# *Concordance of certain 3-braids and Gauss diagrams*

Michael Brandenbursky

Abstract. Let $\beta := \sigma_1\sigma_2^{-1}$ be a braid in $\mathbf{B}_3$, where $\mathbf{B}_3$ is the braid group on 3 strings and $\sigma_1, \sigma_2$ are the standard Artin generators. We use Gauss diagram formulas to show that for each natural number $n$ not divisible by 3 the knot which is represented by the closure of the braid $\beta^n$ is algebraically slice if and only if $n$ is odd. As a consequence, we deduce some properties of Lucas numbers.

## 1. Introduction

Let $\mathrm{Conc}(\mathbf{S}^3)$ denote the abelian group of concordance classes of knots in $\mathbf{S}^3$. Two knots $K_0, K_1 \in \mathbf{S}^3 = \partial\mathbf{B}^4$ are *concordant* if there exists a smooth embedding $c\colon \mathbf{S}^1 \times [0,1] \to \mathbf{B}^4$ such that $c(\mathbf{S}^1 \times \{0\}) = K_0$ and $c(\mathbf{S}^1 \times \{1\}) = K_1$. The knot is called *slice* if it is concordant to the unknot. The addition in $\mathrm{Conc}(\mathbf{S}^3)$ is defined by the connected sum of knots. The inverse of an element $[K] \in \mathrm{Conc}(\mathbf{S}^3)$ is represented by the knot $-K^*$, where $-K^*$ denotes the mirror image of the knot $K$ with the reversed orientation.

Let $\mathrm{AConc}(\mathbf{S}^3)$ denote the algebraic concordance group of knots in $\mathbf{S}^3$. The elements of this group are equivalence classes of Seifert forms $[V_F]$ associated with an arbitrary chosen Seifert surface $F$ of a given knot $K$. The addition in $\mathrm{AConc}(\mathbf{S}^3)$ is induced by direct sum. A knot $K$ is called *algebraically slice* if it has a Seifert matrix which is metabolic. It is a well known fact that every slice knot is algebraically slice. For more information about these groups see [10].

Let $\mathbf{B}_3$ denote the Artin braid group on 3 strings and let $\sigma_1, \sigma_2$ be the standard Artin generators of $\mathbf{B}_3$, i.e. $\sigma_i$ is represented by half-twist of $i+1$-th string over $i$-th string and $\mathbf{B}_3$ has the following presentation

$$\mathbf{B}_3 = \langle \sigma_1, \sigma_2 |\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle\,.$$

2000 *Mathematics Subject Classification.* 57.

*Key words and phrases.* braids, knots, concordance, Gauss diagrams.

In this paper we discuss properties of a family of knots in which every knot is represented by a closure of the braid $\beta^n$, where $\beta = \sigma_1\sigma_2^{-1} \in \mathbf{B}_3$ and $n \neq 0 \bmod 3$. This family of braids is interesting in the following sense: the braid $\beta$ has a minimal length among all non-trivial braids in $\mathbf{B}_3$ whose stable commutator length is zero. Hence by a theorem of Kedra and the author the four ball genus of every knot in this family is bounded by 4, see [2, Section 4.E.].

**Theorem 1.** *Let $n$ be any natural number not divisible by 3. Then the closure of $\beta^n$ is of order 2 in* $\mathrm{AConc}(\mathbf{S}^3)$ *if $n$ is even and the closure of $\beta^n$ is algebraically slice if $n$ is odd.*

We would like to add the following remarks:

- The above theorem is not entirely new. The fact that the closure of $\beta^n$ is a non-slice knot when $n$ is even was proved by Lisca [9] using a celebrated theorem of Donaldson (also [14, Section 6.2] implies the same result). However, our proof of this fact is different. It uses Gauss diagram technique and is simple.
- The main ingredient of our proof is the computation of the Arf invariant. More precisely, we compute $\mathrm{Arf}(\widehat{\beta^n})$ for each $n$ not divisible by 3. Note that if $n$ is divisible by 3 then the closure of $\beta^n$ is a three component totally proper link, and each of the components is a trivial knot. It follows from the result of Hoste [6] that its Arf invariant equals to the forth coefficient of its Conway polynomial. In [1, Corollary 3.5] the author proved that this coefficient can be obtained as a certain count of ascending arrow diagrams with 4 arrows in a Gauss diagram of this link. However, in this case the computation is more involved since there are many ascending arrow diagrams with 4 arrows. It is left to an interested reader.
- It is still unknown whether the induced family of smooth or even algebraic concordance classes is infinite, and these seem to be hard questions.

Let $\{L_n\}_{n=1}^{\infty}$ be a sequence of Lucas numbers, i.e. it is a Fibonacci sequence with $L_1 = 1$ and $L_2 = 3$. Surprisingly, Theorem 1 has a corollary which is the following number theoretic statement.

**Corollary 1.** Let $n \in \mathbf{N}$. Then

(1) $L_{12n\pm 4}$ is equivalent to $5 \bmod 8$ or $7 \bmod 8$

(2) $L_{12n\pm 2} \equiv 3 \bmod 8$

(3) $L_{12n\pm2} - 2$ is a square.

**Remark.** Corollary 1 is not new. All parts of it can be proved directly. However, we think that it is interesting that a number theoretic result can be deduced from a purely topological statement. We would like to point out that the proof (identical to ours) of the fact that $L_{12n\pm2} - 2$ is a square for every $n$ was given first in [14, Section 6.2].

## 2. Proofs

Let us recall the notion of a Gauss diagram.

**Definition 2.1.** Given a classical link diagram $D$, consider a collection of oriented circles parameterizing it. Unite two preimages of every crossing of $D$ in a pair and connect them by an arrow, pointing from the overpassing preimage to the underpassing one. To each arrow we assign a sign (writhe) of the corresponding crossing. The result is called the *Gauss diagram* $G$ corresponding to $D$.

We consider Gauss diagrams up to an orientation-preserving diffeomorphisms of the circles. In figures we will always draw circles of the Gauss diagram with a counter-clockwise orientation. A classical link can be uniquely reconstructed from the corresponding Gauss diagram [4]. We are going to work with based Gauss diagrams, i.e. Gauss diagrams with a base point (different from the endpoints of the arrows) on one of the circles.

**Example 2.2.** Diagram of the trefoil knot together with the corresponding Gauss diagram is shown in Figure 2.1.

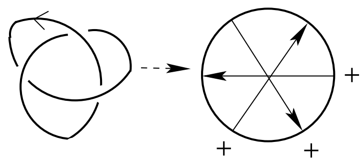


Figure 2.1. Diagrams of the trefoil.

*Proof of Theorem 1.* Recall that a knot is called strongly plus-amphicheiral if it has an orientation-preserving point reflection symmetry. In particular, it is equivalent to its mirror image. Since the closure of $\beta^n$ is a Turk's head knot for each odd $n$, it is strongly plus-amphicheiral. It follows from the work of Long [11] that a strongly plus-amphicheiral knot is algebraically slice and hence the closure of $\beta^n$ is algebraically slice for an odd $n$.

Let us prove the other direction. For a braid $\alpha$ denote by $\widehat{\alpha}$ its closure. Let $i = 1, 2$. By reversing the orientation of all strings in the braid $\beta^{3n+i}$, one immediately sees that the knot $-\widehat{\beta^{3n+i}}$ is equivalent to the knot $\left(\widehat{\beta^{3n+i}}\right)^*$. Hence the knot $\widehat{\beta^{3n+i}}$ is of order at most 2 in $\mathrm{Conc}(\mathbf{S}^3)$ and hence in $\mathrm{AConc}(\mathbf{S}^3)$. To complete the proof we must show that for each odd $n$ the knot $\widehat{\beta^{3n+1}}$ is not algebraically slice and for each even $n$ the knot $\widehat{\beta^{3n+2}}$ is not algebraically slice.

Given knot $K$ let $\mathrm{Arf}(K)$ be the Arf invariant of $K$. Recall that $\mathrm{Arf}(K) := c_2(K) \bmod 2$, where $c_2(K)$ is the coefficient before $z^2$ in the Conway polynomial of $K$. It is known that if $c_2(K) \bmod 2 = 1$, then the knot $K$ is not algebraically slice, see e.g. [3, 8, 10].

**Case 1.** We consider the knot $\widehat{\beta^{3n+1}}$ where $n$ is odd. Suppose that $n = 1$, then $\widehat{\beta^{3n+1}} = \widehat{\beta^4}$ and $\mathrm{Arf}(\widehat{\beta^4}) = 1$. The computation of this fact is simple and is left to the reader. It follows that in order to show that for every odd $n$ one has $\mathrm{Arf}(\widehat{\beta^{3n+1}}) = 1$, it is enough to show that the equality

$$\mathrm{Arf}(\widehat{\beta^{3n+1}}) = \left(\mathrm{Arf}(\widehat{\beta^{3(n-1)+1}}) + 1\right) \bmod 2$$

holds for every $n$.

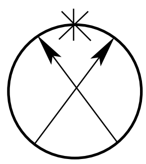

FIGURE 2.2. Arrow diagram of Polyak and Viro.

It follows from the work of Polyak and Viro [13, Corollary of Theorem 1] that one can compute $\mathrm{Arf}(K)$ by counting $\bmod 2$ an arrow diagram, shown in Figure 2.2, in any Gauss diagram of $K$. Let $n$ be any natural number. In Figure 2.3 we show a diagram of a knot $\widehat{\beta^{3n+1}}$ together with a corresponding Gauss diagram $G_n$.

Denote by $A_2(\widehat{\beta^{3n+1}})$ the number, which is a sum $\bmod 2$ of arrow diagrams, shown in Figure 2.2, that appear in $G_n$. Hence $\mathrm{Arf}(\widehat{\beta^{3n+1}}) = A_2(\widehat{\beta^{3n+1}})$. For simplicity we call the arrow diagram presented in Figure 2.2 the diagram of type $A$.

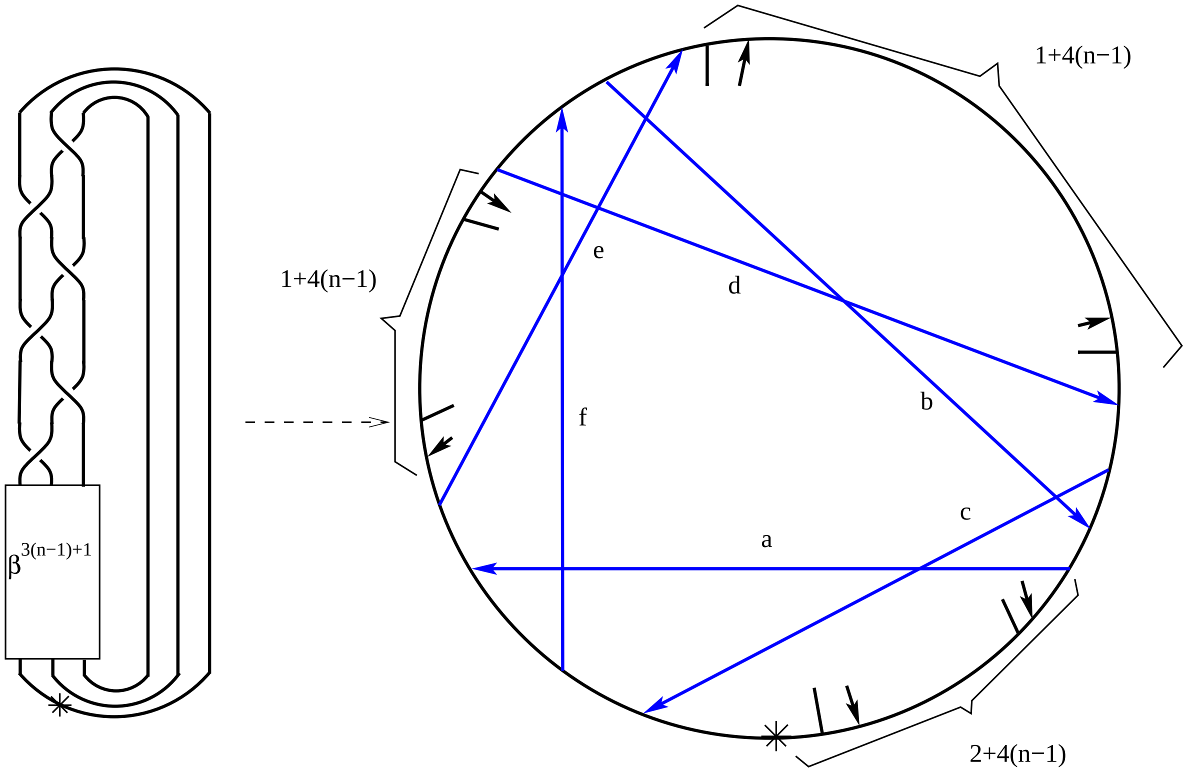


FIGURE 2.3. Knot and Gauss diagrams of $\widehat{\beta^{3n+1}}$.

1. There is only one type $A$ arrow diagram in $G_n$ which involve only blue arrows. It is a diagram whose arrows are labeled by $b$ and $c$.

2. The number of type $A$ arrow diagrams in $G_n$ which involve one black arrow and a blue arrow labeled by $a$ equals to the number of type $A$ arrow diagrams in $G_n$ which involve one black arrow and a blue arrow labeled by $c$.

3. The number of type $A$ arrow diagrams in $G_n$ which involve one black arrow and a blue arrow labeled by $b$ equals to the number of type $A$ arrow diagrams in $G_n$ which involve one black arrow and a blue arrow labeled by $d$.

4. The number of type $A$ arrow diagrams in $G_n$ which involve one black arrow and a blue arrow labeled by $e$ equals to the number of type $A$ arrow diagrams in $G_n$ which involve one black arrow and a blue arrow labeled by $f$. In fact, there are no type $A$ arrow diagrams in $G_n$ which involve one black arrow and a blue arrow labeled by $e$ or by $f$.

5. By removing blue arrows from a Gauss diagram of $\widehat{\beta^{3n+1}}$, we get a Gauss diagram of $\widehat{\beta^{3(n-1)+1}}$.

Claims 1–5 yield the equality

$$A_2(\widehat{\beta^{3n+1}}) = \Big(A_2(\widehat{\beta^{3(n-1)+1}}) + 1\Big) \bmod 2,$$

which concludes the proof of case 1.

**Case 2.** We consider the knot $\widehat{\beta^{3n+2}}$ where $n = 2k$ for $k \geq 0$. Suppose that $n = 0$, then $\widehat{\beta^{3n+2}} = \widehat{\beta^2}$ which is the figure eight knot and so $\mathrm{Arf}(\widehat{\beta^2}) = 1$. It follows that in order to show that for every even $n$ one has $\mathrm{Arf}(\widehat{\beta^{3n+2}}) = 1$, it is enough to show that the equality

$$\mathrm{Arf}(\widehat{\beta^{3n+2}}) = \Big(\mathrm{Arf}(\widehat{\beta^{3(n-1)+2}}) + 1\Big) \bmod 2$$

holds for every $n$.

In Figure 2.4 we show a diagram of a knot $\widehat{\beta^{3n+1}}$ together with a corresponding Gauss diagram $G'_n$. As indicated above $\mathrm{Arf}(\widehat{\beta^{3n+2}}) = A_2(\widehat{\beta^{3n+2}})$.

1. There are three type $A$ arrow diagrams in $G'_n$ which involve only blue arrows. These are diagrams whose arrows are labeled by $a$ and $f$, $b$ and $f$, and $a$ and $e$.

2. The number of type $A$ arrow diagrams in $G'_n$ which involve one black arrow and a blue arrow labeled by $e$ equals to the number of type $A$ arrow diagrams in $G'_n$ which involve one black arrow and a blue arrow labeled by $f$.

3. The number of type $A$ arrow diagrams in $G'_n$ which involve one black arrow and a blue arrow labeled by $d$ equals to the number of type $A$ arrow diagrams in $G'_n$ which involve one black arrow and a blue arrow labeled by $c$.

4. 2. The number of type $A$ arrow diagrams in $G'_n$ which involve one black arrow and a blue arrow labeled by $a$ equals to the number of type $A$ arrow diagrams in $G'_n$ which involve one black arrow and a blue arrow labeled by $b$. In fact, there are no type $A$ arrow diagrams in $G'_n$ which involve one black arrow and a blue arrow labeled by $a$ or by $b$.

5. By removing blue arrows from a Gauss diagram of $\widehat{\beta^{3n+2}}$, we get a Gauss diagram of $\widehat{\beta^{3(n-1)+2}}$.

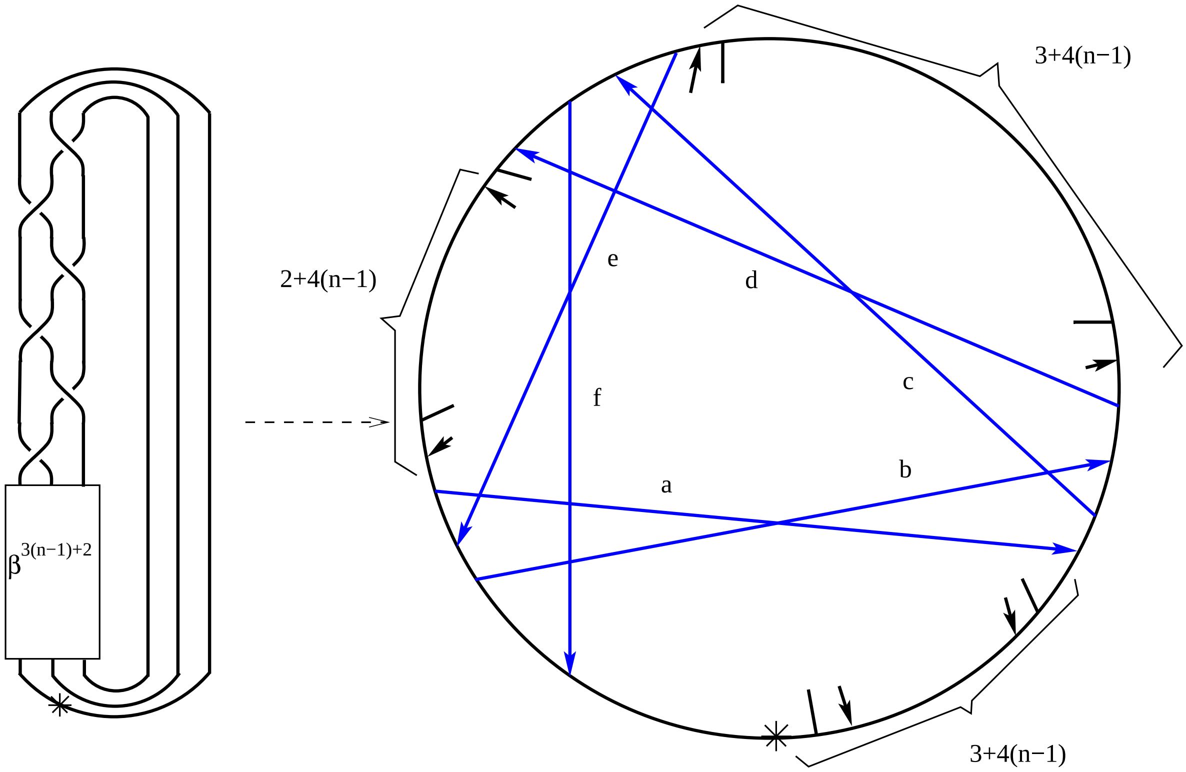


FIGURE 2.4. Knot and Gauss diagrams of $\widehat{\beta^{3n+2}}$.

Claims 1–5 yield the equality

$$A_2(\widehat{\beta^{3n+2}}) = \Big(A_2(\widehat{\beta^{3(n-1)+2}}) + 1\Big) \bmod 2,$$

which concludes the proof of case 2 and the proof of the theorem. □

*Proof of Corollary 1.* It follows from the matrix-tree theorem that the determinant $\det(K)$ of an alternating knot $K$ equals to the number of spanning trees in the associated Tait graph. Note that the knots $\widehat{\beta^{3n+1}}$, $\widehat{\beta^{3n+2}}$ are alternating for each $n$ and hence their Tait graphs are Wheel graphs. It follows from [5] that the number of spanning trees in the Wheel graph on $n+1$ points equals to $L_{2n} - 2$ [1]. Hence

(1) $$\det(\widehat{\beta^n}) = L_{2n} - 2$$

[1]This fact was communicated to the author by Brendan Owens.

for every $n$ not divisible by 3. It follows from Theorem 1 that the knots $\widehat{\beta^{6n+1}}$, $\widehat{\beta^{6n-1}}$ are algebraically slice for each $n$. Since the determinant of an algebraically slice knot is a square we conclude that

- $L_{12n\pm2} - 2$ is a square for every $n$.

In [7, 12] is proved that $\mathrm{Arf}(K) = 0 \Leftrightarrow \det(K) \equiv \pm 1 \bmod 8$. It follows that $L_{12n\pm2}$ is congruent to $1 \bmod 8$ or $3 \bmod 8$. But since a square number can not be congruent to $-1 \bmod 8$ we obtain

- $L_{12n\pm2} - 2 \equiv 3 \bmod 8$.

In the proof of Theorem 1 we showed that the Arf invariant of knots $\widehat{\beta^{6n+2}}$, $\widehat{\beta^{6n-2}}$ equals to 1 for each $n$. It follows from Murasugi result that $\det(\widehat{\beta^{6n\pm2}}) \equiv 3 \bmod 8$ or $\det(\widehat{\beta^{6n\pm2}}) \equiv 5 \bmod 8$. By (1) we get

- $L_{12n\pm4}$ is equivalent to $5 \bmod 8$ or $7 \bmod 8$

which concludes the proof of the corollary. □

**Acknowledgments.** The author would like to thank Steve Boyer, Jarek Kedra, Ana Garcia Lecuona, Paolo Lisca and Brendan Owens for useful conversations. We would like to thank the anonymous referee for careful reading of the manuscript and his/her remarks.

This work was initiated during author's stay in CRM-ISM Montreal. The author was partially supported by the CRM-ISM fellowship. He would like to thank CRM-ISM Montreal for the support and great research atmosphere.

## References

[1] Brandenbursky, M. Link invariants via counting surfaces. *Geom. Dedicata 173* (2014), 243–270.
[2] Brandenbursky, M., and Kędra, J. Concordance group and stable commutator length in braid groups. *Algebr. Geom. Topol. 15*, 5 (2015), 2861–2886.
[3] Cochran, T. D. Concordance invariance of coefficients of Conway's link polynomial. *Invent. Math. 82*, 3 (1985), 527–541.
[4] Goussarov, M., Polyak, M., and Viro, O. Finite-type invariants of classical and virtual knots. *Topology 39*, 5 (2000), 1045–1068.
[5] Hilton, A. J. W. Spanning trees and Fibonacci and Lucas numbers. *Fibonacci Quart. 12* (1974), 259–262.
[6] Hoste, J. The Arf invariant of a totally proper link. *Topology Appl. 18*, 2-3 (1984), 163–177.
[7] Levine, J. Polynomial invariants of knots of codimension two. *Ann. of Math. (2) 84* (1966), 537–554.

[8] Lickorish, W. B. R. *An introduction to knot theory*, vol. 175 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997.

[9] Lisca, P. Private communication.

[10] Livingston, C. A survey of classical knot concordance. In *Handbook of knot theory*. Elsevier B. V., Amsterdam, 2005, pp. 319–347.

[11] Long, D. D. Strongly plus-amphicheiral knots are algebraically slice. *Math. Proc. Cambridge Philos. Soc. 95*, 2 (1984), 309–312.

[12] Murasugi, K. The Arf invariant for knot types. *Proc. Amer. Math. Soc. 21* (1969), 69–72.

[13] Polyak, M., and Viro, O. Gauss diagram formulas for Vassiliev invariants. *Internat. Math. Res. Notices*, 11 (1994), 445ff., approx. 8 pp. (electronic).

[14] Stoimenov, A. Square numbers, spanning trees and invariants of achiral knots. *Comm. Anal. Geom 13*, 2 (2005), 591–631.

Department of Mathematics, Ben Gurion University, Israel

*E-mail address*: brandens@math.bgu.ac.il