Midterm

Mark the correct answer in each part of the following questions.

1. A player draws cards **without replacement** from a full deck of 52 cards until the queen of hearts is obtained. Let X denote the total number of cards drawn in the process and Y the number of princes drawn in the process.

   After the first stage of the experiment is finished, we roll X dice. Let Z be the number of dice showing 6.

   (a) $P(Y = 2) =$

   (i) $1/4$.
   (ii) $1/5$.
   (iii) $\frac{\binom{4}{2}}{2^4}$.
   (iv) $\frac{\binom{5}{2}}{2^5}$.
   (v) None of the above.

   (b) $P(Y = X - 1) =$

   (i) $\frac{1}{52} \cdot \sum_{k=0}^{5} \frac{\binom{5}{k}}{\binom{52}{k}}$.
   (ii) $\frac{1}{52} \cdot \sum_{k=0}^{4} \frac{\binom{4}{k}}{\binom{52}{k}}$.
   (iii) $\sum_{k=0}^{5} \frac{\binom{5}{k}}{\binom{52}{k}} \cdot \frac{1}{52-k}$.
   (iv) $\sum_{k=0}^{4} \frac{\binom{4}{k}}{\binom{52}{k}} \cdot \frac{1}{52-k}$.
   (v) None of the above.

(c) $P(X = 52 | Z = 0) =$

(i) $\frac{(5/6)^{52}}{(1-(5/6)^{51})}$.

(ii) $\frac{(5/6)^{52}}{5\cdot(1-(5/6)^{51})}$.

(iii) $\frac{(5/6)^{52}}{(1-(5/6)^{52})}$.

(iv) $\frac{(5/6)^{52}}{5\cdot(1-(5/6)^{52})}$.

(v) None of the above.

(d) Now suppose that we repeat the first part of the experiment $\binom{52}{4}\cdot 48$ times. Let N be the number of experiments finished after five steps, where first all four princes are drawn (in any order) and then the queen of hearts is drawn. Then $P(N = 5) \approx$

(i) $\frac{e^{-1}}{4!}$.

(ii) $\frac{e^{-1}}{5!}$.

(iii) $\frac{e^{-4}}{4!}$.

(iv) $\frac{e^{-5}}{5!}$.

(v) None of the above.

2. In an infinite experiment, we first toss one coin, then two (other) coins, then three (other) coins, and so forth. Let X_n be the number of coins showing H at the n-th stage, $1 \leq n < \infty$. Define:

$$Y_n = \begin{cases} 0, & X_n \text{ is even,} \\ 1, & X_n \text{ is odd.} \end{cases}$$

(a) For every sufficiently large n,

(i) $F_{Y_n}(0) = 1/2$.

(ii) $F_{Y_n}(0) > 1/2$.

(iii) $F_{Y_n}(0) > 1/2$ if n is even and $F_{Y_n}(0) < 1/2$ if n is odd.

(iv) $F_{Y_n}(0) > 1/2$ if n is odd and $F_{Y_n}(0) < 1/2$ if n is even.

(v) None of the above.

(b) The probability that $X_n \le n-2$ for every $n \ge 3$ is

(i) larger than 1/2.

(ii) exactly 1/2.

(iii) less than 1/2 but strictly positive.

(iv) 0.

(v) None of the above.

(c) Suppose now that we change the experiment by adding a gold coin, as follows. At each stage n, we toss the n regular coins as above, as well as the gold coin. The experiment is stopped at the first stage in which the gold coin shows H. Let Z be the number of coins that showed T at the last stage and W the number of stages. (For example, if in the first 99 stages, the gold coin showed T, and in the 100-th stage it showed H, 25 of the regular coins showed T and the other 75 showed H, then $Z = 25$ and $W = 100$.) Then $P(Z = 1) =$

(i) $\frac{2}{5}$.

(ii) $\frac{3}{7}$.

(iii) $\frac{4}{9}$.

(iv) $\frac{5}{11}$.

(v) None of the above.

(d) Still under the scenario of part (c), we have $P(W = 1|Z = 0) =$

(i) $\frac{3}{4}$.

(ii) $\frac{4}{5}$.

(iii) $\frac{5}{6}$.

(iv) $\frac{6}{7}$.

(v) None of the above.

3. An urn contains 100 white balls and 200 black balls. We draw out of the urn 40 balls **without replacement**. Let W_1 be the number of white balls among the first 20 and W_2 the number of white balls among the next 20.

(a)

(i) Both W_1 and W_2 are hypergeometrically distributed.
(ii) W_1 is hypergeometric, while W_2 is binomial.
(iii) W_1 is hypergeometric, while W_2 is negative binomial.
(iv) W_1 is hypergeometric, while the distribution of W_2 does not belong to any of the special families introduced in class.
(v) None of the above.

(b) Let A be the event whereby the second ball drawn out of the urn is white and B the event whereby the fourth ball is such. Then

(i) A and B are disjoint.
(ii) A and B are non-disjoint, but they are independent.
(iii) $P(A|B) > P(A)$.
(iv) $P(A|B) < P(A)$.
(v) None of the above.

(c) Consider the same experiment, with the following change. We add to the 300 balls in the urn a single red ball. Let W_1' be the number of white balls among the first 20 in the revised experiment.

(i) $P(W_1' = k) = P(W_1 = k)$ for every $0 \leq k \leq 20$.
(ii) $P(W_1' = k) < P(W_1 = k)$ for every $0 \leq k \leq 20$.
(iii) $P(W_1' = k) > P(W_1 = k)$ for every $0 \leq k \leq 20$.
(iv) $P(W_1' = k) < P(W_1 = k)$ for some values of k between 0 and 20 while $P(W_1' = k) > P(W_1 = k)$ for some other values of k.
(v) None of the above.

(d) Suppose now that we repeat the whole original experiment (without the red ball) until the first 20 balls are white and the next 20 balls are black. Let X be the number of repetitions. The distribution of X is

(i) Binomial.
(ii) Hypergeometric.
(iii) Geometric.
(iv) Very close to a Poisson distribution.
(v) None of the above.

Solutions

1. (a) Due to symmetry, the order in which the four princes and the Queen of Hearts are drawn, while disregarding the other 47 cards, is equally likely to be any of the 5! possible orders. Specifically, the probability that the queen is the first, second, third, fourth, or fifth card in this sequence is 1/5 for each position.

   Thus, (ii) is true.

   (b) The event $\{Y = X - 1\}$ occurs if and only if the only cards drawn before the queen of hearts, if any, are princes. Hence this event is the disjoint union of the five events $A_i, 0 \leq k \leq 4$, where A_k is the event that the first k cards are princes and the $(k+1)$-st is the queen of hearts. Now $A_k = B_k \bigcap C_k$, where B_k is the event whereby the first k drawings are of princes and C_k the event whereby the $(k+1)$-st is the queen of hearts. Hence:

$$P(A_k) = P(B_k)P(C_k|B_k) = \frac{\binom{4}{k}}{\binom{52}{k}} \cdot \frac{1}{52 - k}.$$

   Thus, (iv) is true.

   (c) The sample space Ω is the disjoint union of the 52 events Q_k, where Q_k is the event that the queen of hearts is the k-th card to be drawn. By symmetry, all Q_k-s have the same probability 1/52. Under Q_k, the probability that $Z = 0$ is $(5/6)^k$. Consequently, by Bayes' formula:

$$P(X = 52|Z = 0) = \frac{\frac{1}{52} \cdot (5/6)^{52}}{\frac{1}{52} \sum_{k=1}^{52} (5/6)^k}.$$

   After simplification, we obtain:

$$P(X = 52|Z = 0) = \frac{(5/6)^{52}}{5\left(1 - (5/6)^{52}\right)}.$$

   Thus, (iv) is true.

   (d) The event that the first four cards are princes and the fifth is the queen of hearts is just the event A_4 from part (b). According to the calculations there, we have

$$P(A_4) = \frac{\binom{4}{4}}{\binom{52}{4}} \cdot \frac{1}{52-4} = \frac{1}{\binom{52}{4} \cdot 48}.$$

It follows that $N \sim B(m, 1/m)$, where $m = \binom{52}{4} \cdot 48$. By the Poissonian approximation, N is approximately $P(1)$-distributed. Thus, (ii) is true.

2. (a) We have seen in class that X_n is even or odd with probability 1/2 each. In other words,

$$P(Y_n = 0) = P(Y_n = 1) = 1/2.$$

Thus, (i) is true.

(b) We have:

$$P(X_n \le n-2) = 1 - P(X_n = n) - P(X_N = n-1) = 1 - \frac{n+1}{2^n}.$$

Hence the probability in question is

$$p = \prod_{n=3}^{\infty} (1 - (n+1)/2^n).$$

All factors on the right-hand side are less than 1, and the first is 1/2, so that $p < 1/2$. On the other hand, since

$$\sum_{n=3}^{\infty} \frac{n+1}{2^n} < \infty,$$

and none of the factors vanish, the product above is non-zero. Thus, (iii) is true.

(c) Let A_n be the event that the gold coin showed H at the n-th trial, $1 \le n < \infty$. By the law of total probability:

$$\begin{aligned} P(Z=1) &= \textstyle\sum_{n=1}^{\infty} P(A_n)P(Z=1|A_n) \\ &= \textstyle\sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\binom{n}{1}}{2^n} \\ &= \sum_{n=1}^{\infty} \binom{n}{1} \left(\frac{1}{4}\right)^n \\ &= \frac{(1/4)^1}{(1-1/4)^{1+1}} \\ &= \frac{4}{9}. \end{aligned}$$

Thus, (iii) is true.

(d) We have

$$P(W=1, Z=0) = P(W=1)P(Z=0|W=1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

and

$$\begin{aligned} P(Z=0) &= \textstyle\sum_{n=1}^{\infty} P(W=n)P(Z=0|W=n) \\ &= \textstyle\sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{1}{2^n} \\ &= \textstyle\sum_{n=1}^{\infty} \frac{1}{4^n} \\ &= \frac{1}{3}. \end{aligned}$$

Therefore:

$$P(W=1|Z=0) = \frac{P(W=1, Z=0)}{P(Z=0)} = \frac{3}{4}.$$

Thus, (i) is true.

3. (a) For every k between 0 and 20, the number of possibilities of drawing 40 balls, such that $W_1 = k$, is the same as the number of possibilities for which $W_2 = k$. Hence both variables are identically distributed.

   Thus, only (i) is true.

   (b) Under B, there are 299 possibilities for the second ball, in 99 of which it is black and in the other 200 – white. Hence:

$$P(A|B) = \frac{99}{299} < \frac{100}{300} = P(A).$$

   Thus, (iv) is true.

   (c) Intuitively, when the red ball is added, we may expect less white balls to be drawn, so that, for example, $P(W_1 = 0) < P(W_1' = 0)$. Indeed, more formally,

$$P(W_1 = 0) = \frac{\binom{100}{0}\binom{200}{20}}{\binom{300}{20}} = \frac{\binom{200}{20}}{\binom{300}{20}}$$

   and

$$P(W_1' = 0) = \frac{\binom{100}{0}\binom{201}{20}}{\binom{301}{20}} = \frac{\binom{201}{20}}{\binom{301}{20}}.$$

   A routine calculation yields:

$$\frac{P(W_1' = 0)}{P(W_1 = 0)} = \frac{201 \cdot 281}{181 \cdot 301} > 1.$$

   Since the sum of all probabilities, as k ranges from 0 to 20, is 1, we have values of k for which the direction of the inequality is reversed.

   Thus, (iv) is true.

   (d) Defining as a success the event whereby all balls are white, X is the number of trials until the first success in a sequence of independent trials with two possible outcomes – success and failure.

   Thus, (iii) is true.