z$.

Thus for every $x \in X_n(M) \setminus Z \cup (\phi_1)^{-1}(Z)$ we have

$$l_g(\lambda(x, \bar{\phi})) \leq l_g(\gamma(x)) + l_g(\{\phi_t \cdot x\}) + l_g(\gamma(\phi_1(x))) \leq 2D + l_g(\{\phi_t \cdot x\}).$$

Finally, due to Lemma 4, we have $l_g(\{\phi_t \cdot x\}) \leq C \cdot l_{g_0}(\{\phi_t \cdot x\})$ and the proposition follows. \square

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