Final #2

Mark the correct answer in each part of the following questions.

1. All students of BGU have been invited by Restaurants Ranking International to rank all 100 restaurants in Beer Sheva from best to worst. (All participants will get a free meal at their top ranked restaurant.) Assume that students rank the restaurants randomly, each ranking with the same probability.

   (a) The probability that Nurit Cohen and Shoshana Levy will give completely different rankings (i.e., no restaurant will be ranked the same by both students) is approximately

   (i) $\frac{1}{2e}$.
   (ii) $\frac{1}{e}$.
   (iii) $\frac{1}{2}$.
   (iv) $1 - \frac{1}{e}$.
   (v) None of the above.

   (b) The probability that exactly two restaurants will be ranked the same by the two students is approximately

   (i) $\frac{1}{2e}$.
   (ii) $\frac{1}{e}$.
   (iii) $\frac{1}{2}$.
   (iv) $1 - \frac{1}{e}$.
   (v) None of the above.

(c) 10000 students sent their rankings. Let X be the number of students who ranked Chef of the Chefs among the best 20 restaurants. The best lower bound that can be inferred from Chebyshev's inequality regarding $P(1880 \leq X \leq 2120)$ is:

(i) $P(1880 \leq X \leq 2120) \geq \frac{4}{9}$.
(ii) $P(1880 \leq X \leq 2120) \geq \frac{5}{9}$.
(iii) $P(1880 \leq X \leq 2120) \geq \frac{7}{9}$.
(iv) $P(1880 \leq X \leq 2120) \geq \frac{8}{9}$.
(v) None of the above.

2. Reuven participates in a multi-stage game, as follows. At each step, a digit between 0 and 9 is drawn uniformly at random. Reuven guesses which digit has been drawn. If his guess is right, he proceeds to the next stage. At the first time he fails, the game terminates. Reuven's prize is the number formed by the digits he guessed correctly (in shekels), where the first digit is the ones digit, the second – the tens digit, and so forth. (For example, if he first guessed 6, then 0, then 7, then 0, and then 5, and all guesses were correct but the last, he gets 706 shekels. If his first guess is wrong, he gets 0.)

(a) Suppose that Reuven guesses each time a uniformly random digit, independently of former guesses. Let R denote his profit. Then $E(R) =$

(i) $9/2$.
(ii) $10^{9/2}$.
(iii) $\frac{10^{9/2}}{9/2}$.
(iv) ∞.
(v) None of the above.

(b) Shimon participates in the game as well, but his prize S is 2^k, where k is the number of correct guesses. (If he fails right away, he gets $2^0 = 1$ shekel.) Then $E(S) =$

(i) 5/8.
(ii) 7/8.
(iii) 9/8.
(iv) 11/8.
(v) None of the above.

(c) Levy participates in the game also. However, for him we continue drawing digits until the first time he guesses **correctly**. His prize L is the number of **incorrect** guesses prior to his correct guess. Then:

(i) $E(L) = 9$ and $V(L) = 90$.
(ii) $E(L) = 9$ and $V(L) = 100$.
(iii) $E(L) = 10$ and $V(L) = 90$.
(iv) $E(L) = 10$ and $V(L) = 100$.
(v) None of the above.

3. An urn initially contains n blue and n red balls. The balls of each color are enumerated from 1 to n. Assume that n is large. At each step, we take out of the urn a uniformly random ball and return a white ball instead. For each $k \geq 1$, denote by B_k the event that the ball drawn at step k is blue and by R_k the event that the ball is red.

(a) $P(B_{n+1} \cap R_{3n+2}) \approx$

(i) $\frac{1}{8e^2}$.
(ii) $\frac{1}{4e^2}$.
(iii) $\frac{1}{2e^2}$.
(iv) $\frac{1}{e^2}$.
(v) None of the above.

(b) Let X and Y denote the number of blue balls and of red balls in the urn, respectively, after two steps. Then $\rho(X, Y) \approx$

(i) -1.

(ii) $-1/2$.

(iii) $-1/e$.

(iv) 0.

(v) None of the above.

(c) Let Z be the number of steps until the blue ball enumerated by n is drawn. Call a step $k \geq 2$ a *reverse step* if the ball drawn from the urn at step k is the same very white ball placed in the urn at step $k-1$. Let W be the number of reverse steps, out of the n steps from 2 until $n+1$.

(i) Z is negative binomial and W is Poissonian.

(ii) Z is hypergeometric and W is geometric.

(iii) Z is geometric and W is binomial.

(iv) Z is binomial and W is hypergeometric.

(v) None of the above.

(d) Let T be the number of steps until both the blue ball enumerated by n and the red ball enumerated by n are drawn. For every $k \geq 2$:

(i) $P_T(k) = \dfrac{1}{n} \cdot \left(1 - \dfrac{1}{2n-1}\right)^{k-1} \cdot \left(1 - \left(1 - \dfrac{1}{2n-1}\right)^{k-1}\right)$.

(ii) $P_T(k) = \dfrac{1}{n} \cdot \left(1 - \dfrac{1}{2n-1}\right)^{k-1} \cdot \left(1 - \left(1 - \dfrac{1}{2n}\right)^{k-1}\right)$.

(iii) $P_T(k) = \dfrac{1}{n} \cdot \left(1 - \dfrac{1}{2n}\right)^{k-1} \cdot \left(1 - \left(1 - \dfrac{1}{2n-1}\right)^{k-1}\right)$.

(iv) $P_T(k) = \dfrac{1}{n} \cdot \left(1 - \dfrac{1}{2n}\right)^{k-1} \cdot \left(1 - \left(1 - \dfrac{1}{2n}\right)^{k-1}\right)$.

(v) None of the above.

4. A two-stage experiment is held. In the first stage, we toss a coin n times, where n is a large number. Let X denote the number of heads obtained in these tosses.

   In the second stage, if X is even we roll two dice, while if n is odd we roll only one die. Let S be the sum of upfaces in the roll(s).

   (a) $F_S(5) =$

      (i) $\frac{16}{36}$.
      (ii) $\frac{17}{36}$.
      (iii) $\frac{18}{36}$.
      (iv) $\frac{20}{36}$.
      (v) None of the above.

   (b) Suppose n is even. Then $P(X = n|S = 3) =$

      (i) $\frac{1}{2^n}$.
      (ii) $\frac{2}{2^n}$.
      (iii) $\frac{3}{2^n}$.
      (iv) $\frac{4}{2^n}$.
      (v) None of the above.

Solutions

1. (a) Whichever ranking Nurit selects, the required event will occur if Shoshana selects a completely different ranking; that is, she needs to select a ranking that does not agree with Nurit's on any restaurant. The problem is equivalent to the absent-minded secretary problem, and therefore the probability is approximately $1/e$.

   Thus, (ii) is true.

   (b) There are $\binom{n}{2}$ possibilities to choose the restaurants on whose rank Nurit and Shoshana will agree. For each of these pairs, there is a probability of $\frac{1}{n} \cdot \frac{1}{n-1}$ that the two will agree on their ranks. Once they agree on these two restaurants, we need them to disagree on all others, which event is of probability about $1/e$ according to the preceding part. Altogether, the required probability is approximately

   $$\binom{n}{2} \cdot \frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{1}{e} = \frac{1}{2e}.$$

   Thus, (i) is true.

   (c) Clearly, $X \sim B(10000, 0.2)$, so that

   $$E(X) = 10000 \cdot 0.2 = 2000$$

   and

   $$V(X) = 10000 \cdot 0.2 \cdot (1 - 0.2) = 1600.$$

   By Chebyshev's inequality,

   $$P(|X - 2000| > 120) \leq \frac{1600}{120^2} = \frac{1}{9}.$$

   Passing to the complementary event, we obtain

   $$P(1880 \leq X \leq 2120) \geq 1 - \frac{1}{9} = \frac{8}{9}.$$

   Thus, (iv) is true.

2. (a) Denote by $R_k, k \geq 1$, Reuven's prize due to the k-th digit he guesses. Then

$$R = R_1 + R_2 + \cdots ,$$

and therefore

$$E(R) = E(R_1) + E(R_2) + \cdots .$$

Now, he actually gets to guess the k-th digit only if he has guessed correctly all the first $k-1$ digits, which happens with probability $1/10^{k-1}$. If he gets to this stage, then due to this stage he gets on the average $\frac{0+9}{2} \cdot 10^{k-1}$. Hence

$$E(R_k) = \frac{1}{10^{k-1}} \cdot \frac{0+9}{2} \cdot 10^{k-1} = \frac{9}{2}.$$

It follows that the series expressing $E(R)$ diverges to ∞.
Thus, (iv) is true.

(b) The number of correct guesses is $X - 1$, where $X \sim G(9/10)$. Therefore:

$$E(S) = \sum_{k=1}^{\infty} \frac{1}{10^{k-1}} \cdot \frac{9}{10} \cdot 2^{k-1} = \frac{9}{10} \cdot \frac{1}{1 - 2/10} = \frac{9}{8}.$$

Thus, (iii) is true.

(c) Clearly, $L = Y - 1$, where $Y \sim G(1/10)$. Hence:

$$E(L) = E(Y) - 1 = \frac{1}{1/10} - 1 = 9$$

and

$$V(L) = V(Y) = \frac{1 - 1/10}{(1/10)^2} = 90.$$

Thus, (i) is true.

3. (a) The event $B_{n+1} \cap R_{3n+2}$ decomposes to a disjoint union

$$B_{n+1} \cap R_{3n+2} = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} A_{ij},$$

where A_{ij} is the event that blue i is drawn at step $n+1$ and red j at step $3n+2$. For A_{ij} to occur, we need:

- neither blue i nor red j to be drawn in the first n steps;
- blue i to be drawn in the $(n+1)$-st step;
- red j not to be drawn in steps $n+2$ through $3n+1$;
- red j to be drawn in the $(3n+2)$-nd step.

By the multiplication rule:

$$\begin{aligned} P(A_{ij}) &= \left(1 - \frac{2}{2n}\right)^n \cdot \frac{1}{2n} \cdot \left(1 - \frac{1}{2n}\right)^{2n} \cdot \frac{1}{2n} \\ &\approx \frac{1}{e} \cdot \frac{1}{2n} \cdot \frac{1}{e} \cdot \frac{1}{2n} \\ &= \frac{1}{4e^2n^2}. \end{aligned}$$

By symmetry:

$$P(B_{n+1} \cap R_{3n+2}) = n^2 P(A_{11}) \approx \frac{1}{2e^2}.$$

Thus, (ii) is true.

(b) Let X' and Y' denote the number of blue balls and the number of red balls, respectively, drawn in the first two steps. Since $X = n - X'$ and $Y = n - Y'$, we have $\rho(X,Y) = \rho(X',Y')$.

Unless the white ball put in the urn at the first stage is drawn at the second, which happens with probability $1/2n$, we have $X' + Y' = 2$. Hence, X' and Y' are almost linearly negatively related, so we should have $\rho(X',Y') \approx -1$. Formally, we clearly have

$$E(X') = P(B_1) + P(B_2) = \frac{1}{2} + \left(\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{n-1}{2n}\right) = 1 - \frac{1}{4n}.$$

Also,

$$\begin{aligned} E(X'^2) &= \frac{1}{4} \cdot 0^2 + \left(\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{n+1}{2n}\right) \cdot 1^2 + \frac{1}{2} \cdot \frac{n-1}{2n} \cdot 2^2 \\ &= \frac{3}{2} - \frac{3}{4n}. \end{aligned}$$

Hence:

$$V(X') = \frac{3}{2} - \frac{3}{4n} - \left(1 - \frac{1}{4n}\right)^2$$
$$= \frac{1}{2} - \frac{1}{4n} + \frac{1}{16n^2}.$$

Y' has the same expectation and variance as X'. Now, $X'Y' \neq 0$ only if $X' = Y' = 1$. Consequently:

$$E(X'Y') = P(X' = Y' = 1) = 2P(B_1 \bigcap R_2) = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}.$$

Therefore:

$$\text{Cov}(X', Y') = \frac{1}{2} - \left(1 - \frac{1}{4n}\right)^2.$$

Thus, (iii) is true.

(c) Defining the event of drawing the n-th ball as a success, Z is the number of trials until the first success. Hence, $Z \sim G(1/2n)$.

At each step, starting with the second, there is a probability of $1/2n$ for the step to be a reverse step. Moreover, distinct steps are clearly independent. Thus, $W \sim B(n, 1/2n)$.

Thus, (iii) is true.

(d) The event $\{T = k\}$ is the disjoint union of the event where the n-th blue ball is drawn at stage k and the n-th red ball at one of the preceding $k-1$ steps, and the analogous event in opposite colors. By symmetry, the two events are equi-probable. For the first to happen, we need the n-th blue ball to be drawn (for the first time) at stage k, which happens with probability $(1-1/2n)^{k-1} \cdot 1/2n$, and (given that) the n-th red ball to be drawn earlier, which happens with probability $1 - (1 - 1/(2n-1))^{k-1}$. Hence:

$$P(T = k) = 2 \cdot (1 - 1/2n)^{k-1} \cdot 1/2n \cdot (1 - (1 - 1/(2n-1))^{k-1}).$$

Thus, (iii) is true.

4. (a) We have seen in class that the probability of X being even or odd is $1/2$ each. If X is odd, so that we roll one die, the probability of the sum being up to 5 is clearly $5/6$. If it is even, and we roll two dice, the probability is

$$1/36 + 2/36 + 3/36 + 4/36 = 10/36.$$

It follows that:

$$F_S(5) = P(S \leq 5) = \frac{1}{2} \cdot \frac{5}{6} + \frac{1}{2} \cdot \frac{10}{36} = \frac{20}{36} = \frac{5}{9}.$$

Thus, (iv) is true.

(b) We have:

$$P(X = n|S = 3) = \frac{P(X = n)P(S = 3|X = n)}{P(S = 3)}.$$

Since n is even, the second factor in the numerator is $2/6^2$. Therefore:

$$P(X = n|S = 3) = \frac{1/2^n \cdot 2/6^2}{1/2 \cdot 1/6 + 1/2 \cdot 2/6^2} = \frac{1}{2^{n+1}}.$$

Thus, (iv) is true.