ON THE ENTROPY NORM ON THE GROUP OF DIFFEOMORPHISMS OF CLOSED ORIENTED SURFACE

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Abstract. We prove that the entropy norm on the group of diffeomorphisms of a closed orientable surface of positive genus is unbounded.

1. Introduction

Let $\mathbf{M}$ be a smooth compact manifold with some fixed Riemannian metric. Let $f\colon \mathbf{M} \to \mathbf{M}$ be a continuous function. Recall that the topological entropy of f may be defined as follows. Let $\mathbf{d}$ be the metric on $\mathbf{M}$ induced by some Riemannian metric. For $p \in \mathbf{N}$ define a new metric $\mathbf{d}_{f,p}$ on $\mathbf{M}$ by

$$\mathbf{d}_{f,p}(x, y) = \max_{0 \le i \le p} \mathbf{d}(f^i(x), f^i(y)).$$

Let $\mathbf{M}_f(p, \epsilon)$ be the minimal number of ϵ-balls in the $\mathbf{d}_{f,p}$-metric that cover $\mathbf{M}$. The topological entropy $h(f)$ is defined by

$$h(f) = \lim_{\epsilon \to 0} \limsup_{p \to \infty} \frac{\log \mathbf{M}_f(p, \epsilon)}{p},$$

where the base of log is two. It turns out that $h(f)$ does not depend on the choice of Riemannian metric, see [3, 10].

In this note we consider the case when $\mathbf{M}$ is a closed oriented surface $\mathbf{\Sigma}_g$ of genus g. Denote by $\mathrm{Diff}(\mathbf{\Sigma}_g)$ the group of orientation preserving diffeomorphisms of $\mathbf{\Sigma}_g$. Let

$$\mathrm{Ent}(\mathbf{\Sigma}_g) \subset \mathrm{Diff}(\mathbf{\Sigma}_g)$$

be the set of entropy-zero diffeomorphisms. This set is conjugation invariant and it generates $\mathrm{Diff}(\mathbf{\Sigma}_g)$, see Lemma 2.1. In other words, a diffeomorphism of $\mathbf{\Sigma}_g$ is a finite product of entropy-zero diffeomorphisms. One may ask for a minimal decomposition and this question leads to the concept of the entropy norm defined by

$$\|f\|_{\mathrm{Ent}} := \min\{k \in \mathbf{N} \mid f = h_1 \cdots h_k,\ h_i \in \mathrm{Ent}(\mathbf{\Sigma}_g)\}.$$

It is the word norm associated with the generating set $\mathrm{Ent}(\mathbf{\Sigma}_g)$. This set is conjugation invariant, so is the entropy norm. The associated bi-invariant

2000 *Mathematics Subject Classification.* Primary 53; Secondary 57.

Key words and phrases. Entropy, conjugation-invariant norms, quasi-morphisms, mapping class groups.

metric is denoted by $\mathbf{d}_{\mathrm{Ent}}$. It follows from the work of Burago-Ivanov-Polterovich [9] and Tsuboi [17, 18] that for many manifolds all conjugation invariant norms on $\mathrm{Diff}(\mathbf{M})$ are bounded. Hence the entropy norm is bounded in those cases. In particular, it is bounded in case $g = 0$.

Entropy metric may be defined in the same way on the group $\mathrm{Ham}(\mathbf{\Sigma}_g)$ of Hamiltonian diffeomorphisms of $\mathbf{\Sigma}_g$, and on groups $\mathrm{Diff}(\mathbf{\Sigma}_g, \mathrm{area})$ and $\mathrm{Diff}_0(\mathbf{\Sigma}_g, \mathrm{area})$. It is related to the autonomous metric [4, 5, 6, 8, 13]. Recently, the first author in collaboration with Marcinkowski showed that the entropy metric is unbounded on groups: $\mathrm{Ham}(\mathbf{\Sigma}_g)$, $\mathrm{Diff}_0(\mathbf{\Sigma}_g, \mathrm{area})$ and on $\mathrm{Diff}(\mathbf{\Sigma}_g, \mathrm{area})$, see [7]. On the other hand, it is not known, and seems to be a difficult problem, whether $\mathrm{Diff}_0(\mathbf{\Sigma}_g)$ is unbounded in case $g > 0$. In this work we discuss the case of $\mathrm{Diff}(\mathbf{\Sigma}_g)$ where $g > 0$. Our main result is the following

Theorem 1. *Let $\mathbf{\Sigma}_g$ be a closed oriented Riemannian surface of positive genus. Then the diameter of $(\mathrm{Diff}(\mathbf{\Sigma}_g), \mathbf{d}_{\mathrm{Ent}})$ is infinite.*

Remarks.

- The above theorem holds for non-sporadic surfaces with punctures. The proof is exactly the same.
- In [7] the first author in collaboration with Marcinkowski showed that the diameter of $(\mathrm{Diff}(\mathbf{\Sigma}_g, \mathrm{area}), \mathbf{d}_{\mathrm{Ent}})$ is infinite. Our proof of Theorem 1, which is simpler than the one given in [7], is applicable to the case of $\mathrm{Diff}(\mathbf{\Sigma}_g, \mathrm{area})$.
- It would be interesting to know whether the entropy metric, or the autonomous metric are unbounded on $\mathrm{Diff}_0(\mathbf{\Sigma}_g)$ in case $g > 0$.

Acknowledgments. First author was partially supported by Leverhulme Trust Grant RPG-2017-159. Second author was partially supported by the center of advanced studies at Ben Gurion University, by GIF-Young Grant number I-2419-304.6/2016, by ISF Grant number 2095/15 and by DST-INSPIRE, India. He wishes to express his gratitude to the center for the support and excellent working conditions.

2. Preliminaries

Let us start with the following

Lemma 2.1. *Let $\mathbf{\Sigma}_g$ be a closed oriented surface of genus g. Then $\mathrm{Diff}(\mathbf{\Sigma}_g)$ is generated by the set $\mathrm{Ent}(\mathbf{\Sigma}_g)$ of entropy zero diffeomorphisms.*

Proof. The group $\mathrm{Diff}_0(\mathbf{\Sigma}_g)$ is simple and hence is generated by entropy zero diffeomorphisms. It is enough to prove the lemma in case $g > 0$ since $\mathrm{Diff}(\mathbf{\Sigma}_0) = \mathrm{Diff}_0(\mathbf{\Sigma}_0)$. In addition, Dehn twists have entropy zero and they generate $\mathrm{Diff}(\mathbf{\Sigma}_g)/\mathrm{Diff}_0(\mathbf{\Sigma}_g)$ in case $g > 1$. Hence in this case $\mathrm{Diff}(\mathbf{\Sigma}_g)$ is generated by entropy zero diffeomorphisms. In case $g = 1$ we have that

$$\mathrm{Diff}(\mathbf{\Sigma}_1)/\mathrm{Diff}_0(\mathbf{\Sigma}_1) \cong \mathrm{SL}_2(\mathbb{Z}),$$

which in turn is generated by two matrices of finite order. Hence in this case $\mathrm{Diff}(\mathbf{\Sigma}_g)$ is also generated by entropy zero diffeomorphisms. □

Let $\mathbf{\Sigma}_g$ be a closed oriented surface of genus $g > 1$.

2.A. **Translation length in Teichmüller space.** We denote the Teichmüller space associated to $\mathbf{\Sigma}_g$ by $\mathcal{T}(\mathbf{\Sigma}_g)$. We equip $\mathcal{T}(\mathbf{\Sigma}_g)$ with the Teichmüller metric $\mathbf{d}_{\mathcal{T}}$. Let $\mathrm{MCG}(\mathbf{\Sigma}_g)$ be the mapping class group of $\mathbf{\Sigma}_g$, i.e., $\mathrm{MCG}(\mathbf{\Sigma}_g) := \mathrm{Diff}(\mathbf{\Sigma}_g)/\mathrm{Diff}_0(\mathbf{\Sigma}_g)$. Note that it acts naturally on $\mathcal{T}(\mathbf{\Sigma}_g)$. Let $[f] \in \mathrm{MCG}(\mathbf{\Sigma}_g)$. The *translation length* of $[f]$ in $\mathcal{T}(\mathbf{\Sigma}_g)$ is defined by

$$\tau_{\mathcal{T}}([f]) = \lim_{n\to\infty} \frac{\mathbf{d}_{\mathcal{T}}([f]^n(X), X)}{n}$$

where $X \in \mathcal{T}(\mathbf{\Sigma}_g)$. It is independent of the choice of X.

Let $[f] \in \mathrm{MCG}(\mathbf{\Sigma}_g)$ be a pseudo-Anosov element with dilatation $\lambda_{[f]}$. According to Bers [1] proof of Thurston's classification theorem of elements of mapping class group we have:

- there exists $X \in \mathcal{T}(\mathbf{\Sigma}_g)$ such that $\tau_{\mathcal{T}}([f]) = \mathbf{d}_{\mathcal{T}}([f](X), X)$,
- $\tau_{\mathcal{T}}([f]) = \log(\lambda_{[f]})$.

2.B. **Translation length in curve complex.** Given a surface $\mathbf{\Sigma}_g$, we associate to it a simplicial complex as follows: its vertices are free homotopy classes of essential simple closed curves; a collection of $n+1$ vertices form an n-simplex whenever it can be realized by pairwise disjoint closed curves in $\mathbf{\Sigma}_g$. This complex is called the *curve complex* of $\mathbf{\Sigma}_g$ and is denoted by $\mathcal{C}(\mathbf{\Sigma}_g)$. It is known that $\mathcal{C}(\mathbf{\Sigma}_g)$ is connected. We consider the path metric on the 1-skeleton of $\mathcal{C}(\mathbf{\Sigma}_g)$ and denote it by $\mathbf{d}_{\mathcal{C}}$.

Mapping class group $\mathrm{MCG}(\mathbf{\Sigma}_g)$ acts by isometry on $\mathcal{C}(\mathbf{\Sigma}_g)$. Given a mapping class $[f] \in \mathrm{MCG}(\mathbf{\Sigma}_g)$, the *translation length* of $[f]$ in $\mathcal{C}(\mathbf{\Sigma}_g)$ is defined by

$$\tau_{\mathcal{C}}([f]) = \lim_{n\to\infty} \frac{\mathbf{d}_{\mathcal{C}}([f]^n(\alpha), \alpha)}{n}$$

where α is a vertex in $\mathcal{C}(\mathbf{\Sigma}_g)$. The translation length is independent of α and is non-zero if and only if $[f]$ is a pseudo-Anosov mapping class [15].

2.C. **Bestvina-Fujiwara quasimorphisms.** Let G be a group. Recall that a function $\psi : G \to \mathbb{R}$ is called a quasimorphism if there exists $D > 0$ such that

$$|\psi(ab) - \psi(a) - \psi(b)| < D$$

for all $a, b \in G$. A quasimorphism ψ is called homogeneous if $\psi(a^n) = n\psi(a)$ for all $n \in \mathbb{Z}$ and all $a \in G$. Given a quasimorphism ψ we can always construct a homogeneous quasimorphism $\overline{\psi}$ by setting

$$\overline{\psi}(a) := \lim_{p\to\infty} \frac{\psi(a^p)}{p}$$

In [2], Bestvina and Fujiwara constructed infinitely many homogeneous quasimorphisms on $\mathrm{MCG}(\boldsymbol{\Sigma}_g)$. Let us recall their construction.

Let w be a finite oriented path in $\mathcal{C}(\boldsymbol{\Sigma}_g)$. Denote the length of a path ω by $|\omega|$. For any finite path σ in $\mathcal{C}(\boldsymbol{\Sigma}_g)$, we define

$$|\sigma|_\omega := \{\text{the number of non-overlapping copies of } \omega \text{ in } \sigma\}.$$

Fix a positive integer $W < |\omega|$. Given any two vertices $\alpha, \beta \in \mathcal{C}(\boldsymbol{\Sigma}_g)$, define

$$c_{\omega,W}(\alpha, \beta) = \mathbf{d}_{\mathcal{C}}(\alpha, \beta) - \inf(|\sigma| - W|\sigma|_\omega),$$

where the infimum is taken over all paths σ between α and β.

It turns out that the function $\psi_\omega : \mathrm{MCG}(\boldsymbol{\Sigma}_g) \to \mathbb{R}$ defined by

$$\psi_\omega([f]) = c_{\omega,W}(\alpha, [f](\alpha)) - c_{\omega^{-1},W}(\alpha, [f](\alpha)),$$

where α is a vertex of $\mathcal{C}(\boldsymbol{\Sigma}_g)$, is a quasimorphism [2]. The induced homogeneous quasimorphism is denoted by $\overline{\psi}_\omega$. We denote by $Q_{BF}(\mathrm{MCG}(\boldsymbol{\Sigma}_g))$the space of homogeneous quasimorphisms on $\mathrm{MCG}(\boldsymbol{\Sigma}_g)$ which is spanned by Bestvina-Fujiwara quasimorphisms. In [2] it is proved that $Q_{BF}(\mathrm{MCG}(\boldsymbol{\Sigma}_g))$is infinite dimensional whenever $\boldsymbol{\Sigma}_g$ is a non-sporadic surface.

3. Proof of the main result

Let us start with the following well-known

Lemma 3.1. *Let G be a group generated by set S and let $\psi : G \to \mathbb{R}$ be a non-trivial homogeneous quasimorphism which vanishes on S. Then the induced word norm $\|\cdot\|_S$ is unbounded.*

For the reader convenience we present its proof.

Proof. Let $g \in G$ such that $\psi(g) \neq 0$. Then $g = s_1 \cdot \ldots \cdot s_{\|g\|_S}$. It follows that $|\psi(g)| \leq \|g\|_S D_\psi$. Hence for each n we get $\|g^n\|_S \geq n|\psi(g)|/D_\psi$ and the proof follows. $\square$

Now we prove Theorem 1.

Case 1. Let $g = 1$ and denote $\mathbf{T} := \mathbf{\Sigma}_1$. Let us consider homomorphism $F : \mathrm{Diff}(\mathbf{T}) \to \mathrm{SL}_2(\mathbb{Z})$ induced by the action of a diffeomorphism on the first homology $H_1(\mathbf{T}, \mathbb{Z})$. It is known that F is surjective (see [11, Theorem 2.5]). By [14, Theorem 1], $\log(\mathrm{spec}(f)) \leq h(f)$ where $\mathrm{spec}(f)$ is the modulus of the largest eigenvalue of $F(f)$. Therefore if f has entropy zero then the modulus of the eigenvalues of $F(f)$ is at most one.

There are three types of elements in $\mathrm{SL}_2(\mathbb{Z})$: *periodic* (trace<2), *parabolic* (trace=2) and *hyperbolic* (trace>2). Therefore if $F(f)$ is hyperbolic then $\mathrm{spec}(f) > 1$ and hence $h(f) > 0$. Hence if f is an entropy zero diffeomorphism, then $F(f)$ is either parabolic or periodic.

The value of any homogeneous quasimorphism on a periodic element is zero. It follows from the work of Polterovich and Rudnick [16, Proposition 3] that there exists a non-trivial homogeneous quasimorphism on $\mathrm{SL}_2(\mathbb{Z})$ which vanishes on parabolic elements. Therefore there exists a non-trivial homogeneous quasimorphim on $\mathrm{Diff}(\mathbf{T})$ whose restriction on entropy-zero diffeomorphisms is zero. Hence by Lemma 3.1 the entropy norm on $\mathrm{Diff}(\mathbf{T})$ is unbounded.

Case 2. Let $g > 1$. Given a homeomorphism f of a surface $\mathbf{\Sigma}_g$ define

$$H(f) = \inf\{h(f') : f' \text{ is isotopic to } f\}$$

The topological entropy of $[f] \in \mathrm{MCG}(\mathbf{\Sigma}_g)$ is defined to be $H(f)$.

Lemma 3.2. *Each quasimorphism in* $Q_{BF}(\mathrm{MCG}(\mathbf{\Sigma}_g))$*is Lipschitz with respect to the topological entropy.*

Proof. Let $\psi \in Q_{BF}(\mathrm{MCG}(\mathbf{\Sigma}_g))$. If $[f]$ is reducible then $\psi([f]) = 0$ for all $\psi \in Q_{BF}(\mathrm{MCG}(\mathbf{\Sigma}_g))$. Therefore it is enough to consider only pseudo-Anosov elements of $\mathrm{MCG}(\mathbf{\Sigma}_g)$. Since $\psi \in Q_{BF}(\mathrm{MCG}(\mathbf{\Sigma}_g))$, then $\psi = \sum_i^k a_i \overline{\psi}_{w_i}$, where $a_1, \ldots, a_k \in \mathbb{R}$ and $w_1, \ldots, w_k$ are some paths in $\mathcal{C}(S)$. It follows from the definition of $\overline{\psi}_{w_i}$ that $\overline{\psi}_{w_i}([f]) \leq \tau_{\mathcal{C}}([f])$ for each $[f] \in \mathrm{MCG}(\mathbf{\Sigma}_g)$ and each $i \in \{1, \ldots, k\}$. Therefore we have

$$|\psi([f])| \leq (\sum_{i=1}^{k} |a_i|)\tau_{\mathcal{C}}([f]).$$

By setting $C_\psi := \sum_{i=1}^k |a_i|$ we get $|\psi([f])| \leq C_\psi \tau_{\mathcal{C}}([f])$.

Let $sys : \mathcal{T}(\mathbf{\Sigma}_g) \to \mathcal{C}(\mathbf{\Sigma}_g)$ be the systole function, i.e., $X \in \mathcal{T}(\mathbf{\Sigma}_g)$ goes to a vertex in $\mathcal{C}(\mathbf{\Sigma}_g)$ which corresponds to a simple closed curve of minimal length in X. By [15] there exist $K, C > 0$ such that for all $X, Y \in \mathcal{T}(\mathbf{\Sigma}_g)$

$$\mathbf{d}_{\mathcal{C}}(sys(X), sys(Y)) \leq K\mathbf{d}_{\mathcal{T}}(X, Y) + C.$$

It is immediate that $[f]^n(sys(X)) = sys([f]^n(X))$ for every $[f] \in \mathrm{MCG}(\mathbf{\Sigma}_g)$.

Let $[f] \in \mathrm{MCG}(\mathbf{\Sigma}_g)$ be a pseudo-Anosov element with dilatation $\lambda_{[f]}$. It follows from Bers [1] proof of Thurston's theorem that $\tau_{\mathcal{T}}([f]) = \log \lambda_{[f]}$. Therefore

$$\begin{aligned}\frac{\tau_{\mathcal{C}}([f])}{\tau_{\mathcal{T}}([f])} &= \lim_{n\to\infty} \frac{\frac{\mathbf{d}_{\mathcal{C}}(sys(X),[f]^n(sys(X)))}{n}}{\frac{\mathbf{d}_{\mathcal{T}}(X,[f]^n(X))}{n}} \\ &= \lim_{n\to\infty} \frac{\frac{\mathbf{d}_{\mathcal{C}}(sys(X),sys([f]^n(X)))}{n}}{\frac{\mathbf{d}_{\mathcal{T}}(X,[f]^n(X))}{n}} \\ &\leq \lim_{n\to\infty} \frac{K\mathbf{d}_{\mathcal{T}}(X,[f]^n(X))+C}{\mathbf{d}_{\mathcal{T}}(X,[f]^n(X))} = K\end{aligned}$$

Thus

$$\tau_{\mathcal{C}}([f]) \leq K\tau_{\mathcal{T}}([f]).$$

It follows that for each $\psi \in Q_{BF}(\mathrm{MCG}(\mathbf{\Sigma}_g))$we have

$$|\psi([f])| \leq C_\psi \tau_{\mathcal{C}}([f]) \leq C_\psi K \tau_{\mathcal{T}}([f]) = C_\psi K \log \lambda_{[f]}.$$

By Thuston's result [12, Proposition 10.13], $\log \lambda_{[f]} = H(f)$. Hence

$$|\psi([f])| \leq C_\psi K H(f)$$

and the proof of the lemma follows. $\square$

Let $\Pi : \mathrm{Diff}(\mathbf{\Sigma}_g) \to \mathrm{MCG}(\mathbf{\Sigma}_g)$ be the quotient map and let $\psi \in Q_{BF}(\mathrm{MCG}(\mathbf{\Sigma}_g))$. It follows from the proof of Lemma 3.2 that for each $f \in \mathrm{Diff}(\mathbf{\Sigma}_g)$ we have

$$|\psi\Pi(f)| \leq C_\psi K H(f) \leq C_\psi K h(f).$$

Hence for each non-trivial $\psi \in Q_{BF}(\mathrm{MCG}(\mathbf{\Sigma}_g))$the homogeneous quasimorphism

$$\psi\Pi : \mathrm{Diff}(\mathbf{\Sigma}_g) \to \mathbb{R}$$

is non-trivial and Lipschitz with respect to the topological entropy. It follows that it vanishes on the set of entropy-zero diffeomorphisms. Hence by Lemma 3.1 the entropy norm on $\mathrm{Diff}(\mathbf{\Sigma}_g)$ is unbounded. $\square$

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