Mahler measures, complex and p-adic

Joint work with Christopher Deninger from the University of Münster.
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Definition of Mahler's measure

For a Laurent polynomial

$$P(z_1, z_2, \dots, z_n) = \sum a_I z_1^{i_1} \cdots z_n^{i_n} \in \mathbb{C}[z_1^{\pm}, \dots, z_n^{\pm}]$$

The Mahler measure of P is given by

$$m(P) = (2\pi i)^{-n} \int_{\mathbb{T}^n} \log |P(z_1, \dots, z_n)| \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}$$

where $\mathbb{T} = \{z \in \mathbb{C} : \ |z| = 1\}$

Change of variables $\Longrightarrow$

$$m(P) = \int_0^1 \cdots \int_0^1 \log|P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})| d\theta_1 \cdots d\theta_n$$

Case $n = 1$

$$\tilde{P}(z) = \frac{P}{z^{\mathrm{ord}_0(P)}}$$

$$m(P) = \frac{1}{2\pi i}\int_{\mathbb{T}} \log|P(z)|\frac{dz}{z} = \log|\tilde{P}(0)| - \sum_{\substack{0<|b|<1\\ P(b)=0}} \log|b|$$

Jensen's formula

History - Lehmer's work

Case $n = 1$ - Lehmer (Annals of Math. 1933).
Motivation - Finding large prime numbers:
Suppose $P(x) = \prod (x - \alpha_i) \in \mathbb{Z}[x]$, $|\alpha_i| \neq 1$.
Set $\Delta_n(P) = \prod (\alpha_i^n - 1) \in \mathbb{Z}$
Example: $P(x) = x - 2$,
$\Delta_n(P) = 2^n - 1$ (Mersenne primes)
Measure for growth of $\Delta_n(P)$:

$$\lim_{n \to \infty} \left| \frac{\Delta_{n+1}(P)}{\Delta_n(P)} \right| = \prod \max\{1, |\alpha_i|\}$$

$$= M(P) := \exp(m(P)) \, .$$

Slower growth implies a larger chance for finding primes.
Lehmer's best example:

$$G(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1,$$

$$M(G) = 1.176\ldots$$

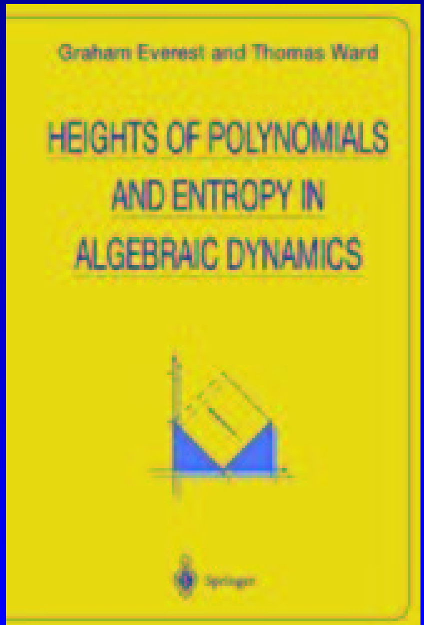
This is still the smallest value > 1 known today

Lehmer's conjecture: 0 is not an accumulation point for Mahler measures of integral polynomials in one variable.

History - Mahler's work

Mahler (1960's) compared $m(P)$ with other measures on polynomials, e.g., L^1 and L^∞ norms, and also provided the integral formula.

Account of history and elementary properties:
Everest and Ward, Height of Polynomials and Entropy in Algebraic Dynamics, Springer (1999)



Ties with Dynamical systems

To $P \in \mathbb{Z}[z_1^{\pm}, \dots, z_n^{\pm}]$ we associate

$$X_P = \mathrm{Hom}(\mathbb{Z}[z_1^{\pm}, \dots, z_n^{\pm}]/P, \mathbb{T})$$

X_P has a $\mathbb{Z}^n$-action: $(k_1, \dots, k_n)$ acts via multiplication by $z_1^{k_1} \cdots z_n^{k_n}$.

Example: $P(z) = a_d z^d + \cdots a_0 \in \mathbb{Z}[z]$.

$$X_P = \{(x_k) \in \mathbb{T}^{\mathbb{Z}} : a_0 z_k + a_1 z_{k+1} + \cdots + a_d z_{k+d} = 0, \text{ all } k\}$$

$\mathbb{Z}$-action via shift.

Ties with Dynamical systems

Theorem (Lind, Schmidt and Ward 1990) The topological entropy of X_P is exactly $m(P)$.

Recall *Topological Entropy*

X - compact topological space.

$T : X \to X$ - continuous map.

$\mathcal{U}$ - open cover of X.

$N(\mathcal{U}) = \min\{|\mathcal{V}| : \ \mathcal{V}$ is a subcover of $\mathcal{U}\}$.

$\mathcal{U} \vee \mathcal{V} := \{U \cap V : \ U \in \mathcal{U}, \ V \in \mathcal{V}\}$ - join of $\mathcal{U}$ and $\mathcal{V}$.

Definition: The topological entropy of T is

$$\sup_{\mathcal{U}} \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U} \vee T^{-1}\mathcal{U} \vee \cdots \vee T^{-(n-1)}\mathcal{U}) \ .$$

Theorem (Lind, Schmidt, Ward 1990) The set of all possible entropies of $\mathbb{Z}^n$-actions via automorphisms on compact groups is either $[0, \infty]$ or equal to the set of Mahler measures of polynomials in n variables, depending on Lehmer's conjecture.

Essential spanning forests

Γ - graph with vertex set $\mathbb{Z}^n$, invariant under shifts.
$k(y)$ - number of edges connecting y and 0.
D - number of edges coming out of a vertex (assumed finite).
$P(z_1, \ldots z_n) = D - \sum k(y) z^y$.
Definition: An essential spanning forest is a subgraph on all the vertices, with no cycles and having only infinite connected components.
The essential spanning forest dynamical system:
X - set of all essential spanning forests.
$\mathbb{Z}^n$-action - via shifts.
Theorem (Burton & Pemantle 1993, Solomyak 1998)
The entropy of X is $m(P)$.

Variation of Mahler measures

Rodriguez-Villegas 1998

Consider $P(x,y) \in \mathbb{C}[x^{\pm}, y^{\pm}]$.

Set $P_k(x,y) = k - P(x,y)$, $\lambda = \frac{1}{k}$

$P_k = \frac{1}{\lambda}(1 - \lambda P)$

$$m(P_k) = (2\pi i)^{-2} \int_{\mathbb{T}^2} \log \left|\frac{1}{\lambda}(1 - \lambda P(x,y))\right| \frac{dx}{x}\frac{dy}{y}$$

$$\int_{\mathbb{T}^2} x^k y^m \frac{dx}{x}\frac{dy}{y} = \begin{cases} (2\pi i)^2 & \text{if } k = m = 0 \\ 0 & \text{otherwise,} \end{cases}$$

so $m(P_k) = \operatorname{Re}\tilde{m}(\lambda)$

$$\tilde{m}(\lambda) = -\log\lambda - \sum_{n=1}^{\infty} \frac{a_n}{n}\lambda^n\,,$$

where a_n is the constant coefficient of $P(x, y)^n$.

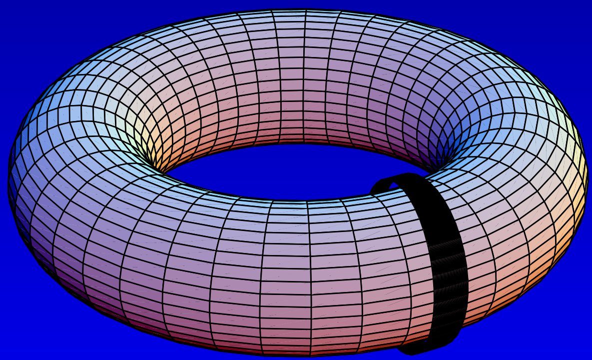
E.g., if $P(x,y) = x + y + x^{-1} + y^{-1}$, then

$$\tilde{m}(\lambda) = -\log(\lambda) - \sum_{n=1}^{\infty} \frac{1}{2n} \binom{2n}{n}^2 \lambda^{2n}$$

Set $a_0 = 1$. Then

$$-\lambda \frac{d}{d\lambda}\tilde{m}(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n = \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2} \frac{1}{1-\lambda P(x,y)}\frac{dx}{x}\frac{dy}{y}$$

This is a period for the family of curves
$C_\lambda := \{(x,y) \in \mathbb{C}^2 : \ 1 - \lambda P(x,y) = 0\}$



It satisfies a differential equation with algebraic coefficients, the *Picard-Fuchs* equation.

In the example: $u_0(\lambda) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 \lambda^{2n}$.

Under substitution $\lambda^2 = \mu$ we get the equation

$$\mu(16\mu - 1)\frac{d^2u_0}{d\mu^2} + (32\mu - 1)\frac{du_0}{d\mu} + 4u_0 = 0\ .$$

L-functions

Arithmetic Geometric objects $X \Longrightarrow L$-function $L(X,s) : \{s \in \mathbb{C} : \ \mathrm{Re}\, s > \alpha\} \to \mathbb{C}$.
Examples:

- a Dirichlet character is a function $\chi : \mathbb{Z} \longrightarrow \mathbb{C}$, multiplicative, periodic of period N (N is called the conductor of χ)

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \,.$$

E.g., if $\chi = 1$ we get Riemann's zeta function.

It is easy to see that

$$L(\chi, s) = \prod_{p \text{ prime}} \left(1 - \chi(p) p^{-s}\right)^{-1} .$$

L-functions of elliptic curves

- For an elliptic curve $y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Z}$, set

$$a_p = p + 1 - \text{number of solutions of } y^2 = x^3 + ax + b \text{ modulo } p$$

$$L(E, s) = \prod_{p \text{ prime}} \left(1 - a_p p^{-s} + p^{-2s}\right)^{-1}$$

Taniyama-Shimura conjecture (implies Fermat) says $L(E, s)$ has analytic continuation to all of $\mathbb{C}$.

Mahler and L-functions

Let χ_3 be the Dirichlet character of conductor 3 with $\chi_3(1) = 1$ and $\chi_3(2) = -1$.

Theorem (Smyth 1981)

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_3, 2) \; ,$$

$$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3) \; .$$

Theorem (Schmidt)

$$m((x+y)^2+3) = \frac{2}{3}\log(3) + \frac{\sqrt{3}}{\pi} L(\chi_3, 2) \; .$$

Deninger's Explanation

consider X an algebraic variety over $\mathbb{Q}$.
This roughly means: X is a set in $\mathbb{C}^n$ defined by polynomial equations and inequations with coefficients in $\mathbb{Q}$.

$K_n(X)$ - algebraic K-theory groups of X =?

Beilinson's "regulator"

$$r_{\mathcal{D}} : K_n(X) \to H^i_{\mathcal{D}}(X, \mathbb{R}(2i-n)) \ .$$

We are interested in the case:
$\dim X = n$,
$r_{\mathcal{D}} : K_{n+1}(X) \to H^{n+1}_{\mathcal{D}}(X, \mathbb{R}(n+1))$.
In $K_{n+1}(X)$ we have symbols $\{f_0, \dots, f_n\}$, where f_i are invertible algebraic functions on X.

$$r_{\mathcal{D}}(\{f_0, \dots, f_n\}) = \sum_{i=0}^{n} \frac{(-1)^i}{(n+1)!} \sum_{\sigma \in \mathbb{S}_{n+1}} \operatorname{sgn}(\sigma)$$

$$\log|f_{\sigma(0)}| \frac{d\bar{f}_{\sigma(1)}}{\bar{f}_{\sigma(1)}} \wedge \cdots \wedge \frac{d\bar{f}_{\sigma(i)}}{\bar{f}_{\sigma(i)}} \wedge \frac{df_{\sigma(i+1)}}{f_{\sigma(i+1)}} \wedge \cdots \wedge \frac{df_{\sigma(n)}}{f_{\sigma(n)}}$$

Suppose P does not vanish on $\mathbb{T}^n$. Let

$$X = (\mathbb{C} - \{0\})^n - \{(z_1, \dots, z_n) \in \mathbb{C}^n : \ P(z_1, \dots, z_n) = 0\}$$

Then $\mathbb{T}^n \subset X$. The functions $z_1, \dots, z_n$, P are invertible on X, hence $\{z_1, \dots, z_n, P\} \in K_{n+1}(X)$

Deninger's theorem 1997

Note that on $\mathbb{T}$:

- $\log|z_i| = 0$,
- $\bar{z}_i = \dfrac{1}{z_i}$ so $\dfrac{d\bar{z}_i}{\bar{z}_i} = -\dfrac{dz_i}{z_i}$.

Theorem (Deninger 1997)

$$\int_{\mathbb{T}^n} r_{\mathcal{D}}(\{P, z_1, \dots, z_n\}) = (2\pi i)^n m(P)$$

A better formula is obtained as follows:
$Z = (\mathbb{C} - \{0\})^n - X$
$A = Z \cap \mathbb{T}^{n-1} \times \{|z_n| \leq 1\}$.
$\{z_1, \dots, z_n\} \in K_n(Z)$.
$P^*(z_1, \dots z_{n-1}) := P(z_1, \dots, z_{n-1}, 0)$
Theorem (Deninger 1997) Under certain assumptions

$$m(P^*) - m(P) = \left(\frac{(-1)}{2\pi i}\right)^{n-1} \int_A r_{\mathcal{D}}(\{z_1, \dots, z_n\}) \,. \tag{1}$$

Beilinson's conjecture

Beilinson's conjecture: "A determinant with entries like (1) is related to a special value of an L-function"
Consequence: If

- $\{z_1, \ldots, z_n\}$ extends to a "compactification" Y of Z,
- The determinant happens to be 1×1,

Then we get a relation with the L-function of Y.

p-adic Mahler measures

- Analogue of Beilinson regulator $=$ Syntomic regulator (Fontaine, Messing, Gros, Nizioł, B.)
- Analogue of integration on the complex torus $=$ One of
  - Multidimensional Shnirelman integration;
  - Integration on the complex torus imported via the theory of p-adic periods

p-adic numbers

$\mathbb{Q}_p$ = completion of $\mathbb{Q}$ with respect to the absolute value

$$|p^n \frac{r}{s}|_p = p^{-n} \quad r, s \text{ prime to } p$$

$\mathbb{C}_p$ = completion of the algebraic closure of $\mathbb{Q}_p$.

Shnirelman integration

$\mathbb{T}_p = \{x \in \mathbb{C}_p : \ |x|_p = 1\}$
$f : \mathbb{T}_p^n \to \mathbb{C}_p$.
Definition of Shnirelman's integral:

$$\int_{\mathbb{T}_p^n} f(z) \frac{dz_1}{z_1} \ldots \frac{dz_n}{z_n} \ := \ \lim_{\substack{N\to\infty \\ (N,p)=1}} \frac{1}{N^n} \sum_{\zeta \in \mu_N^n} f(\zeta) \ .$$

Similarity with usual integration:

- Looks like Riemann sums;
- Residue theorem:

$$f(z) = \sum_{I \in \mathbb{Z}^n} a_I z_1^{i_1} \ldots z_n^{i_n} \Rightarrow \int_{\mathbb{T}_p^n} f(z) \frac{dz_1}{z_1} \ldots \frac{dz_n}{z_n} = a_0 \ .$$

p-adic Mahler measure I

Assume $P \in \mathbb{C}_p[z_1^{\pm}, \dots, z_n^{\pm}]$ does not vanish on $\mathbb{T}_p^n$

$$m_p(P) = \int_{\mathbb{T}_p^n} \log_p P(z) \frac{dz_1}{z_1} \dots \frac{dz_n}{z_n}$$

$\log_p : \mathbb{C}_p \to \mathbb{C}_p = p$-adic version of the logarithm.
For $n = 1$

$$m_p(P) = \log_p \tilde{P}(0) - \sum_{\substack{0<|b|_p<1 \\ P(b)=0}} \log_p b$$

p-adic L-functions

They are p-adic functions interpolating special values of complex L-functions.
Example: Kubota-Leopoldt p-adic L-function - Interpolating special values of Riemann's ζ:

$$\zeta(1-k) = -\frac{B_k}{k}$$

B_k - Bernoulli numbers, $\dfrac{t}{e^t - 1} = \sum \dfrac{B_k}{k!} t^k$

Set $\zeta^*(s) = (1 - p^{-s})\zeta(s)$.
Kummer congruences:

$$k_1 \equiv k_2 \pmod{(p-1)p^n} \Longrightarrow$$

$$\zeta^*(1-k_1) \equiv \zeta^*(1-k_2) \pmod{p^{n+1}}$$

Consequence: Existence of p-adic L-function L_p

It satisfies for example: $L_p(1-n) = \zeta^*(1-n)$ if $p-1|n$.

Importing the complex torus to the p-adic world.
For X a variety over $\mathbb{Q}$:

$$H_i(X(\mathbb{C}),\mathbb{Q}) \to H_i^{et}(\bar{X},\mathbb{Q}_p) \to H_i^{dR}(X/\mathbb{Q}) \otimes B_{dR}$$

$B_{dR} =$ "field of p-adic periods"- a mysterious field containing $\mathbb{C}_p$.
These maps depend on σ - a choice of embeddings

$$\mathbb{C} \hookleftarrow \bar{Q} \hookrightarrow \mathbb{C}_p$$

So $\mathbb{T}^n \in H_i(X(\mathbb{C}), \mathbb{Q})$ is “imported” to the p-adic world.

$$\mathbb{T}^n \in H_i^{dR}(X/\mathbb{Q}) \otimes B_{dR}$$

p-adic Mahler II

$$X = (\mathbb{A}^1 - \{0\})^n$$
$$- \{(z_1, \dots, z_n) : \; P(z_1, \dots, z_n) = 0\}$$

Syntomic regulator:

$$r_{syn}\{P, z_1, \dots, z_n\} \in H^n_{dR}(X/\mathbb{Q}) \otimes \mathbb{Q}_p$$

Definition The (p, σ)-Mahler measure of P is

$$\langle r_{syn}\{P, z_1, \dots, z_n\}, \mathbb{T}^n \rangle \in B_{dR}$$

For $n = 1$

$$m_{p,\sigma}(P) = \log_p \tilde{P}(0) - \sum_{\substack{0<|b|_\infty<1 \\ P(b)=0}} \log_p b$$

In some cases we can tie this to special values of p-adic L-functions.