Finite and p-adic polylogarithms

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Overview

- Complex polylogarithms are interesting functions satisfying many functional equations
- Finite polylogarithms are certain polynomials that experimentally satisfy functional equations similar to those coming from derivatives of complex polylogarithms.
- The functional equations for the finite polylog can be derived from those of the complex polylog using the p-adic polylog.

The complex polylogarithm

$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} z^k/k^n$

Example: $\mathrm{Li}_1(z) = -\log(1-z)$

Can be continued to a multi-valued function.

Satisfies the differential equation

$$\frac{d}{dz}\mathrm{Li}_n(z) = \frac{1}{z}\mathrm{Li}_{n-1}(z) \qquad (1)$$

The Bloch-Wigner-Ramakrishnan function

A single valued real valued version of Li_n:

$$D_n(z) = R_n\left(\sum_{k=0}^{n-1} \frac{2^k B_k}{k!} \log^k |z| \,\mathrm{Li}_{n-k}(z)\right)$$

B_k are the Bernoulli numbers:

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k t^k, \quad B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}.$$

and

$$R_n = \begin{cases} \mathrm{Re} & 2 \mid n \\ \mathrm{Im} & 2 \nmid n \end{cases}$$

Example:

$$D(z) = D_2(z) = \mathrm{Im}(\mathrm{Li}_2(z) + \log(1-z)\log|z|)$$

Functional equations

D_n satisfy many functional equations

Example: $\log(xy) = \log(x) + \log(y)$

Example: The 5-term relation:

$$D(x) + D(y) + D\left(\frac{1-x}{1-xy}\right) + D(1-xy) + D\left(\frac{1-y}{1-xy}\right) = 0$$

and many others.

Functional equations of polylogarithms are very mysterious and are supposed to reflect very deep number theoretic facts.

Finite polylogarithms

$\mathrm{li}_n(z) \in \mathbb{Z}/p[z]$ defined (Elbaz-Vincent and Gangl) by

$$\mathrm{li}_n(z) := \sum_{k=1}^{p-1} z^k/k^n$$

Kontsevich introduced li_1, proved that it satisfied the 4-term relation

$$\mathrm{li}_1(x+y) = \mathrm{li}_1(y) + (1-y)\,\mathrm{li}_1\left(\frac{x}{1-y}\right) + y\,\mathrm{li}_1\left(\frac{-x}{y}\right)$$

and noticed that the same equation is satisfied by the (real valued) function

$$DL_2(x) := x\log(x) + (1-x)\log(1-x) = x(1-x)\frac{d}{dx}D_2(x)$$

The functional equation is known as the **fundamental equation of information theory**

Elbaz-Vincent and Gangl discovered similar relation between $\mathrm{li}_2(z)$ and $DL_3(x)$.

Kontsevich conjectured that these relations can be explained using the p-adic polylogarithm.

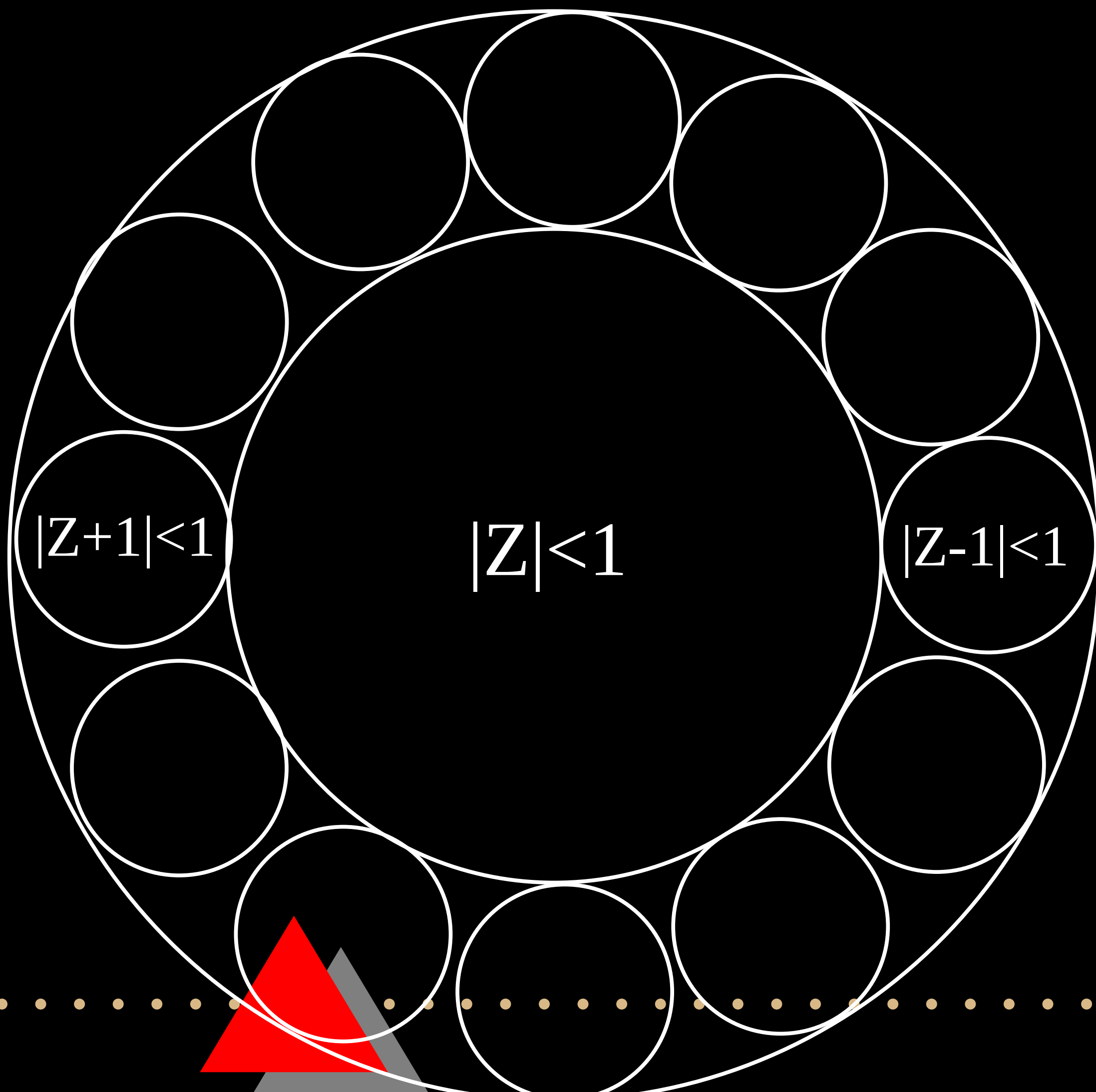
The problem of p-adic integration

Coleman (82) developed a way of solving equations like (1) p-adically.

Example: How to solve $\frac{d}{dz}y = \frac{1}{z}$?

What is the problem and what is the solution?

$\mathbb{Q}_p$ splits to a sum of circles $|z - \alpha| < 1$



On $|z-\alpha|<1$, $|\alpha|=1$ the solution is

$$\begin{aligned}\log(z) &= \log(\alpha(1+\left(\frac{z}{\alpha}\right)-1)) \\ &= \log(\alpha)+\log(1+\left(\frac{z}{\alpha}\right)-1) \\ &= \log(\alpha)-\sum_{n=1}^{\infty}\frac{(1-\frac{z}{\alpha})^n}{n}\end{aligned}$$

$|1-z/\alpha|<1$ so this series converges p-adically.

But what is $\log(\alpha)$? All we can really say is that the solution is

$$C - \sum_{n=1}^{\infty} \frac{\left(1 - \frac{z}{\alpha}\right)^n}{n}$$

For some constant C and there is a different constant for each circle.

Coleman's solution - continuation along Frobenius

We ask for "Frobenius equivariance" of the integral.

In the example this means: Since

$$\frac{d(z^p)}{z^p} = p\frac{dz}{z}$$

We should expect $y(z^p) = py(z)$.

If $\alpha^p = \alpha$ this implies $y(\alpha) = 0$. This suffices to determine y.

p-adic polylogarithms

Using this method Coleman defines

$$\mathrm{Li}_n(z) : \mathbb{Q}_p \longrightarrow \mathbb{Q}_p$$

There exist functional equations.

Example:

$$D_2(z) = \mathrm{Li}_2(z) + \frac{1}{2}\log(z)\log(1-z)$$

D_2 also satisfied the 5-term relation.

Wojtkoviac's general principle: Each functional equation of complex D_n gives the same equation for p-adic D_n.

The main theorem

Define

$$L_n(z) = \sum_{m=0}^{n-1} \frac{(-1)^m}{m!} \mathrm{Li}_{n-m}(z) \log^m(z)$$

$$F_n(z) = -nL_n(z) - L_{n-1}(z)\log(z)$$

and

$$DF_n(z) = z(1-z)\frac{d}{dz}F_n(z)$$

$$X = \{z \in \mathbb{Z}_p : \quad |z| = |z-1| = 1\}$$

Theorem (B.) for every $p > n+1$,

$$DF_n(X) \subset p^{n-1}\mathbb{Z}_p$$

and for $z \in X$

$$p^{1-n} DF_n(z) \equiv \mathrm{li}_{n-1}(z) \pmod{p}$$

Distributions and measures

Definition: a distribution on $\mathbb{Z}_p$ is a finite additive measure μ from the collection of subsets of the form $a+p^n\mathbb{Z}_p$ to $\mathbb{Q}_p$.

To define it suffices to define $\mu(a+p^n\mathbb{Z}_p)$ s.t.,

$$\mu(a+p^n\mathbb{Z}_p) = \sum_{b\equiv a \pmod{p^n}} \mu(b+p^{n+1}\mathbb{Z}_p)$$

A distribution is a measure if its set of values is bounded.

Integration with respect to a measure

If μ is a measure one can define an integral

$$\int_{\mathbb{Z}_p} f(x)d\mu(x), \quad f:\mathbb{Z}_p \to \mathbb{Q}_p \text{ continuous}$$

Example: $z \in \mathbb{Q}_p$ such that $z^{p^n} \neq 1$.

Define: $\mu_z(a + p^n\mathbb{Z}_p) = \dfrac{z^a}{1 - z^{p^n}}$.

This is a distribution:

$$\sum_{k=0}^{p-1} \frac{z^{a+kp^n}}{1-z^{p^{n+1}}} = \frac{z^a}{1-z^{p^{n+1}}} \sum_{k=0}^{p-1} \left(z^{p^n}\right)^k$$

$$= \frac{z^a}{1-z^{p^{n+1}}} \cdot \frac{1-z^{p^{n+1}}}{1-z^{p^n}} = \frac{z^a}{1-z^{p^n}}$$

The relation with the p-adic polylogarithm

Let $\mathrm{Li}_n^{(p)}(z) := \mathrm{Li}_n(z) - \mathrm{Li}_n(z^p)/p^n$.

Notice:

$$\mathrm{Li}_n^{(p)}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} - \sum_{k=1}^{\infty} \frac{z^{pk}}{(pk)^n} = \sum_{p\nmid k} \frac{z^k}{k^n}$$

Theorem (Coleman) if $|z-1| \geq 1$, then

$$\mathrm{Li}_n^{(p)}(z) = \int_{\mathbb{Z}_p^\times} x^{-n} d\mu_z(x) \, .$$

This implies that $\mathrm{Li}_n^{(p)}(z)$ is congruent modulo p to

$$\sum_{a=1}^{p-1} a^{-n}\mu_z(a+p\mathbb{Z}_p) = \sum_{a=1}^{p-1} a^{-n}\frac{z^a}{1-z^p}\,.$$

From knowing $\mathrm{Li}_n^{(p)}$ we get Li_n at α s.t. $\alpha^p = \alpha$.

The rest is computation.