Final #2

Mark the correct answer for each part of each question.

1. Ravit and Sarit compete on the title of Queen of Hearts, as follows. Each of them in turn (Ravit first, then Sarit, then Ravit again, and so forth) draws a random card from a full deck. The draws are **with replacement**. The first to draw the queen of hearts wins the game and gets as a prize $\binom{X}{2}$ shekels from the other player, where X is the number of draws in the game. Denote by R the winnings of Ravit and by S the winnings of Sarit. (For example, if Ravit wins in her first draw then $X = 1$ and $R = \binom{1}{2} = 0, S = -R = 0$; if Ravit does not succeed in her first attempt and Sarit does, then $X = 2, S = \binom{2}{2} = 1, R = -1$.)

   (a) If it is known that Ravit won, then the probability that she won at stage 13 is:

   (i) $\dfrac{51^{12} \cdot 101}{52^{14}}$.

   (ii) $\dfrac{2 \cdot 51^{12}}{52^{14}}$.

   (iii) $\dfrac{51^{12} \cdot 103}{52^{14}}$.

   (iv) $\dfrac{2 \cdot 51^{12}}{52^{13}}$.

   (v) None of the above.

(b) $E(R) =$

(i) $\frac{-51 \cdot 52}{103^3}$.

(ii) $\frac{-52^2}{103^3}$.

(iii) $\frac{51 \cdot 52}{103^3}$.

(iv) $\frac{52^2}{103^3}$.

(v) None of the above.

(c) Markov's Inequality yields: $P(X \geq 10^6) \leq$

(i) $\frac{51}{10^6}$.

(ii) $\frac{52}{10^6}$.

(iii) $\frac{103}{10^6}$.

(iv) $\frac{104}{10^6}$.

(v) None of the above.

Remark: We mean the best bound that can be obtained. For example, if (i) is correct, then (ii)-(iv) are correct as well, but only (i) should be marked as the correct answer.

(d) We repeat the game 156 times. The probability that, out of these 156 times, Ravit wins in the first draw exactly 4 times, is approximately:

(i) $\frac{27}{8e^3}$.

(ii) $\frac{33}{8e^3}$.

(iii) $\frac{35}{8e^3}$.

(iv) $\frac{39}{8e^3}$.

(v) None of the above.

(e) Now suppose that the draws in the game are **without replacement** and that the referee gets one shekel per round. Let J be the total winnings of the referee. (For example, if the queen of hearts is the last card to be drawn, then $J = 52$.) Then $V(J) =$

(i) $\frac{2701}{12}$.

(ii) $\frac{2703}{12}$.

(iii) $\frac{2705}{12}$.

(iv) $\frac{2707}{12}$.

(v) None of the above.

2. A two-stage experiment is held. In the first stage, a die is rolled until it lands on 6. Denoting by Y the number of rolls, we then roll the die again Y times for the second stage. Let S_1 and S_2 denote the sum of outcomes in the first and the second stages, respectively. (For example, if it showed $3, 1, 6$ in the first three rolls and $2, 5, 2$ in the next three, then $Y = 3, S_1 = 10, S_2 = 9$.)

   (a) $E(S_1) =$

   (i) 18.

   (ii) 20.

   (iii) 21.

   (iv) 24.

   (v) None of the above.

   (b) $V(S_2) =$

   (i) 355.

   (ii) 365.

   (iii) 375.

   (iv) 385.

   (v) None of the above.

   (c) The probability that the sequence of outcomes in the first stage is identical to the sequence of outcomes in the second (for example, in the first stage the outcomes are $3, 2, 2, 6$ and in the second again $3, 2, 2, 6$) is:

(i) $\frac{1}{28}$.

(ii) $\frac{1}{29}$.

(iii) $\frac{1}{30}$.

(iv) $\frac{1}{31}$.

(v) None of the above.

(d) $F_{S_1,S_2}(7.5, 10.5) =$

(i) $\frac{1}{6} + \frac{31}{6^4}$.

(ii) $\frac{1}{6} + \frac{32}{6^4}$.

(iii) $\frac{1}{6} + \frac{33}{6^4}$.

(iv) $\frac{1}{6} + \frac{34}{6^4}$.

(v) None of the above.

(e) Let X_1 denote the number of even outcomes in the first stage (including the outcome of 6 at the end of this stage) and X_2 the number of even outcomes in the second. Then:

(i) X_1 is geometrically distributed and X_2 is binomially distributed.

(ii) X_1 is geometrically distributed but X_2 is not binomially distributed.

(iii) Both X_1 and X_2 are geometrically distributed.

(iv) Both X_1 and X_2 are binomially distributed.

(v) None of the above.

3. An urn contains n balls, enumerated by $1, 2, \ldots, n$. Two balls are drawn randomly **without replacement**. Let X denote the number on the first and Y the number on the second.

(a) $\rho(X, Y) =$

(i) $\dfrac{-1}{n-1}$.

(ii) $\dfrac{-1}{\sqrt{n^2-1}}$.

(iii) $\dfrac{-1}{\sqrt{n^2+1}}$.

(iv) $\dfrac{-1}{n+1}$.

(v) None of the above.

(b) $P(X = Y + 1 \mid X > Y) =$:

(i) $\dfrac{1}{n}$.

(ii) $\dfrac{2}{n}$.

(iii) $\dfrac{3}{n}$.

(iv) $\dfrac{4}{n}$.

(v) None of the above.

(c) Suppose that $n = 3$. We want to bound $P(X \geq 3)$ from above. Using Chernoff's bound with $t = \log 3$, we obtain

(i) $P(X \geq 3) \leq \dfrac{11}{27}$.

(ii) $P(X \geq 3) \leq \dfrac{12}{27}$.

(iii) $P(X \geq 3) \leq \dfrac{13}{27}$.

(iv) $P(X \geq 3) \leq \dfrac{14}{27}$.

(v) None of the above.

Solutions

1. (a) Clearly, X is $G(1/52)$-distributed. The event that Ravit wins at stage 13 means that $X = 13$, the probability of which is $(51/52)^{12} \cdot 1/52$. Ravit wins if X is odd, the probability for which is

$$\sum_{k=0}^{\infty} \left(\frac{51}{52}\right)^{2k} \cdot \frac{1}{52} = \frac{1}{52} \cdot \frac{1}{1-(51/52)^2} = \frac{52}{103}.$$

Hence the required probability is

$$\frac{(51/52)^{12} \cdot 1/52}{52/103} = \frac{51^{12} \cdot 103}{52^{14}}.$$

Thus, (iii) is true.

(b) The contribution to R is positive if the process ends at an odd stage (Ravit wins) and is negative if it ends at an even stage (Sarit wins). Therefore

$$\begin{aligned} E(R) &= \sum_{n=1}^{\infty} (-1)^{n-1} \binom{n}{2} \left(\frac{51}{52}\right)^{n-1} \cdot \frac{1}{52} \\ &= -\frac{1}{51} \sum_{n=1}^{\infty} \binom{n}{2} \left(\frac{-51}{52}\right)^{n} \\ &= -\frac{1}{51} \cdot \frac{(51/52)^2}{(1-(-51/52))^3} \\ &= -\frac{51 \cdot 52}{103^3}. \end{aligned}$$

Thus, (i) is true.

(c) Since $X \sim G(1/52)$, we have $E(X) = 52$. By Markov's inequality,

$$P(X \geq 10^6) \leq \frac{E(X)}{10^6} = \frac{52}{10^6}.$$

Thus, (ii) is true.

(d) In each play, the probability that Ravit wins on the first draw is $1/52$. Hence, denoting by N the number of times she wins over 156 independent repetitions, we have $N \sim B(156, 1/52)$. The latter distribution is approximately $P(\lambda)$, where

$$\lambda = 156 \cdot \frac{1}{52} = 3.$$

It follows that

$$P(N = 4) \approx e^{-3} \cdot \frac{3^4}{4!} = \frac{27}{8e^3}.$$

Thus, (i) is true.

(e) Without replacement, the Queen of Hearts is equally likely to appear in any of the 52 positions, so $J \sim U[1, 52]$. Hence

$$V(J) = \frac{52^2 - 1}{12} = \frac{2703}{12}.$$

Thus, (ii) is true.

2. (a) Given Y, the first sum S_1 consists of $Y - 1$ outcomes $U_i, 1 \leq i \leq Y - 1$, that are distributed $U[1, 5]$ and an additional 6. Since $Y \sim G(1/6)$:

$$\begin{aligned} E(S_1) &= E(E(S_1 \mid Y)) \\ &= E\left(E\left(\textstyle\sum_{i=1}^{Y-1} U_i + 6\right)\right) \\ &= E(3(Y-1) + 6) \\ &= 3E(Y) + 3 \\ &= 21. \end{aligned}$$

Thus, (iii) is true.

(b) Given Y, the sum S_2 consists of Y outcomes U_i' that are distributed $U[1,6]$ each. Therefore:

$$
\begin{aligned}
V(S_2) &= E(V(S_2 \mid Y)) + V(E(S_2 \mid Y)) \\
&= E\left(V\left(\sum_{i=1}^{Y} U_i'\right)\right) + V\left(E\left(\sum_{i=1}^{Y} U_i'\right)\right) \\
&= E\left(Y \cdot \frac{6^2-1}{12}\right) + V\left(Y \cdot \frac{7}{2}\right) \\
&= 6 \cdot \frac{6^2-1}{12} + \frac{1-1/6}{(1/6)^2} \cdot \left(\frac{7}{2}\right)^2 \\
&= \frac{35}{2} + 30 \cdot \left(\frac{7}{2}\right)^2 \\
&= 385.
\end{aligned}
$$

Thus, (iv) is true.

(c) The probability that the sequences are identical, conditioned on Y, is $1/6^k$. Therefore the required probability p is:

$$
\begin{aligned}
p &= \sum_{k=1}^{\infty} P(Y=k)\left(\frac{1}{6}\right)^k \\
&= \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{k-1} \frac{1}{6} \cdot \left(\frac{1}{6}\right)^k \\
&= \frac{1}{36} \sum_{k=1}^{\infty} \left(\frac{5}{36}\right)^{k-1} \\
&= \frac{1}{36} \cdot \frac{1}{1-5/36} \\
&= \frac{1}{31}.
\end{aligned}
$$

Thus, (iv) is true.

(d) We require $S_1 \le 7.5$ and $S_2 \le 10.5$. This happens if and only if either (i) $Y = 1$ (in which case $S_1 = 6$ and $S_2 \le 6$), or (ii) $Y = 2$ with the first outcome 1 (so $S_1 = 7$) and in stage two the sum of the two outcomes does not exceed 10. It follows that:

$$F_{S_1,S_2}(7.5, 10.5) = \frac{1}{6} + \frac{1}{6} \cdot \left(1 - \frac{3}{6^2}\right) = \frac{1}{6} + \frac{33}{6^4}.$$

Thus, (iii) is true.

(e) When considering X_1, we may ignore odd outcomes. Considering an outcome of 2 or 4 as a failure and of 6 as a success, we see that $X_1 \sim G(1/3)$. As to X_2, since it is unbounded it is not binomial, and since it may be 0 it cannot be geometric.

Thus, (ii) is true.

3. (a) Since $X, Y \sim U[1, n]$, we have

$$E(X) = E(Y) = \frac{n+1}{2}$$

and

$$V(X) = V(Y) = \frac{n^2 - 1}{12}.$$

Now:

$$\begin{aligned}
E(XY) &= E(E(XY \mid Y)) \\
&= E(Y \cdot E(X \mid Y)) \\
&= E\left(Y \cdot \frac{n(n+1)/2 - Y}{n-1}\right) \\
&= \frac{n(n+1)}{2(n-1)} \cdot \frac{n+1}{2} - \frac{1}{n-1} \cdot \frac{(n+1)(2n+1)}{6} \\
&= \frac{(n+1)(3n+2)}{12},
\end{aligned}$$

and therefore:

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \frac{(n+1)(3n+2)}{12} - \frac{(n+1)^2}{4} \\ &= -\frac{n+1}{12}.\end{aligned}$$

Finally:

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{-\dfrac{n+1}{12}}{\dfrac{n^2-1}{12}} = -\frac{1}{n-1}.$$

Thus, (i) is true.

(b) The event $\{X = Y+1\}$ consists of the points $(2,1),(3,2),\ldots,(n,n-1)$, namely $n-1$ points out of $n(n-1)$. By symmetry, $P(X > Y) = 1/2$. It follows that:

$$P(X = Y+1 \mid X > Y) = \frac{\frac{n-1}{n(n-1)}}{1/2} = \frac{2}{n}.$$

Thus, (ii) is true.

(c) For $t = \log 3$ we have:

$$\begin{aligned}P(X \geq 3) &\leq \frac{E\big(e^{tX}\big)}{e^{3t}} \\ &= \frac{\frac{1}{3}\left(e^t + e^{2t} + e^{3t}\right)}{e^{3t}} \\ &= \frac{\frac{1}{3}\left(3+9+27\right)}{27} \\ &= \frac{13}{27}.\end{aligned}$$

Thus, (iii) is true.